**P-bursts in lργ-spaces**

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Abstract: In [3], the author introduced linear partition γ-codes (or lργ-codes) as a natural generalization of error control codes endowed with the Rosenbloom-Tsfasman(RT) metric [8] to block coding and obtained various lower and upper bounds for the detection and correction of random block errors. However, a practical situation is when block errors are not randomly scattered but are confined to consecutive blocks. These situations arise for example, in computer and communication systems, mobile systems, aircrafts and satellites where the presence of strong electromegnatic waves in the environment or the bombardment of energetic particles such as cosmic particles on a RAM chip of a computer is highly likely to upset data in consecutive blocks. Motivated by this, in this paper, we propose a new model of block errors viz. Partition-burst errors (or P-burst errors) and obtain bounds for the correction and detection of the same in lργ-codes.

Key words: Block codes, γ-metric, bursts

INTRODUCTION

π-codes introduced by K. Feng et al. [1] is a natural generalization of Hamming metric to block space $F_q^n = \bigoplus_{i=1}^s F_q^{n_i}$ where $n = \sum_{i=1}^s n_i$, $n_1 \leq n_2 \leq \cdots \leq n_s$.

Also, linear partition error control codes in the γ-metric [3] is a natural generalization of error control codes endowed with the Rosenbloom-Tsfasman(RT) metric [8] and has applications in different area of combinatorial/discrete mathematics, e.g. in the theory of uniform distributions, experimental designs, cryptography etc. Since we know that the RT-metric [4, 5, 6, 8] is stronger than the Hamming metric, therefore linear partition γ-codes (or lργ-codes) introduced by the author in [3] are stronger than the π-codes [1]. In [3], the author obtained various upper and lower bounds for the detection and correction of random block errors where random block errors are independent block errors that can occur anywhere within the codeword. However, a practical situation is when block errors are not randomly scattered but are confined to consecutive blocks. These situations arise for example, in computer and communication systems, mobile systems, aircrafts and satellites where the presence of strong electromegnatic waves in the environment or the bombardment of energetic particles such as cosmic particles on a RAM chip of a computer is highly likely to upset data in consecutive blocks. Motivated by this, in this paper, we propose a new model of block errors viz. Partition-burst errors or P-burst errors and obtain bounds for the correction and detection of the same in lργ-codes.

2. DEFINITIONS AND NOTATIONS

Let $q, n$ be positive integers with $q = p^m$, a power of a prime number $p$. Let $F_q$ be the finite field having $q$ elements. A partition $P$ of the positive integer $n$ is defined as:

$$P : n = n_1 + n_2 + \cdots + n_s,$$

$$1 \leq n_1 \leq n_2 \leq \cdots \leq n_s, s \geq 1.$$

The partition $P$ is denoted $s$

$$P : n = [n_1][n_2] \cdots [n_s].$$

In the case, when

$$P : n = [n_1]^{r_1}[n_2]^{r_2} \cdots [n_t]^{r_t},$$

we write

$$P : n = [n_1]^r_1[n_2]^r_2 \cdots [n_t]^r_t,$$

where

$$n_1 < n_2 < \cdots < n_t.$$

Further, given a partition $P : n = [n_1][n_2] \cdots [n_s]$ of a positive integer $n$, the linear space $F_q^n$ over $F_q$ can be viewed as the direct sum

$$F_q^n = F_q^{n_1} \oplus F_q^{n_2} \oplus \cdots \oplus F_q^{n_s},$$

or equivalently

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_s,$$

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where \( V = F_q^n \) and \( V_i = F_q^{n_i} \) for all \( i \leq i \leq s \).

Consequently, each vector \( v \in F_q^n \) can be uniquely written as \( v = (v_1, v_2, \ldots, v_s) \) where \( v_i \in V_i = F_q^{n_i} \) for all \( 1 \leq i \leq s \). Here \( v_i \) is the \( i \)-th block of block size \( n_i \) of the vector \( v \).

