
Q-POLYNOMIAL DISTANCE-REGULAR GRAPHS WITH $a_1 = 0$

Štefko Miklavič

ISSN 1318-4865

November 19, 2002

Ljubljana, November 19, 2002
Q-POLYNOMIAL DISTANCE-REGULAR
GRAPHS WITH $a_1 = 0$

Štefko Miklavič
Nova Gorica Polytechnic
Vipavska 13, POB 301
5001 Nova Gorica, Slovenia
stefko.miklavic@p-ng.si

November 15, 2002

Abstract

We will show that every Q-polynomial distance-regular graph with
diameter $d \geq 3$ and intersection number $a_1 = 0$ is 1-homogeneous in the
sense of Nomura.

Key words: distance-regular graphs, Q-polynomial distance-regular graphs,
1-homogeneous distance-regular graphs.

1 Introduction

Let $\Gamma$ denote a Q-polynomial distance-regular graph with diameter $d \geq 3$
and intersection numbers $c_i, a_i, b_i$. We will show that if $a_1 = 0$ then $\Gamma$ is 1-
homogeneous in the sense of Nomura [7]. We will also give explicit formulae for
the parameters of the equitable partition of the vertices of $\Gamma$, corresponding to
the distance from a pair of adjacent vertices $x$ and $y$. To obtain our results, we
use Terwilliger’s “balanced set” characterization of the Q-polynomial property
[8].
The Hermitean forms graphs with $r = 2$ (see Brouwer et al. [2, p. 285]) provide examples of \(Q\)-polynomial distance-regular graphs with \(a_1 = 0\).

After some preliminaries in the next section, we will discuss the homogeneous and the \(Q\)-polynomial property in Section 3 and Section 4. We will prove the main theorem and some of its consequences in Section 5 and Section 6. In Section 7 we will discuss the Hermitean forms graphs.

## 2 Preliminaries

In this section, we review some definitions and basic concepts. See the book of Brouwer et al. [2] for more background information.

Throughout this paper, \(\Gamma\) will denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set \(V \Gamma\), edge set \(E \Gamma\), path length distance function \(\partial\), and diameter \(d := \max \{\partial(x, y)\mid x, y \in V \Gamma\}\). For a vertex \(x \in V \Gamma\) define \(\Gamma_i(x)\) to be the set of vertices at distance \(i\) from \(x\). We abbreviate \(\Gamma(x) := \Gamma_1(x)\). The graph \(\Gamma\) is said to be distance-regular whenever for all integers \(h, i, j (0 \leq h, i, j \leq d)\), and all \(x, y \in V \Gamma\) with \(\partial(x, y) = h\), the number

\[
p^h_{ij} := |\{z \mid z \in V \Gamma, \partial(x, z) = i, \partial(y, z) = j\}|
\]

is independent of \(x, y\). The constants \(p^h_{ij} (0 \leq h, i, j \leq d)\) are known as the intersection numbers of \(\Gamma\). For convenience, set \(c_i := p^1_{i-1} (1 \leq i \leq d)\), \(a_i := p^0_i (0 \leq i \leq d)\), \(b_i := p^{i+1}_i (0 \leq i \leq d - 1)\), \(k_i := p^0_i (0 \leq i \leq d)\), and \(c_0 = b_d = 0\). We observe \(a_0 = 0, c_1 = 1\). Moreover,

\[
c_i + a_i + b_i = k (0 \leq i \leq d),
\]

where \(k := k_1\). From now on we assume \(\Gamma\) is distance-regular with diameter \(d \geq 3\).

In the following two lemmas we cite some well known facts about the intersection numbers; see for example Brouwer et al. [2, p. 127, 134]

\textbf{Lemma 2.1} Let \(\Gamma\) denote a distance-regular graph with diameter \(d \geq 3\). Then for all integers \(h, i, j (0 \leq h, i, j \leq d)\) the following (i), (ii) hold.

(i) If one of \(h, i, j\) is greater than the sum of the other two, then \(p^h_{ij} = 0\).

(ii) If one of \(h, i, j\) is equal to the sum of the other two, then \(p^h_{ij} \neq 0\).

\[\square\]
Lemma 2.2 Let $\Gamma$ denote a distance-regular graph with diameter $d \geq 3$. Then the following (i)-(iii) hold.

(i) $k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i}$ $(0 \leq i \leq d)$,

(ii) $p_{i,i-1}^1 = \frac{c_i k_i}{k}$ $(1 \leq i \leq d)$,

(iii) $p_{ii}^1 = \frac{a_i k_i}{k}$ $(0 \leq i \leq d)$.

