

INTUITIONISTIC FUZZY H_ν -IDEALS

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The notion of intuitionistic fuzzy sets was introduced by Atanassov as a generalization of the notion of fuzzy sets. In this paper, we apply the concept of intuitionistic fuzzy sets to H_ν -rings. We introduce the notion of an intuitionistic fuzzy H_ν -ideal of an H_ν -ring and then some related properties are investigated. We state some characterizations of intuitionistic fuzzy H_ν -ideals. Also we investigate some natural equivalence relations on the set of all intuitionistic fuzzy H_ν -ideals of an H_ν -ring.

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1. Introduction and preliminaries

Hyperstructure theory was born in 1934 when Marty [11] defined hypergroups as a generalization of groups. This theory has been studied in the following decades and nowadays by many mathematicians. A recent book [3] contains a wealth of applications. There are applications to the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, combinatorics, codes, artificial intelligence, and probabilities. Vougiouklis in the fourth Algebraic Hyperstructures and Applications Congress (1990) [15] introduced the notion of H_ν -structures. The H_ν -structures are hyperstructures where the equality is replaced by the nonempty intersection. The main tool in the study of H_ν -structure is the fundamental structure which is the same as in the classical hyperstructures. In this paper, we deal with H_ν -rings. H_ν -rings are the largest class of algebraic systems that satisfy ring-like axioms. In [4], Darafsheh and Davvaz defined the H_ν -ring of fractions of a commutative hyperring which is a generalization of the concept of ring of fractions. For the notion of an H_ν -near-ring module, you can see [7]. In [13], Spartalis studied a wide class of H_ν -rings resulting from an arbitrary ring by using the P -hyperoperations. In [18], Vougiouklis introduced the classes of H_ν -rings useful in the theory of representations.

A *hyperstructure* is a nonempty set H together with a map $*$: $H \times H \rightarrow \mathcal{P}^*(H)$ called *hyperoperation*, where $\mathcal{P}^*(H)$ denotes the set of all nonempty subsets of H . The image of the pair (x, y) is denoted by $x * y$. If $x \in H$ and $A, B \subseteq H$, then by $A * B$, $A * x$, and $x * B$,

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we mean

$$A * B = \bigcup_{a \in A, b \in B} a * b, \quad A * x = A * \{x\}, \quad x * B = \{x\} * B. \quad (1.1)$$

A hyperstructure $(H, *)$ is called an H_V -semigroup if

$$(x * (y * z)) \cap ((x * y) * z) \neq \emptyset \quad \forall x, y, z \in H. \quad (1.2)$$

Defintion 1.1. An H_V -ring is a system $(R, +, \cdot)$ with two hyperoperations satisfying the following ring-like axioms:

(i) $(R, +, \cdot)$ is an H_V -group, that is,

$$\begin{aligned} ((x + y) + z) \cap (x + (y + z)) &\neq \emptyset \quad \forall x, y, z \in R, \\ a + R &= R + a = R \quad \forall a \in R; \end{aligned} \quad (1.3)$$

(ii) (R, \cdot) is an H_V -semigroup;

(iii) (\cdot) is *weak distributive* with respect to $(+)$, that is, for all $x, y, z \in R$,

$$\begin{aligned} (x \cdot (y + z)) \cap (x \cdot y + x \cdot z) &\neq \emptyset, \\ ((x + y) \cdot z) \cap (x \cdot z + y \cdot z) &\neq \emptyset. \end{aligned} \quad (1.4)$$

An H_V -ring $(R, +, \cdot)$ is called *dual H_V -ring* if $(R, \cdot, +)$ is an H_V -ring. If both operations $(+)$ and (\cdot) are weak commutative, then R is called a *weak commutative dual H_V -ring*.

We see that H_V -rings are a nice generalization of rings. For more definitions, results, and applications on H_V -rings, see [4, 5, 7, 8, 13–15, 17, 18].

Example 1.2 (cf. Vougiouklis [18]). Let $(H, *)$ be an H_V -group, then for every hyperoperation (\circ) such that $\{x, y\} \subseteq x \circ y$ for all $x, y \in H$, the hyperstructure $(H, *, \circ)$ is a dual H_V -ring.