**Definition 1.** Let \( v = (v_1, v_2, \ldots, v_s) \in F_q^n = F_q^{n_1} \oplus F_q^{n_2} \oplus \cdots \oplus F_q^{n_s} \) be an s-block vector of length \( n \) over \( F_q \) corresponding to the partition \( P : n = [n_1][n_2] \cdots [n_s] \) of \( n \). We define the \( \gamma \)-weight of the block vector \( v \) as

\[
\gamma(v) = \max_{i=1}^{s} \{ |v_i| \neq 0 \}.
\]

The \( \gamma \)-distance \( d_\gamma(u, v) \) between two s-block vectors of length \( n \) viz. \( u = (u_1, u_2, \ldots, u_s) \) and \( v = (v_1, v_2, \ldots, v_s) \), \( u_i, v_i \in F_q^{n_i} (1 \leq i \leq s) \) corresponding to the partition \( P \) is defined as

\[
d_\gamma(u, v) = \gamma(v - u) = \max_{i=1}^{s} \{ |v_i - u_i| \neq 0 \}.
\]

Then \( d_\gamma(u, v) \) is a metric on \( F_q^n = F_q^{n_1} \oplus F_q^{n_2} \oplus \cdots \oplus F_q^{n_s} \).

**Note.** Once the partition \( P \) is specified, we will denote the \( \gamma \)-weight \( \gamma(v) \) by \( \gamma(v) \) and \( \gamma \)-distance \( d_\gamma \) by \( d_\gamma \) respectively.

**Definition 2.** A linear partition \( \gamma \)-code (or \( lp \gamma \)-code) \( V \) of length \( n \) corresponding to the partition \( P : n = [n_1][n_2] \cdots [n_s], n_1 \leq n_2 \leq \cdots \leq n_s \) is a \( F_q \)-linear subspace of \( F_q^n = F_q^{n_1} \oplus F_q^{n_2} \oplus \cdots \oplus F_q^{n_s} \) equipped with the \( \gamma \)-metric and is denoted as \([n, k, d_\gamma; P] \) code where \( k = \dim_{F_q}(V) \) and \( d_\gamma = d_\gamma(V) = \min \gamma \)-distance of the code \( V \).

**Remark 3.**

1. For \( P : n = [1]^n \), the \( \gamma \)-metric (or \( \gamma \)-weight) reduces to the \( \rho \)-metric (or \( \rho \)-weight) respectively [8].

2. For a partition \( P : n = [n_1][n_2] \cdots [n_s] \) of the positive integer \( n \), the \( \gamma \)-distance (or \( \gamma \)-weight) is always greater than or equal to the \( \pi \)-distance (or \( \pi \)-weight) [1] respectively, i.e.

\[
\gamma \text{-distance} \geq \pi \text{-distance}
\]

and

\[
\gamma \text{-weight} \geq \pi \text{-weight}.
\]

**3. P-BURSTS IN \( lp \gamma \)-SPACES**

We define \( P \)-bursts in \( lp \gamma \)-spaces as follows:

**Definition 4.** Let \( n \) be a positive integers and \( P : n = [n_1][n_2] \cdots [n_s], n_1 \leq n_2 \leq \cdots \leq n_s \) be a partition of \( n \). A \( P \)-burst of block length \( b (1 \leq b \leq s) \) is a vector

\[
v = (v_1, v_2, \ldots, v_s) \in F_q^n = \bigoplus_{i=1}^{s} F_q^{n_i}
\]

such that all the nonzero blocks in \( v \) are confined to some \( b \) consecutive block positions, the first and last of which are nonzero.

**Definition 5.** A \( P \)-burst of block length \( b \) or less \((1 \leq b \leq s)\) is a \( P \)-burst of block length \( t \) where \( 1 \leq t \leq b \leq s \).

**Remark 6.** For \( P : n = [1]^n \), the definition of \( P \)-burst reduces to that of classical burst given by Fire [2].

**Example 7.** Let \( q = 2, n = 9 \) and \( P : 9 = [1]^4[2][3] \).

Here \( s = 6 \). The vectors \( v_1 = (001\overline{1}0\overline{1}0\overline{1}0\overline{1}0\overline{1}) \) and \( v_2 = (00\overline{1}0\overline{1}0\overline{1}0\overline{1}0\overline{1}) \) and \( v_3 = (1\overline{1}0\overline{1}0\overline{1}0\overline{1}0\overline{1}) \) are all \( P \)-bursts of block length 4 or less.