Let $\Gamma$ denote a distance-regular graph with diameter $d \geq 3$. For each integer $i$ $(0 \leq i \leq d)$, the $i$th distance matrix $A_i$ has rows and columns indexed by $VT$, and $x, y$ entry

$$
(A_i)_{xy} = \begin{cases} 
1 & \text{if } \partial(x, y) = i, \\
0 & \text{if } \partial(x, y) \neq i 
\end{cases} \quad (x, y \in VT). 
$$

(3)

Then

$$A_0 = I, \quad (4)$$

$$A_0 + A_1 + \cdots + A_d = J \quad (J = \text{ all 1's matrix}), \quad (5)$$

$$A_i^t = A_i \quad (0 \leq i \leq d), \quad (6)$$

and

$$A_i A_j = \sum_{h=0}^d p_{ij}^h A_h \quad (0 \leq i, j \leq d). \quad (7)$$

By (4), (6) and (7), the matrices $A_0, A_1, \ldots, A_d$ form a basis for a commutative semi-simple $\mathbb{R}$-algebra $M$, known as the Bose-Mesner algebra. By Godsil [4, Theorem 12.2.1], the algebra $M$ has a second basis $E_0, E_1, \ldots, E_d$ such that

$$E_0 = |VT|^{-1} J, \quad (8)$$

$$E_i E_j = \delta_{ij} E_i \quad (0 \leq i, j \leq d), \quad (9)$$

$$E_0 + E_1 + \cdots + E_d = I, \quad (10)$$

$$E_i^t = E_i \quad (0 \leq i \leq d). \quad (11)$$
The $E_0, E_1, \ldots, E_d$ are known as the primitive idempotents of $\Gamma$, and $E_0$ is the trivial idempotent.

Set $A := A_1$, and define the real numbers $\theta_i$ ($0 \leq i \leq d$) by

$$A = \sum_{i=0}^{d} \theta_i E_i.$$  \hspace{1cm} (12)

Then $AE_i = E_i A = \theta_i E_i$ ($0 \leq i \leq d$), and $\theta_0 = k$. The scalars $\theta_0, \theta_1, \ldots, \theta_d$ are distinct, since $A$ generates $M$ [1, p. 197]. The $\theta_0, \theta_1, \ldots, \theta_d$ are known as the eigenvalues of $\Gamma$.

For notational convenience, we identify $VT$ with the standard orthonormal basis in the Euclidean space $V$, $\langle \cdot, \cdot \rangle$, where $V = \mathbb{R}^{VT}$ (column vectors), and where $\langle \cdot, \cdot \rangle$ is the dot product

$$\langle u, v \rangle = u^t v \quad (u, v \in VT).$$

Observe $M$ acts on $V$ by left multiplication. The Euclidean space $V$, $\langle \cdot, \cdot \rangle$ is known as the standard module of $\Gamma$.

In the following lemma, we cite some well known results about primitive idempotents.

**Lemma 2.3** (Terwilliger [8, Lemma 1.1]) Let $\Gamma$ denote a distance-regular graph with diameter $d \geq 3$. Pick any $\theta, \theta_0, \theta_1, \ldots, \theta_d \in \mathbb{R}$, and set

$$E := |VT|^{-1} \sum_{i=0}^{d} \theta_i A_i.$$  \hspace{1cm} (13)

Then the following (i)-(iii) are equivalent:

(i) $\theta$ is an eigenvalue of $\Gamma$, and $E$ is the associated primitive idempotent.

(ii) For all $x, y \in VT$,

$$\langle Ex, Ey \rangle = |VT|^{-1} \theta_i \quad \text{whenever } \partial(x,y) = i,$$

and

$$\sum_{z \in VT \atop \partial(x, z) = 1} Ez = \theta Ex.$$
(iii) The intersection numbers of \( \Gamma \) satisfy

\[
c_i \theta_{i-1}^* + a_i \theta_i^* + b_i \theta_{i+1}^* = \theta \theta_i^* \quad (0 \leq i \leq d),
\]

and \( \theta_0^* = \text{rank } E \).

If (i)-(iii) hold, we call the sequence \( \theta_0^*, \theta_1^*, \ldots, \theta_d^* \) the dual eigenvalue sequence associated with \( \theta, E \). The sequence is trivial whenever \( E = E_0 \) (in which case \( \theta_0^* = \theta_1^* = \cdots = \theta_d^* = 1 \)).

### 3 The 1-homogeneous property

In this section, we recall the 1-homogeneous property.