Example 1.3 (cf. Dramalidis [8]). On the set \mathbb{R}^n , where \mathbb{R} is the set of real numbers, we define three hyperoperations:

$$\begin{aligned} x \uplus y &= \{r(x + y) \mid r \in [0, 1]\}, \\ x \otimes y &= \{x + r(y - x) \mid r \in [0, 1]\}, \\ x \square y &= \{x + ry \mid r \in [0, 1]\}. \end{aligned} \quad (1.5)$$

Then the hyperstructure $(\mathbb{R}^n, *, \circ)$, where $*, \circ \in \{\uplus, \otimes, \square\}$, is a weak commutative dual H_V -ring.

Defintion 1.4. Let R be an H_V -ring. A nonempty subset I of R is called a *left* (resp., *right*) H_V -ideal if the following axioms hold:

- (i) $(I, +)$ is an H_V -subgroup of $(R, +)$,
- (ii) $R \cdot I \subseteq I$ (resp., $I \cdot R \subseteq I$).

2. Fuzzy sets and intuitionistic fuzzy sets

The concept of a fuzzy subset of a nonempty set was first introduced by Zadeh [19].

Let X be a nonempty set, a mapping $\mu : X \rightarrow [0, 1]$ is called a fuzzy subset of X . The complement of μ , denoted by μ^c , is the fuzzy set of X given by $\mu^c(x) = 1 - \mu(x)$ for all $x \in X$.

Note that using fuzzy subsets, we can introduce on any ring the structure of H_ν -ring.

Example 2.1 (cf. Davvaz [5]). Let $(R, +, \cdot)$ be an ordinary ring and let μ be a fuzzy subset of R . We define hyperoperations $\uplus, \otimes, *$ on R as follows:

$$\begin{aligned} x \uplus y &= \{t \mid \mu(t) = \mu(x + y)\}, \\ x \otimes y &= \{t \mid \mu(t) = \mu(x \cdot y)\}, \\ x * y &= y * x = \{t \mid \mu(x) \leq \mu(t) \leq \mu(y)\} \quad (\text{if } \mu(x) \leq \mu(y)). \end{aligned} \quad (2.1)$$

Then $(R, *, *)$, $(R, *, \otimes)$, $(R, *, \uplus)$, $(R, \uplus, *)$, and (R, \uplus, \otimes) are H_ν -rings.

Rosenfeld [12] applied the concept of fuzzy sets to the theory of groups and defined the concept of fuzzy subgroups of a group. Since then, many papers concerning various fuzzy algebraic structures have appeared in the literature. In [5–7], Davvaz applied the concept of fuzzy set theory in the algebraic hyperstructures, in particular in [5] he defined the concept of fuzzy H_ν -ideal of an H_ν -ring which is a generalization of the concept of fuzzy ideal.

Defintion 2.2. Let $(R, +, \cdot)$ be an H_ν -ring and μ a fuzzy subset of R . Then μ is said to be a *left* (resp., *right*) *fuzzy H_ν -ideal* of R if the following axioms hold:

- (1) $\min\{\mu(x), \mu(y)\} \leq \inf\{\mu(z) \mid z \in x + y\}$ for all $x, y \in R$,
- (2) for all $x, a \in R$, there exists $y \in R$ such that $x \in a + y$ and

$$\min\{\mu(a), \mu(x)\} \leq \mu(y), \quad (2.2)$$

- (3) for all $x, a \in R$ there exists $z \in R$ such that $x \in z + a$ and

$$\min\{\mu(a), \mu(x)\} \leq \mu(z), \quad (2.3)$$

- (4) $\mu(y) \leq \inf\{\mu(z) \mid z \in x \cdot y\}$ (resp., $\mu(x) \leq \inf\{\mu(z) \mid z \in x \cdot y\}$) for all $x, y \in R$.