We now obtain a necessary bound analogous to Fire’s bound [2] for the correction of all \( P \)-bursts of block length \( b \) or less \((1 \leq b \leq s)\) in \( lp \gamma \)-codes.

**Theorem 8.** Let \( n \) be a positive integers and \( P : n = [n_1][n_2] \cdots [n_s], n_1 \leq n_2 \leq \cdots \leq n_s \) be a partition of \( n \). The number of parity check digits required in an \([n, k; P] \) \( lp \gamma \)-code over \( F_q \) that corrects all \( P \)-bursts of block length \( b \) or less is at least

\[
[\log_q(B_q(b, [n_1][n_2] \cdots [n_s])),]
\]

where

\[
B_q(b, [n_1][n_2] \cdots [n_s]) = 1 + \sum_{i=1}^{s} (q^{n_i} - 1) + \sum_{j=2}^{b} \sum_{l=1}^{s-j+1} q^{(i+j-2)\left(\prod_{l=i+1}^{i+j-1} q^{n_e}\right)\times (q^{n_1}+j-1 - 1)}
\]

is the number of \( P \)-bursts of block length \( b \) or less including the vector of all zeros over \( F_q \) corresponding to the partition \( P : n = [n_1][n_2] \cdots [n_s] \) and \([x]\) denotes the smallest integer greater than or equal to \( x \).

**Proof.** We enumerate the number of \( P \)-bursts of block length \( b \) or less as follows:

The number of \( P \)-bursts of block length 1 \( = \sum_{i=1}^{s} (q^{n_i} - 1) \).

The number of \( P \)-bursts of block length \( j (2 \leq j \leq b) \)

\[
= \sum_{l=1}^{s-j+1} (q^{n_1} - 1) \times (q^{n_1}+j-1 - 1).
\]
Thus, the total number of $P$-bursts of block length $b$ or less including the pattern of all zeros is given by
\[ 1 + \text{number of } P \text{-bursts of block length } 1 + \]
\[ + \sum_{j=2}^{b} \text{number of } P \text{-bursts of block length } j \]
\[ = B_q(b, [n_1][n_2] \cdots [n_s]) \]
given by (1).

The result now follows from the fact that in order to correct all $P$-bursts of block length $b$ or less, the number of correctable $P$-bursts enumerated in (1) must belong to distinct cosets of the standard array and the number of available cosets is $q^{n-k}$.

The necessary bound for the correction of weighted $P$-bursts runs as follows:

**Theorem 9.** Let $n$ be a positive integer and $P : n = [n_1][n_2] \cdots [n_s], n_1 \leq n_2 \leq \cdots \leq n_s$ be a partition of $n$. The number of parity check digits required in an $[n,k;P]$ $(p\gamma)$-code over $\mathbb{F}_q$ that corrects all $P$-bursts of block length $b$ or less with $\gamma$-weight $w$ or less $(1 \leq w \leq s)$ is at least
\[ \lceil \log_q(B_q(b, w, [n_1][n_2] \cdots [n_s])) \rceil, \]
where
\[ B_q(b, w, [n_1][n_2] \cdots [n_s]) \]
\[ = 1 + \sum_{i=1}^{w} (q^{n_i} - 1) + \sum_{j=2}^{b} \sum_{l=1}^{w-j+1} \]
\[ \left( q^{n_i-l} \prod_{l=i+1}^{i+j-2} q^{n_r} \right) \times \left( q^{n_i+j-1} - 1 \right). \]

**Proof.** The proof is similar to the proof of Theorem 8.

**Example 10.** Let $q = 2, b = 2$ and $n = 6$. Let $P : 6 = [n_1][n_2][n_3][n_4] = [1][1][2][2]$ be a partition of $n = 6$ with $s = 4$. The number of $P$-bursts of block length 2 or less over $\mathbb{F}_2$ corresponding to the partition $P$ is given by
\[ B_2(2, [n_1][n_2][n_3][n_4]) = B_2(2, [1][1][2][2]) \]
\[ = 1 + \sum_{i=1}^{4} (2^{n_i} - 1) \]
\[ + \sum_{l=1}^{3} (2^{n_i} - 1)(2^{n_{i+1}} - 1) \]
\[ = 1 + (1 + 1 + 3 + 3) + \]
\[ + (1 + 3 + 9) = 22. \]