**Definition 3.1** Let \( \Gamma \) denote a distance-regular graph with diameter \( d \geq 3 \) and let \( x, y \) denote adjacent vertices in \( V \Gamma \). For all integers \( i \) and \( j \) we define \( D_i^j = D_i^j(x, y) \) by

\[
D_i^j = \Gamma_i(x) \cap \Gamma_j(y).
\]

We observe \( D_i^j = \emptyset \) unless \( 0 \leq i, j \leq d \). Moreover if \( 0 \leq i, j \leq d \) then \( |D_i^j| = p_i^j \).

**Lemma 3.2** Let \( \Gamma \) denote a distance-regular graph with diameter \( d \geq 3 \) and let \( x, y \) denote adjacent vertices in \( V \Gamma \). Then the following (i), (ii) hold.

(i) For \( 0 \leq i, j \leq d \), if \( |i - j| > 1 \) then \( D_i^j = \emptyset \). If \( |i - j| = 1 \) then \( D_i^j \neq \emptyset \).

(ii) For \( 0 \leq i \leq d \) we have \( D_i^0 = \emptyset \) if and only if \( a_i = 0 \).

**Proof.** Immediate from Lemma 2.1 and Lemma 2.2.

We visualize the \( D_i^j \) as follows in Figure 1.

**Lemma 3.3** (Jurisíc et al. [5, Lemma 2.11]) Let \( \Gamma \) denote a distance-regular graph with diameter \( d \geq 3 \). Fix adjacent vertices \( x, y \in V \Gamma \), and pick an integer \( i \) (\( 1 \leq i \leq d \)). Then with reference to Definition 3.1, the following (i), (ii) hold.
Figure 1: The distance partition corresponding to a pair of adjacent vertices $x$ and $y$. Observe that $D^i_{i-1} \cup D^i_i \cup D^i_{i+1} = \Gamma_i(x)$ and $D^i_{i-1} \cup D^i_i \cup D^i_{i+1} = \Gamma_i(y)$.

(i) Each $z \in D^i_{i-1}$ (resp. $D^i_{i+1}$) is adjacent to
(a) precisely $c_{i-1}$ vertices in $D^i_{i-1}$ (resp. $D^i_{i+1}$),
(b) precisely $c_i - c_{i-1} - |\Gamma(z) \cap D^i_{i-1}|$ vertices in $D^i_{i-1}$ (resp. $D^i_{i+1}$),
(c) precisely $a_{i-1} - |\Gamma(z) \cap D^i_{i-1}|$ vertices in $D^i_{i-1}$ (resp. $D^i_{i+1}$),
(d) precisely $b_i$ vertices in $D^i_{i+1}$,
(e) precisely $a_i - a_{i-1} + |\Gamma(z) \cap D^i_{i-1}|$ vertices in $D^i_{i+1}$.

(ii) Each $z \in D^i_i$ is adjacent to
(a) precisely $c_i - |\Gamma(z) \cap D^i_{i-1}|$ vertices in $D^i_{i-1}$,
(b) precisely $c_i - |\Gamma(z) \cap D^i_{i+1}|$ vertices in $D^i_{i+1}$,
(c) precisely $b_i - |\Gamma(z) \cap D^i_{i-1}|$ vertices in $D^i_{i-1}$,
(d) precisely $b_i - |\Gamma(z) \cap D^i_{i+1}|$ vertices in $D^i_{i+1}$,
(e) precisely $a_i - b_i - c_i + |\Gamma(z) \cap D^i_{i-1}| + |\Gamma(z) \cap D^i_{i+1}|$ vertices in $D^i_{i+1}$.

An equitable partition of a graph is a partition $\pi = \{C_1, C_2, \ldots, C_s\}$ of its vertex set into nonempty cells, such that for all $i, j$ ($1 \leq i, j \leq s$) the number $c_{ij}$ of neighbours, which a vertex in the cell $C_i$ has in the cell $C_j$, is independent of the choice of the vertex in $C_i$. We call the $c_{ij}$ the corresponding parameters.

Let $\Gamma$ denote a distance-regular graph with diameter $d \geq 3$. The graph $\Gamma$ is said to be $1$-homogeneous, whenever for all pairs $x, y$ of adjacent vertices, the partition of $V \Gamma$ given by $\{D_j^i(x, y) | 0 \leq i, j \leq d, \ D_j^i(x, y) \neq \emptyset\}$ is equitable, and moreover the corresponding parameters are independent of the choice of $x, y$.

Recall $\Gamma$ is bipartite whenever the intersection number $a_i = 0$ for $0 \leq i \leq d$. We say $\Gamma$ is almost bipartite whenever $a_i = 0$ for $0 \leq i \leq d - 1$ and $a_d \neq 0$.

**Corollary 3.4** Let $\Gamma$ denote a distance-regular graph with diameter $d \geq 3$. Assume $\Gamma$ is bipartite or almost bipartite. Then $\Gamma$ is $1$-homogeneous.
Proof. Immediate from of Lemma 3.3.