Example 2.3 (cf. Davvaz [5]). Let $(R, +, \cdot)$ be an ordinary ring and let μ be a fuzzy ideal of R . We consider the H_ν -ring (R, \uplus, \otimes) defined in Example 2.1. Then μ is a left fuzzy H_ν -ideal of (R, \uplus, \otimes) .

The concept of intuitionistic fuzzy set was introduced by Atanassov [1] as a generalization of the notion of fuzzy set. Some fundamental operations on intuitionistic fuzzy sets are defined by Atanassov in [2]. In [9], Kim et al. introduced the notion of an intuitionistic fuzzy subquasigroup of a quasigroup. Also in [10], Kim and Jun introduced the concept of intuitionistic fuzzy ideals of semirings.

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Defintion 2.4. An intuitionistic fuzzy set A of a nonempty set X is an object having the form

$$A = \{(x, \mu_A(x), \lambda_A(x)) \mid x \in X\}, \quad (2.4)$$

where the functions $\mu_A : X \rightarrow [0, 1]$ and $\lambda_A : X \rightarrow [0, 1]$ denote the degree of membership (namely, $\mu_A(x)$) and the degree of nonmembership (namely, $\lambda_A(x)$) of each element $x \in X$ to the set A , respectively, and $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$ for all $x \in X$.

Defintion 2.5. For every two intuitionistic fuzzy sets A and B , define the following operations:

- (1) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\lambda_A(x) \geq \lambda_B(x)$ for all $x \in X$,
- (2) $A^c = \{(x, \lambda_A(x), \mu_A(x)) \mid x \in X\}$,
- (3) $A \cap B = \{(x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\lambda_A(x), \lambda_B(x)\}) \mid x \in X\}$,
- (4) $A \cup B = \{(x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\lambda_A(x), \lambda_B(x)\}) \mid x \in X\}$,
- (5) $\square A = \{(x, \mu_A(x), \mu_A^c(x)) \mid x \in X\}$,
- (6) $\diamond A = \{(x, \lambda_A^c(x), \lambda_A(x)) \mid x \in X\}$.

For the sake of simplicity, we will use the symbol $A = (\mu_A, \lambda_A)$ for the intuitionistic fuzzy set $A = \{(x, \mu_A(x), \lambda_A(x)) \mid x \in X\}$.

Defintion 2.6. Let $(R, +, \cdot)$ be an ordinary ring. An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in R is called a *left* (resp., *right*) *intuitionistic fuzzy ideal* of R if

- (1) $\min\{\mu_A(x), \mu_A(y)\} \leq \mu_A(x - y)$ for all $x, y \in R$,
- (2) $\mu_A(y) \leq \mu_A(x \cdot y)$ (resp., $\mu_A(x) \leq \mu_A(x \cdot y)$) for all $x, y \in R$,
- (3) $\lambda_A(x - y) \leq \max\{\lambda_A(x), \lambda_A(y)\}$ for all $x, y \in R$,
- (4) $\lambda_A(x \cdot y) \leq \lambda_A(y)$ (resp., $\lambda_A(x \cdot y) \leq \lambda_A(x)$) for all $x, y \in R$.

3. Intuitionistic fuzzy H_v -ideals

In what follows, let R denote an H_v -ring, and we start by defining the notion of intuitionistic fuzzy H_v -ideals.

Defintion 3.1. An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ in R is called a *left* (resp., *right*) *intuitionistic fuzzy H_v -ideal* of R if

- (1) $\min\{\mu_A(x), \mu_A(y)\} \leq \inf\{\mu_A(z) \mid z \in x + y\}$ for all $x, y \in R$,
- (2) for all $x, a \in R$, there exist $y, z \in R$ such that $x \in (a + y) \cap (z + a)$ and

$$\min\{\mu_A(a), \mu_A(x)\} \leq \min\{\mu_A(y), \mu_A(z)\}, \quad (3.1)$$

- (3) $\mu_A(y) \leq \inf\{\mu_A(z) \mid z \in x \cdot y\}$ (resp., $\mu_A(x) \leq \inf\{\mu_A(z) \mid z \in x \cdot y\}$) for all $x, y \in R$,
- (4) $\sup\{\lambda_A(z) \mid z \in x + y\} \leq \max\{\lambda_A(x), \lambda_A(y)\}$ for all $x, y \in R$,
- (5) for all $x, a \in R$, there exist $y, z \in R$ such that $x \in (a + y) \cap (z + a)$ and

$$\max\{\lambda_A(y), \lambda_A(z)\} \leq \max\{\lambda_A(a), \lambda_A(x)\}, \quad (3.2)$$