These 22 $P$-bursts of block length 2 or less over $\mathbb{F}_2$ in a 4-block vector of length $n = 6$ corresponding to the partition $P : 6 = [1][1][2][2]$ are given by
\[ v_0 = (0:0:00:00), \]
\[ v_1 = (1:0:00:00), \]
\[ v_2 = (0:1:00:00), \]
\[ v_3 = (0:0:11:00), \]
\[ v_4 = (0:0:10:00), \]
\[ v_5 = (0:0:11:00), \]
\[ v_6 = (0:0:10:01), \]
\[ v_7 = (0:0:00:10), \]
\[ v_8 = (0:0:00:11), \]
\[ v_9 = (1:1:00:00), \]
\[ v_{10} = (0:1:01:00), \]
\[ v_{11} = (0:1:10:00), \]
\[ v_{12} = (0:1:11:00), \]
\[ v_{13} = (0:0:11:01), \]
\[ v_{14} = (0:0:10:01), \]
\[ v_{15} = (0:0:11:01), \]
\[ v_{16} = (0:0:01:10), \]
\[ v_{17} = (0:0:10:10), \]
\[ v_{18} = (0:0:11:10), \]
\[ v_{19} = (0:0:11:10), \]
\[ v_{20} = (0:0:11:11), \]
\[ v_{21} = (0:0:11:11). \]

In view of Theorem 9, the $(p\gamma)$-code of length $n = 6$ corresponding to the partition $P : 6 = [1][1][2][2]$ over $\mathbb{F}_2$ that corrects all $P$-bursts of block length 2 or less must have at least $\lceil \log_2(22) \rceil = 5$ parity checks. Table 1 gives various partitions of $n = 6$ and the number of parity check digits required to correct all $P$-bursts of block length 2 or less over $\mathbb{F}_2$ for $q = 2$ corresponding to the given partitions.
There shall always exist an $n$-dimensional code over $\mathbb{F}_q$ where $P : n = [n_1][n_2] \cdots [n_s]$, $n_1 \leq n_2 \leq \cdots \leq n_s$ that corrects all $P$-bursts of block length $b$ or less ($b < s/2$) satisfying the inequality

$$q^{n-k} \geq \left( q^{n_1} - 1 \right) \times \left( B_q(b, [n_1][n_2]) \cdots [n_s-b]) \right)$$

where $B_q(b, [n_1][n_2]) \cdots [n_s-b])$ is given by (1).

**Proof.** The existence of such an $lp\gamma$-code over $\mathbb{F}_q$ will be proven by constructing an $(n-k) \times n$ parity check matrix $H$ for the desired code. To correct all $P$-burst of block length $b$ or less, it is necessary and sufficient that no code vector consists of the sum or difference of two $P$-bursts of block length $b$ or less. Thus, it is necessary and sufficient that no linear combination involving two sets of $b$ or fewer consecutive blocks of columns of $H$ is zero. So, we construct the parity check matrix $H = [H_1, H_2, \cdots , H_{m-1}]$ where $H_i = (H_1^{(i)}, H_2^{(i)}, \cdots , H_{m}^{(i)})$ is the $i^{th}$ block of columns of $H$ as follows:

Choose any nonzero block of $n_1$ columns of length $(n-k)$ as the first block $H_1$ of the parity check matrix $H$. After having selected the first $(m-1)$ blocks viz. $H_1, H_2, \cdots , H_{m-1}$ of block sizes $n_1, n_2, \cdots , n_{m-1}$ respectively, we lay down the following condition to add the $m^{th}$ block $H_m$ ($2 \leq m \leq s$) of block size $n_m$ as follows:

The $m^{th}$ block $H_m = (H_1^{(m)}, H_2^{(m)}, \cdots , H_{m}^{(m)})$ may be added to $H$ provided that

$$(\alpha_{m-b+1}H_{m-b+1} + + \alpha_{m-b+2}H_{m-b+2} + + \cdots + \alpha_{m-1}H_{m-1} + \alpha_mH_m) + + (\beta_iH_i + \beta_{i+1}H_{i+1} + + \cdots + \beta_{i+b-1}H_{i+b-1}) \neq 0,$$

where $i$ is a positive integer lying between $1$ and $m - 2b + 1$ and

$$\alpha_i = (\alpha_1^{(i)}, \alpha_2^{(i)}, \cdots , \alpha_n^{(i)}) \in \mathbb{F}_q^n$$

for all $m - b + 1 \leq i \leq m$,

$$\beta_i = (\beta_1^{(i)}, \beta_2^{(i)}, \cdots , \beta_n^{(i)}) \in \mathbb{F}_q^n$$

for all $1 \leq i \leq m - i + b - 1$.

Condition (4) clearly ensures that there will be no code vector that can be expressed as a sum or a difference of two vectors each of which is a $P$-burst of block length $b$ or less.

To calculate the number of coefficient tuples $\beta_i$s is equivalent to enumerating the number of $P$-bursts of block length $b$ or less in a vector consisting of $(m-b)$ block with block sizes $n_1, n_2, \cdots , n_{m-b}$ respectively. The number of such $P$-bursts is given by $B_q(b, [n_1][n_2] \cdots [n_{m-b}])$. We note that the number $B_q(b, [n_1][n_2] \cdots [n_{m-b}])$ include the case when all $\beta_i$s are zero tuples.

The coefficient tuples $\alpha_i$s ($m - b + 1 \leq l \leq m$)

$$\sum_{l=1}^{m} \sum_{n_l} q^{l-m+b+1} - 1 \times \left( B_q(b, [n_1][n_2] \cdots [n_{m-b}]) \right)$$

can be chosen in $(q^{l-m+b+1} - 1)$ ways. Thus the total number of choices of coefficient tuples $\alpha_i$s and $\beta_i$s is given by

$$\sum_{l=1}^{m} q^{l-m+b+1} - 1 \times \left( B_q(b, [n_1][n_2] \cdots [n_{m-b}]) \right).$$

If the number of possible vectors of length $(n-k)$ is greater than or equal to number enumerated in (5), then the $m^{th}$ block of block size $n_m$ can be added to $H$ i.e.
$m^{th}$ can be added to $H$ provided
\[ q^{n-k} \geq \left( q^{d=m-b+1} - 1 \right) \times \left( B_q(b, [n_1][n_2] \cdots [n_{m-b}]) \right). \] (6)

The inequality (3) now follows from the fact that there are $s$ blocks in $H$ of sizes $n_1 \leq n_2 \leq \cdots \leq n_s$ and therefore taking $m = s$ in (6).

The sufficient bound for the correction of weighted $P$-bursts runs as follows:

**Theorem 12.** There shall always exist an $[n, k; P]$ $lp\gamma$-code over $F_q$ where $P : n = [n_1][n_2] \cdots [n_s], n_1 \leq n_2 \leq \cdots \leq n_s$ that correct all $P$-bursts of block length $b$ or less ($b < s/2$) with $\gamma$-weight $w$ or less ($1 \leq w \leq s$) satisfying the inequality
\[ q^{n-k} \geq \left( q^{d=w-b+1} - 1 \right) \times \left( B_q(b, w, [n_1][n_2] \cdots [n_{w-b}]) \right) \] (7)

where $B_q(b, w, [n_1][n_2] \cdots [n_{w-b}])$ is given by (2).