4 The $Q$-polynomial property

Let $\Gamma$ denote a distance-regular graph with diameter $d \geq 3$. The Krein parameters $q_{ij}^h (0 \leq h, i, j \leq d)$ of $\Gamma$ are defined by

$$E_i \circ E_j = |V\Gamma|^{-1} \sum_{h=0}^{d} q_{ij}^h E_h \quad (0 \leq i, j \leq d),$$

(14)

where $\circ$ denotes entrywise multiplication. We say $\Gamma$ is $Q$-polynomial (with respect to the given ordering $E_0, E_1, \ldots, E_d$ of the primitive idempotents), whenever for all distinct integers $i, j$ ($0 \leq i, j \leq d$),

$$q_{ij}^1 \neq 0 \text{ if and only if } |i - j| = 1.$$

Let $E$ denote a nontrivial primitive idempotent of $\Gamma$. We say $\Gamma$ is $Q$-polynomial with respect to $E$ whenever there exists an ordering $E_0, E_1 = E, \ldots, E_d$ of the primitive idempotents of $\Gamma$, with respect to which $\Gamma$ is $Q$-polynomial.

We have the following useful lemmas about $Q$-polynomial distance-regular graphs.

Lemma 4.1 (Brouwer et al. [2, Thm. 8.1.1]) Let $\Gamma$ denote a distance-regular graph with diameter $d \geq 3$. Let $E$ denote a nontrivial primitive idempotent of $\Gamma$ and let $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ denote the corresponding dual eigenvalue sequence. Suppose $\Gamma$ is $Q$-polynomial with respect to $E$. Then $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ are mutually distinct.

Lemma 4.2 (Caughman [3, Lemma 8.2]) Let $\Gamma$ denote a distance-regular graph with diameter $d \geq 3$. Let $E$ denote a nontrivial primitive idempotent of $\Gamma$ and let $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ denote the corresponding dual eigenvalue sequence. Assume $\Gamma$ is $Q$-polynomial with respect to $E$. Then

$$(\theta_2^* - \theta_i^*)(\theta_0^* - \theta_i^*) = (\theta_1^* - \theta_{i+1}^*)(\theta_1^* - \theta_{i-1}^*)$$

(15)

for $1 \leq i \leq d - 1$.  

1
Lemma 4.3 (Terwilliger [8, Thm. 3.3]) Let $\Gamma$ denote a distance-regular graph with diameter $d \geq 3$. Let $E$ denote a nontrivial primitive idempotent of $\Gamma$ and let $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ denote the corresponding dual eigenvalue sequence. Then the following (i), (ii) are equivalent:

(i) $\Gamma$ is $Q$-polynomial with respect to $E$.

(ii) $\theta_0^* \neq \theta_i^*$ ($1 \leq i \leq d$), and for all integers $h, i, j$ ($1 \leq h \leq d$), $0 \leq i, j \leq d$ and for all vertices $x, y \in V\Gamma$ with $\partial(x, y) = h$ the following hold:
\[
\sum_{z \in V\Gamma \atop \partial(x, z) = i} E_z - \sum_{z \in V\Gamma \atop \partial(y, z) = j} E_z \in \text{span} \{Ex - Ey\}.
\]

Suppose (i), (ii) hold. Then for all integers $h, i, j$ ($1 \leq h \leq d$), $0 \leq i, j \leq d$ and for all $x, y \in V\Gamma$ such that $\partial(x, y) = h$,
\[
\sum_{z \in V\Gamma \atop \partial(x, z) = i} E_z - \sum_{z \in V\Gamma \atop \partial(y, z) = j} E_z = \frac{h^i}{\theta_0^* - \theta_i^*} (Ex - Ey).
\]

5 The main theorem

In this section we prove that if $\Gamma$ is $Q$-polynomial with $a_1 = 0$, then $\Gamma$ is 1-homogeneous.

Theorem 5.1 Let $\Gamma$ denote a distance-regular graph with diameter $d \geq 3$ and intersection number $a_1 = 0$. Let $E$ denote a nontrivial primitive idempotent of $\Gamma$ and let $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ denote the corresponding dual eigenvalue sequence. Assume $\Gamma$ is $Q$-polynomial with respect to $E$. Then with reference to Definition 3.1 the following (i)-(iii) hold.