- (6) $\sup\{\lambda_A(z) \mid z \in x \cdot y\} \leq \lambda_A(y)$ (resp., $\sup\{\lambda_A(z) \mid z \in x \cdot y\} \leq \lambda_A(x)$) for all $x, y \in R$.

Example 3.2. Let μ be a left fuzzy H_v -ideal of (R, \uplus, \otimes) defined in Example 2.3. Then, as it is not difficult to see, $A = (\mu_A, \mu_A^c)$ is a left intuitionistic fuzzy H_v -ideal of (R, \uplus, \otimes) .

Here we present all the proofs for left H_v -ideals. For right H_v -ideals, similar results hold as well.

LEMMA 3.3. *If $A = (\mu_A, \lambda_A)$ is a left intuitionistic fuzzy H_v -ideal of R , then so is $\square A = (\mu_A, \mu_A^c)$.*

Proof. It is sufficient to show that μ_A^c satisfies the conditions (4), (5), (6) of Definition 3.1. For $x, y \in R$, we have

$$\min \{\mu_A(x), \mu_A(y)\} \leq \inf \{\mu_A(z) \mid z \in x + y\}, \quad (3.3)$$

and so

$$\min \{1 - \mu_A^c(x), 1 - \mu_A^c(y)\} \leq \inf \{1 - \mu_A^c(z) \mid z \in x + y\}. \quad (3.4)$$

Hence

$$\min \{1 - \mu_A^c(x), 1 - \mu_A^c(y)\} \leq 1 - \sup \{\mu_A^c(z) \mid z \in x + y\}, \quad (3.5)$$

which implies that

$$\sup \{\mu_A^c(z) \mid z \in x + y\} \leq 1 - \min \{1 - \mu_A^c(x), 1 - \mu_A^c(y)\}. \quad (3.6)$$

Therefore

$$\sup \{\mu_A^c(z) \mid z \in x + y\} \leq \max \{\mu_A^c(x), \mu_A^c(y)\}, \quad (3.7)$$

in this way, Definition 3.1(4) is verified.

Now, let $a, x \in R$, then there exist $y, z \in R$ such that $x \in (a + y) \cap (z + a)$ and

$$\min \{\mu_A(a), \mu_A(x)\} \leq \min \{\mu_A(y), \mu_A(z)\}. \quad (3.8)$$

So

$$\min \{1 - \mu_A^c(a), 1 - \mu_A^c(x)\} \leq \min \{1 - \mu_A^c(y), 1 - \mu_A^c(z)\}. \quad (3.9)$$

Hence

$$\max \{\mu_A^c(y), \mu_A^c(z)\} \leq \max \{\mu_A^c(a), \mu_A^c(x)\}, \quad (3.10)$$

and Definition 3.1(5) is satisfied.

For the condition (6), let $x, y \in R$, then since μ_A is a left fuzzy H_v -ideal of R , we have

$$\mu_A(y) \leq \inf \{\mu_A(z) \mid z \in x \cdot y\}, \quad (3.11)$$

and so

$$1 - \mu_A^c(y) \leq \inf \{1 - \mu_A^c(z) \mid z \in x \cdot y\}, \quad (3.12)$$

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which implies that

$$\sup\{\mu_A^c(z) \mid z \in x \cdot y\} \leq \mu_A^c(y). \quad (3.13)$$

Therefore Definition 3.1(6) is satisfied. \square

LEMMA 3.4. *If $A = (\mu_A, \lambda_A)$ is a left intuitionistic fuzzy H_v -ideal of R , then so is $\diamond A = (\lambda_A^c, \lambda_A)$.*

The proof is similar to the proof of Lemma 3.3.