**Proof.** The proof is similar to the proof of Theorem 11. $\square$

**Example 13.** Let $q = 2, b = 2$ and $n = 6$. Let $P : 6 = [n_1][n_2] \cdots [n_s] = [1]^6$ be a partition of $n = 6$ with $s = 6$. We compute the right hand side of (3) for these values of parameters i.e.
\[
\text{R.H.S. of (3)} = \left( 2^{l=6} - 1 \right) \times \left( B_2(2, [n_1][n_2] \cdots [n_4]) \right) \\
= (2^{n_2+n_3} - 1) \times \\
\times (1 + 4 + \sum_{i=1}^{3} (2^{n_i} - 1)(2^{n_i+1} - 1)) \\
= 24.
\]

Therefore on taking $n - k = 5$, inequality (3) is satisfied and hence there exists a $[6, 1; P]$ where $P : [6] = [1]^6$ $lp\gamma$-code over $F_2$ correcting all $P$-bursts of block length 2 or less. Consider the following $5 \times 6$ parity check matrix $H$ constructed by the algorithm discussed in Theorem 11:
\[
H = \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}_{5 \times 6}
\]

The $lp\gamma$-code $V$ with $H$ as the parity check matrix corrects all $P$-bursts of block length 2 or less. This is justified by Table 2 which shows that syndromes of all $P$-bursts of block length 2 or less are all distinct.

**Table 2**

<table>
<thead>
<tr>
<th>P-bursts of block length 2 or less</th>
<th>Syndromes</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0:0:0:0:0:0)</td>
<td>(00000)</td>
</tr>
<tr>
<td>(1:0:0:0:0:0)</td>
<td>(10000)</td>
</tr>
<tr>
<td>(0:1:0:0:0:0)</td>
<td>(01000)</td>
</tr>
<tr>
<td>(0:0:1:0:0:0)</td>
<td>(00100)</td>
</tr>
<tr>
<td>(0:0:0:1:0:0)</td>
<td>(00010)</td>
</tr>
<tr>
<td>(0:1:1:0:0:0)</td>
<td>(11100)</td>
</tr>
<tr>
<td>(0:1:0:1:0:0)</td>
<td>(01110)</td>
</tr>
<tr>
<td>(1:1:0:0:0:0)</td>
<td>(11000)</td>
</tr>
<tr>
<td>(0:0:1:1:0:0)</td>
<td>(11110)</td>
</tr>
<tr>
<td>(0:0:0:1:1:0)</td>
<td>(00110)</td>
</tr>
<tr>
<td>(0:0:0:0:1:1)</td>
<td>(10010)</td>
</tr>
</tbody>
</table>

**Example 14.** Let $q = 2, b = 1$ and $n = 6$. Let $P : 6 = [2]^3$ be a partition of $n = 6$. Thus $n_1 = n_2 = n_3 = 2$ and $s = 3$. On computing the right hand side of (3) for these values of parameters, we get
\[
\text{R.H.S. of (3)} = 3 \times B_2(1, [n_1][n_2]) \\
= 3(1 + (2^{n_1} - 1) + (2^{n_2} - 1)) \\
= 21.
\]

Therefore, on choosing $n - k = 5$, inequality (3) is satisfied and hence there exists a $[6, 1; P]$ where $P : [6] = [2]^3$ $lp\gamma$-code over $F_2$ correcting all $P$-bursts of block length 1 or less. Consider the following $5 \times 6$ parity
check matrix $H$ constructed by the algorithm discussed in Theorem 11

$$
H = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}_{5 \times 6}
$$

The $l(pγ)$-code $V$ with $H$ as the parity check matrix corrects all $P$-bursts of block length 1 or less as seen from the Table 3 which shows that syndromes of all $P$-bursts of block length 1 or less are all distinct.

**Table 3**

<table>
<thead>
<tr>
<th>$P$-bursts of block length 1 or less</th>
<th>Syndromes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(00;0000)$</td>
<td>$(00000)$</td>
</tr>
<tr>
<td>$(10;0000)$</td>
<td>$(10000)$</td>
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<tr>
<td>$(11;0000)$</td>
<td>$(11000)$</td>
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<tr>
<td>$(00;1000)$</td>
<td>$(00100)$</td>
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<tr>
<td>$(00;0100)$</td>
<td>$(00010)$</td>
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<tr>
<td>$(00;1100)$</td>
<td>$(00110)$</td>
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<tr>
<td>$(00;0010)$</td>
<td>$(00001)$</td>
</tr>
<tr>
<td>$(00;0001)$</td>
<td>$(01110)$</td>
</tr>
<tr>
<td>$(00;0000)$</td>
<td>$(01111)$</td>
</tr>
</tbody>
</table>