(i) For all integers $i$ ($2 \leq i \leq d$), and for all $z \in D_i^{i-1} \cup D_{i-1}^i$,
\[
|\Gamma(z) \cap D_i^{i-1}| = a_{i-1} \frac{(\theta_i^* - \theta_1^*)(\theta_i^* - \theta_{i-1}^*) + (\theta_i^* - \theta_2^*)(\theta_i^* - \theta_{i-1}^*)}{(\theta_0^* - \theta_{i-1}^*)(\theta_i^* - \theta_{i-1}^*)}.
\]
(ii) For all integers $i$ $(2 \leq i \leq d)$ such that $a_i \neq 0$, and for all $z \in D_i^i$,
\[|\Gamma(z) \cap D_{i-1}^i| = a_i \frac{\theta_{i-1}^i \theta_{i-1}^i - \theta_1^i}{(\theta_0^i - \theta_1^i)(\theta_{i-1}^i - \theta_i^i)}.\]

(iii) For all integers $i$ $(2 \leq i \leq d - 1)$ such that $a_i \neq 0$, and for all $z \in D_i^i$,
\[|\Gamma(z) \cap D_{i+1}^i| = b_i \frac{\theta_{i+1}^i \theta_{i+1}^i - \theta_1^i}{(\theta_0^i - \theta_1^i)(\theta_{i+1}^i - \theta_i^i)}.\]

We remark the denominators in the above quotients are nonzero by Lemma 4.1.

**Proof:** We split the proof into several cases.

**Case 1.** Let $z$ be a vertex in $D_2^1 \cup D_1^1$. Then $z$ has no neighbours in $D_1^1$ since $D_1^1 = \emptyset$.

**Case 2.** Let $z$ be a vertex in $D_2^2$. Then $z$ has no neighbours in $D_1^1$ since $D_1^1 = \emptyset$. We abbreviate $\tau = |\Gamma(z) \cap D_3^1|$ and $\eta = |\Gamma(z) \cap D_2^1|$. We observe $\tau + \eta = b_2$. By Lemma 4.3 we have
\[\sum_{\substack{w \in V \Gamma \\partial(x,w) = 1 \\partial(z,w) = 3}} E_w - \sum_{\substack{w \in V \Gamma \\partial(x,w) = 1 \\partial(z,w) = 3}} E_w = b_2 \frac{\theta_3^i - \theta_1^i}{\theta_0^i - \theta_2^i}(Ex - Ez).\]  

Observe that $\{w \in V \Gamma | \partial(x,w) = 1, \partial(z,w) = 3\} \subseteq D_2^2$. Taking the inner product of (17) with $Ey$ using Lemma 2.3 (ii), we get (after multiplying by $|V\Gamma|$)
\[\eta \theta_3^i + \tau \theta_3^i - b_2 \theta_2^i = b_2 \frac{\theta_3^i - \theta_1^i}{\theta_0^i - \theta_2^i}(\theta_1^i - \theta_2^i).\]

Evaluating the above line using $\eta = b_2 - \tau$, we obtain
\[\tau = b_2 \frac{(\theta_1^i - \theta_2^i)(\theta_3^i - \theta_1^i)}{(\theta_0^i - \theta_2^i)(\theta_3^i - \theta_2^i)}.\]

**Case 3.** Let $z \in D_i^{i-1}$ $(3 \leq i \leq d)$. We abbreviate $\rho = |\Gamma(z) \cap D_{i-1}^i|$ and $\zeta = |\Gamma(z) \cap D_i^{i-1}|$. We observe $\rho + \zeta = a_{i-1}$. By Lemma 4.3 we have
\[\sum_{\substack{w \in V \Gamma \\partial(x,w) = i-1 \\partial(z,w) = 1}} E_w - \sum_{\substack{w \in V \Gamma \\partial(x,w) = i-1 \\partial(z,w) = 1}} E_w = a_{i-1} \frac{\theta_{i-1}^i - \theta_1^i}{\theta_0^i - \theta_{i-1}^i}(Ex - Ez).\]
Taking the inner product of (18) with $Ey$ using Lemma 2.3, we get (after multiplying by $|V\Gamma|$)

$$\rho \theta_{i-1}^* + \zeta \theta_i^* - a_{i-1} \theta_2^* = a_{i-1} \frac{\theta_{i-1}^* - \theta_i^*}{\theta_0^* - \theta_{i-1}^*} (\theta_1^* - \theta_i^*).$$

Evaluating the above line using $\zeta = a_{i-1} - \rho$ we obtain

$$\rho = a_{i-1} \frac{(\theta_1^* - \theta_i^*) (\theta_0^* - \theta_{i-1}^*) + (\theta_2^* - \theta_i^*) (\theta_0^* - \theta_{i-1}^*)}{(\theta_0^* - \theta_{i-1}^*) (\theta_1^* - \theta_i^*)}.$$  

The cases $z \in D_{i-1}^i$ ($3 \leq i \leq d$) are treated similarly.