**Remark 15.** Let $b = bg$ be the largest value of $b$ satisfying inequality (3), then for $b = bg + 1$, the opposite inequality is satisfied and we have the following theorem:

**Theorem 16.** There exists an $[n, k; P]$ $l(pγ)$-code over $F_q$ where $P : n = [n_1][n_2] \cdots [n_s], n_1 \leq n_2 \leq \cdots \leq n_s$ that corrects any single $P$-burst of block length $bg < s/2$ or less for which the following inequality is satisfied:

$$
q^{n-k} < \left( q^{l(s-bg)} - 1 \right) \times \\
\times B_q(bg + 1, [n_1][n_2] \cdots [n_{s-bg-1}]),
$$

where $B_q(bg+1, [n_1][n_2] \cdots [n_{s-bg-1}])$ is given by (1).

\[\text{4. NECESSARY AND SUFFICIENT CONDITIONS FOR THE DETECTION AND CORRECTION OF } P\text{-BURSTS IN } l(pγ)\text{-CODES}\]

In this section, we first obtain necessary and sufficient conditions for the detection of $P$-bursts in $l(pγ)$-codes and then obtain necessary condition for the simultaneous detection and correction of the same.

**Theorem 17.** An $[n, k; P]$ $l(pγ)$-code over $F_q$ where $P : n = [n_1][n_2] \cdots [n_s], n_1 \leq n_2 \leq \cdots \leq n_s$ that detects all $P$-bursts of block length $b$ or less must have at least

$$
\sum_{l=s-b+1}^{s} n_l \text{ parity check digits}.
$$

**Proof.** Let $V$ be an $[n, k; P]$ $l(pγ)$-code over $F_q$ where $P : n = [n_1][n_2] \cdots [n_s], n_1 \leq n_2 \leq \cdots \leq n_s$. Consider $\zeta$ to be the collection of all those vectors of $F_q^n = F_q^n + F_q^{n_2} + \cdots + F_q^{n_s}$ which have all their nonzero components (if at all they have) confined to last $b$ block positions. Then $\zeta = \emptyset$ as null vector belongs to it. We claim that no two vectors in $\zeta$ can belong to the same coset of the standard array. Let, if possible, $a, b \in \zeta$ such that $a, b \in$ same coset of the standard array. Then

$$
a - b \in V.
$$

But by the nature of the elements of $\zeta$, $a, b \in \zeta \Rightarrow a - b \in \zeta$ and, therefore, $a - b$ is a $P$-burst of block length $b$ or less. Since no $P$-burst of block length $b$ or less is a codeword, therefore,

$$
a - b \notin V.
$$

(8) and (9) lead to a contradiction. Therefore, no two members in $\zeta$ can be in the same coset of the standard array. Since the number of available cosets $= q^{n-k}$ and

$$
\sum_{l=s-b+1}^{s} n_l
$$

number of elements in $\zeta = q^{l(s-b+1)}$. Therefore, we must have

$$
q^{n-k} \geq \sum_{l=s-b+1}^{s} n_l
$$

(10)

The converse of Theorem 17 is also true and is proven below:

**Theorem 18.** There shall always exist an $[n, k; P]$ $l(pγ)$-code over $F_q$ where $P : n = [n_1][n_2] \cdots [n_s], n_1 \leq n_2 \leq \cdots \leq n_s$ that detects all $P$-bursts of block length $b$ or less provided

$$
n - k \geq \sum_{l=s-b+1}^{s} n_l.
$$

(10)
Proof. The theorem is proved by constructing an \((n-k) \times n\) parity check matrix \(H = [H_1, H_2, \ldots, H_n]\) for the desired code. Choose the \(n_1\) columns in the first block \(H_1\) to be any nonzero \((n-k)\)-tuples. After having chosen the first \((j-1)\) blocks suitably viz. \(H_1, H_2, \ldots, H_{j-1}\) of block sizes \(n_1, n_2, \ldots, n_{j-1}\) respectively, we lay down the following condition to add the \(j^{th}\) block \(H_j\) of length \(n_j\) as follows:

The \(j^{th}\) block \(H_j\) to be added must be such that

\[
\begin{align*}
\lambda_j.H_j + \lambda_{j-1}.H_{j-1} + \cdots + \lambda_{j-b+1}.H_{j-b+1} & \neq 0, \quad (11)
\end{align*}
\]

where \(\lambda_l = (\lambda_l^{(1)}, \lambda_l^{(2)}, \ldots, \lambda_l^{(n)}) \in F_q^n\) for all \(j-b+1 \leq l \leq j\).