**Case 4.** Let $z \in D_i^i$ ($3 \leq i \leq d$). We abbreviate $\sigma = |\Gamma(z) \cap D_{i-1}^i|$ and $\delta = |\Gamma(z) \cap D_{i+1}^i|$. We observe $\sigma + \delta = c_i$. By Lemma 4.3 we have

$$\sum_{\delta(x,w) = i \atop \delta(x,v) = i-1} Ew - \sum_{\delta(x,w) = i \atop \delta(x,v) = i+1} Ew = c_i \frac{\theta_{i-1}^* - \theta_i^*}{\theta_0^* - \theta_i^*} (Ex - Ez). \quad (19)$$

Again, taking the inner product of (19) with $Ey$ using Lemma 2.3 we get

$$\delta \theta_i^* + \sigma \theta_{i-1}^* - c_i \theta_2^* = c_i \frac{\theta_{i-1}^* - \theta_i^*}{\theta_0^* - \theta_i^*} (\theta_1^* - \theta_i^*).$$

Evaluating the above line using $\delta = c_i - \sigma$ we obtain

$$\sigma = c_i \frac{(\theta_1^* - \theta_i^*) (\theta_{i-1}^* - \theta_0^*) + (\theta_2^* - \theta_i^*) (\theta_0^* - \theta_i^*)}{(\theta_0^* - \theta_i^*) (\theta_{i-1}^* - \theta_i^*)}.$$  

**Case 5.** Let $z \in D_i^i$ ($3 \leq i \leq d - 1$). We abbreviate $\tau = |\Gamma(z) \cap D_{i+1}^i|$ and $\gamma = |\Gamma(z) \cap D_{i+1}^i|$. We observe $\tau + \gamma = b_i$. By Lemma 4.3 we have

$$\sum_{\delta(x,w) = i \atop \delta(x,v) = i+1} Ew - \sum_{\delta(x,w) = i \atop \delta(x,v) = i+1} Ew = b_i \frac{\theta_{i+1}^* - \theta_i^*}{\theta_0^* - \theta_i^*} (Ex - Ez). \quad (20)$$

Again, taking the inner product of (20) with $Ey$ using Lemma 2.3 we get

$$\gamma \theta_i^* + \tau \theta_{i+1}^* - b_i \theta_2^* = b_i \frac{\theta_{i+1}^* - \theta_i^*}{\theta_0^* - \theta_i^*} (\theta_1^* - \theta_i^*).$$
Evaluating the above line using $\gamma = b_i - \tau$ we obtain

$$\tau = b_i \frac{(\theta^*_i - \theta^*_i)(\theta^*_i + \theta^*_i + \theta^*_i)(\theta^*_i - \theta^*_i)}{(\theta^*_i - \theta^*_i)(\theta^*_i + \theta^*_i)}.$$

\[\square\]

**Lemma 5.2** Let $\Gamma$ denote a distance-regular graph with diameter $d \geq 3$ and intersection number $a_1 = 0$. Let $E$ denote a non trivial primitive idempotent of $\Gamma$ and let $\theta_0^*, \theta_1^*, \ldots, \theta_a^*$ denote the corresponding dual eigenvalue sequence. Assume $\Gamma$ is $Q$-polynomial with respect to $E$. Then with reference to Definition 3.1 the following (i), (ii) hold.

(i) For all integers $i$ ($2 \leq i \leq d - 1$) such that $a_i \neq 0$, and for all $z \in D^i_i$,

$$|\Gamma(z) \cap D^i_{i-1}| = c_i \frac{(\theta^*_i - \theta^*_{i-1})(\theta^*_i - \theta^*_{i+1})}{(\theta^*_0 - \theta^*_i)(\theta^*_i - \theta^*_i)}.$$

(ii) For all integers $i$ ($2 \leq i \leq d - 1$) such that $a_i \neq 0$, and for all $z \in D^i_i$,

$$|\Gamma(z) \cap D^i_{i+1}| = b_i \frac{(\theta^*_i - \theta^*_{i+1})(\theta^*_{i-1} - \theta^*_i)}{(\theta^*_0 - \theta^*_i)(\theta^*_i - \theta^*_i)}.$$

**Proof.** Simplify the formulae in Theorem 5.1 (i), (ii) using Lemma 4.2. \[\square\]

**Corollary 5.3** Let $\Gamma$ denote a distance-regular graph with diameter $d \geq 3$ and intersection number $a_1 = 0$. If $\Gamma$ is $Q$-polynomial then $\Gamma$ is 1-homogeneous.

**Proof.** Immediate from Theorem 5.1 and Lemma 3.3. \[\square\]

**Remark:** The converse of the above corollary is not true. Indeed the Coxeter graph is a 1-homogeneous distance-regular graph with $d = 4$ and $a_1 = 0$, but it is not $Q$-polynomial, see Brouwer et al. [2, Section 6.10].

### 6 Comments on the intersection numbers

Let $\Gamma$ denote a $Q$-polynomial distance-regular graph with diameter $d \geq 3$ and intersection number $a_1 = 0$. In this section we show that one of the following hold: (i) $\Gamma$ is bipartite; (ii) $\Gamma$ is almost bipartite; or (iii) $a_i \neq 0$ for $2 \leq i \leq d$.

We will use the following two lemmas.
Lemma 6.1 Let $\Gamma$ denote a $Q$-polynomial distance-regular graph with diameter $d \geq 3$ and intersection number $a_1 = 0$. Suppose there exists an integer $i$ ($2 \leq i \leq d - 1$) such that $a_i \neq 0$. Then with reference to Definition 3.1 the following (i)-(iii) hold.

(i) For all $z \in D_i$ we have $|\Gamma(z) \cap D_{i+1}^{i+1}| \neq 0$.

(ii) $D_{i+1}^{i+1} \neq \emptyset$.

(iii) $a_{i+1} \neq 0$.

Proof. (i) Suppose $|\Gamma(z) \cap D_{i+1}^{i+1}| = 0$. By Lemma 5.2 (ii) we have either $\theta_i^* = \theta_{i+1}^*$ or $\theta_i^* = \theta_{i-1}^*$. But this is in contradiction with Lemma 4.1.

(ii), (iii) Immediate from (i) above.

Lemma 6.2 Let $\Gamma$ denote a $Q$-polynomial distance-regular graph with diameter $d \geq 3$ and intersection number $a_1 = 0$. Suppose there exists an integer $i$ ($3 \leq i \leq d - 1$) such that $a_i \neq 0$. Then with reference to Definition 3.1 the following (i)-(iii) hold.

(i) For all $z \in D_i$ we have $|\Gamma(z) \cap D_{i-1}^{i-1}| \neq 0$.

(ii) $D_{i-1}^{i-1} \neq \emptyset$.

(iii) $a_{i-1} \neq 0$.

Proof. Similar to the proof of Lemma 6.1.

Corollary 6.3 Let $\Gamma$ denote a $Q$-polynomial distance-regular graph with diameter $d \geq 3$ and intersection number $a_1 = 0$. Then exactly one of the following (i)-(iii) hold.

(i) $\Gamma$ is bipartite,

(ii) $\Gamma$ is almost bipartite,

(iii) $a_i \neq 0$ for $2 \leq i \leq d$.

Proof. Immediate from Lemma 6.1 and Lemma 6.2.
7 An example

In this section, we recall the Hermitean forms graph.

Let $\Gamma$ be a Hermitean forms graph for a prime power $r$ and let $d$ denote the diameter of $\Gamma$. Then the intersection numbers of $\Gamma$ are given by

\[ b_i = \frac{r^{2d} - r^{2i}}{r + 1} \quad 0 \leq i \leq d - 1, \]

\[ c_i = \frac{r^{i+1} - r^{i} - (-1)^i}{r + 1} \quad 1 \leq i \leq d, \]

see Brouwer et al. [2, Thm. 9.5.7]. Therefore we have

\[ a_i = \frac{r^{2d} - r^{2i-1} - (-r)^{i-1} - 1}{r + 1} \quad 0 \leq i \leq d. \]

Let $\theta$ denote the minimal eigenvalue of $\Gamma$. By Brouwer et al. [2, Cor. 8.4.2] $\theta$ is given by

\[ \theta = -\frac{r^{2d-1} + 1}{r + 1}. \]

Let $E$ denote the primitive idempotent of $\Gamma$ corresponding to $\theta$ and let $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ denote the corresponding dual eigenvalue sequence. By Brouwer et al. [2, Cor. 8.4.2], $\Gamma$ is $Q$-polynomial with respect to $E$. Moreover, by Brouwer et al. [2, Cor. 8.4.2] and since the $Q$-polynomial structure is self-dual ([2, Cor. 8.4.4]) we have

\[ \theta_i^* = \frac{b_i}{(-r)^i} + \frac{(-r)^i - 1}{r + 1} = \frac{(-r)^{2d-1}}{r + 1} \quad (0 \leq i \leq d). \]

For the rest of this section assume $r = 2$. In this case we have $a_1 = 0$; therefore $\Gamma$ is 1-homogeneous by Corollary 5.3. The parameters of the corresponding equitable partition are given by the following theorem. We remark $a_2 = 3$. Since $a_2 \neq 0$ we find $a_i \neq 0$ for $2 \leq i \leq d$ in view of Corollary 6.3 (iii).