This condition ensures that no linear combination involving \(b\) or fewer consecutive blocks of columns of \(H\) will be zero meaning thereby that no \(P\)-burst of block length \(b\) or less will be a codeword. The number of distinct linear combinations occurring on the left hand side of (11) is

\[
\sum_{q^{j-b+1} - 1}^{j} n_l = 1.
\]

Therefore, \(j^{th}\) block \(H_j\) can be added to \(H\) provided the number of nonzero \((n-k)\)-tuples is greater than or equal to this number i.e.

\[
q^{n-k} - 1 \geq \left( q^{j-b+1} - 1 \right)
\]

or

\[
n_k - 1 \geq \sum_{l=j-b+1}^{j} n_l.
\]

The result now follows by taking \(j = s\) as there are \(s\) blocks in \(H\) and we get (10).

\(\square\)

Remark 19. On combining Theorems 17 and 18 we get the following:

"For the detection of all \(P\)-bursts of block length \(b\) or less and simultaneously detecting all \(P\)-bursts of block length \(t\) or less \((t \geq b)\), the code \(V\) must have at least \(\sum_{l=s-2b+1}^{s} n_l\) parity check digits. Furthermore, in order to correct all \(P\)-bursts of block length \(b\) or less and simultaneously detect all \(P\)-bursts of block length \(t\) or less \((t \geq b)\), the code \(V\) must have at least \(\sum_{l=s-(b+t)+1}^{s} n_l\) parity check digits."

Proof. Consider an \(s\)-block vector of length \(n\) which is a \(P\)-burst of block length \(2b\) or less. Such a vector can always be expressed as a sum or difference of two vectors each of which is a \(P\)-burst of block length \(b\) or less. Since the \(lp\gamma\)-code \(V\) corrects all \(P\)-bursts of block length \(b\) or less, therefore, all \(P\)-bursts of block length \(b\) or less must be the coset leaders of the different cosets of the standard array i.e. all \(P\)-bursts each of block length \(b\) or less must belong to distinct cosets of the standard array. This implies that the difference or sum of two \(P\)-bursts each of block length \(b\) or less can not be a code vector. This further implies that the vector under discussion which is a \(P\)-burst of block length \(2b\) or less expressible as a sum or difference of two \(P\)-bursts of block length \(b\) or less can not be a code vector. Hence using Theorem 17, the \(lp\gamma\) code \(V\) must have at least \(\sum_{l=s-2b+1}^{s} n_l\) parity check digits.

Again, consider a \(P\)-burst of block length \(b + t\) or less. Since the code corrects all \(P\)-bursts of block length \(b\) or less and simultaneously detects all \(P\)-bursts of block length \(t\) or less \((t \geq b)\), therefore all correctable or detectable vectors must belong to the distinct cosets of the standard array unless the error vector is the same. Since a \(P\)-burst of block length \(b + t\) can be expressed as a sum or difference of two vectors one of which is a \(P\)-burst of block length \(b\) or less and the other one is a \(P\)-burst of block length \(t\) or less, therefore, the vector which is a \(P\)-burst of block length \(b + t\) or less can not be a code vector. Accordingly, the code must have at least \(\sum_{l=s-(b+t)+1}^{s} n_l\) parity check digits. \(\square\)

5. CONCLUSION

In this paper, we have introduced the notion of \(P\)-bursts in the block space

\[
F_q^n = F_q^{n_1} \oplus \cdots \oplus F_q^{n_s}
\]

corresponding to the partition \(P : n - \sum_{l=s-(b+t)+1}^{s} n_l\) of the positive integer \(n\). We have obtained the various necessary and sufficient conditions on the parameters of \(lp\gamma\)-codes in terms of lower and upper bounds for the detection and correction of \(P\)-bursts with and without weight constraint.

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