**Theorem 7.1** Let $\Gamma$ denote a Hermitean forms graph with $r = 2$ and diameter $d \geq 3$. Then with reference to Definition 3.1, the following (i), (ii) hold.

13
(i) For $1 \leq i \leq d$, each $z \in D_{i-1}^i$ (resp. $D_{i}^{i-1}$) is adjacent to

(a) precisely $0$ vertices in $D_{i-1}^{i-1}$,
(b) precisely $2^{i-2}(2^{i-1} - (-1)^{i-1})/3$ vertices in $D_{i-2}^{i-1}$ (resp. $D_{i-1}^{i-2}$),
(c) precisely $2^{i-2}(2^{i-1} + (-1)^{i-1})$ vertices in $D_{i-1}^{i-1}$ (resp. $D_{i-1}^{i-1}$),
(d) precisely $(2^{2i-3} - (-2)^{i-2} - 1)/3$ vertices in $D_{i-1}^{i-1}$ (resp. $D_{i-1}^{i-1}$),
(e) precisely $(2^{2i} - 2^{2i})/3$ vertices in $D_{i-1}^{i+1}$ (resp. $D_{i+1}^{i-1}$),
(f) precisely $2^{i-2}(2^{i-1} - (-1)^{i-1})$ vertices in $D_{i}^{i-1}$.

(ii) For $2 \leq i \leq d$, each $z \in D_{i}^{i}$ is adjacent to

(a) precisely $(-1)^i2^{i-1}((-2)^{i-2} - 1)/3$ vertices in $D_{i-1}^{i-1}$,
(b) precisely $(2^{2i} - 2^{2i})/(3$ vertices in $D_{i+1}^{i-1}$,
(c) precisely $2^{2i-3}$ vertices in $D_{i-1}^{i-1}$,
(d) precisely $2^{2i-3}$ vertices in $D_{i-1}^{i-1}$,
(e) precisely $0$ vertices in $D_{i-1}^{i}$,
(f) precisely $0$ vertices in $D_{i-1}^{i}$,
(g) precisely $(2^{2i-3} - (-2)^{i-1} - 1)/3$ vertices in $D_{i}^{i+1}$.

Proof. From Theorem 5.1 we get the following formulae.

(i) For all integers $i$ ($2 \leq i \leq d$) and for all $z \in D_{i}^{i-1} \cup D_{i-1}^{i}$, we have $|\Gamma(z) \cap D_{i-1}^{i-1}| = 0$.

(ii) For all integers $i$ ($2 \leq i \leq d$) and for all $z \in D_{i}^{i}$, we have $|\Gamma(z) \cap D_{i-1}^{i-1}| = a_i((-2)^{i-2} - 1)/((-2)^i - 1)$.

(iii) For all integers $i$ ($2 \leq i \leq d-1$) and for all $z \in D_{i}^{i}$, we have $|\Gamma(z) \cap D_{i+1}^{i} | = b_i$.

The present theorem now follows by Lemma 3.3 and from the formulae of the intersection numbers $b_i$ and $c_i$.

We conclude the paper with some conjectures.

**Conjecture 7.1** Let $\Gamma$ denote a distance-regular graph with diameter $d \geq 3$ and intersection numbers $a_1 = 0$, $a_2 \neq 0$. Let $\theta$ denote a nontrivial eigenvalue of $\Gamma$ and let $E$ denote the corresponding primitive idempotent. Assume $\Gamma$ is $Q$-polynomial with respect to $E$. Then $\theta$ is the minimal eigenvalue of $\Gamma$.

**Conjecture 7.2** Let $\Gamma$ denote a distance-regular graph with diameter $d \geq 3$ and intersection numbers $a_1 = 0$, $a_2 \neq 0$. Assume $\Gamma$ has classical parameters $(d, b, \alpha, \beta)$, see Brouwer et al. [2, Sec. 6.1]. Then $b = -2$. 

14
Conjecture 7.3 Let $\Gamma$ denote a distance-regular graph with diameter $d \geq 3$ and intersection numbers $a_1 = 0$, $a_2 \neq 0$. Assume $\Gamma$ has classical parameters. Let $i$ denote an integer ($2 \leq i \leq d$) and let $x, y$ denote vertices of $\Gamma$ at distance $\partial(x, y) = i$. Then $x, y$ are contained in a weak-geodetically closed subgraph of $\Gamma$ which has diameter $i$. See [6], [9], [10], [11], [12] for more information on this topic.

Acknowledgement: The author would like to thank Paul Terwilliger for his valuable guidance and suggestions during the preparation of this manuscript.

References