Combining the above two lemmas, it is not difficult to see that the following theorem is valid.

THEOREM 3.5. *$A = (\mu_A, \lambda_A)$ is a left intuitionistic fuzzy H_v -ideal of R if and only if $\square A$ and $\diamond A$ are left intuitionistic fuzzy H_v -ideals of R .*

COROLLARY 3.6. *$A = (\mu_A, \lambda_A)$ is a left intuitionistic fuzzy H_v -ideal of R if and only if μ_A and λ_A^c are left fuzzy H_v -ideals of R .*

Defintion 3.7. For any $t \in [0, 1]$ and fuzzy set μ of R , the set

$$U(\mu; t) = \{x \in R \mid \mu(x) \geq t\} \quad (\text{resp., } L(\mu; t) = \{x \in R \mid \mu(x) \leq t\}) \quad (3.14)$$

is called an *upper* (resp., *lower*) t -level cut of μ .

THEOREM 3.8. *If $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy H_v -ideal of R , then for every $t \in \text{Im}(\mu_A) \cap \text{Im}(\lambda_A)$, the sets $U(\mu_A; t)$ and $L(\lambda_A; t)$ are H_v -ideals of R .*

Proof. Let $t \in \text{Im}(\mu_A) \cap \text{Im}(\lambda_A) \subseteq [0, 1]$ and let $x, y \in U(\mu_A; t)$. Then $\mu_A(x) \geq t$ and $\mu_A(y) \geq t$, and so $\min\{\mu_A(x), \mu_A(y)\} \geq t$. It follows from Definition 3.1(1) that $\inf\{\mu_A(z) \mid z \in x + y\} \geq t$. Therefore for all $z \in x + y$, we have $z \in U(\mu_A; t)$, so $x + y \subseteq U(\mu_A; t)$. Hence for all $a \in U(\mu_A; t)$, we have $a + U(\mu_A; t) \subseteq U(\mu_A; t)$ and $U(\mu_A; t) + a \subseteq U(\mu_A; t)$. Now, let $x \in U(\mu_A; t)$, then there exist $y, z \in R$ such that $x \in (a + y) \cap (z + a)$ and $\min\{\mu_A(x), \mu_A(a)\} \leq \min\{\mu(y), \mu(z)\}$. Since $x, a \in U(\mu_A; t)$, we have $t \leq \min\{\mu_A(x), \mu_A(a)\}$, and so $t \leq \min\{\mu_A(y), \mu_A(z)\}$, which implies that $y \in U(\mu_A; t)$ and $z \in U(\mu_A; t)$. This proves that $U(\mu_A; t) \subseteq a + U(\mu_A; t)$ and $U(\mu_A; t) \subseteq U(\mu_A; t) + a$.

Now, for every $x \in R$ and $y \in U(\mu_A; t)$, we show that $x \cdot y \subseteq U(\mu_A; t)$. Since A is a left intuitionistic fuzzy H_v -ideal of R , we have

$$t \leq \mu_A(y) \leq \inf\{\mu_A(z) \mid z \in x \cdot y\}. \quad (3.15)$$

Therefore, for every $z \in x \cdot y$, we get $\mu_A(z) \geq t$, which implies that $z \in U(\mu_A; t)$, so $x \cdot y \subseteq U(\mu_A; t)$.

If $x, y \in L(\lambda_A; t)$, then $\max\{\lambda_A(x), \lambda_A(y)\} \leq t$. It follows from Definition 3.1(4) that $\sup\{\lambda_A(z) \mid z \in x + y\} \leq t$. Therefore for all $z \in x + y$, we have $z \in L(\lambda_A; t)$, so $x + y \subseteq L(\lambda_A; t)$. Hence for all $a \in L(\lambda_A; t)$, we have $a + L(\lambda_A; t) \subseteq L(\lambda_A; t)$ and $L(\lambda_A; t) + a \subseteq L(\lambda_A; t)$. Now, let $x \in L(\lambda_A; t)$, then there exist $y, z \in R$ such that $x \in (a + y) \cap (z + a)$ and $\max\{\lambda_A(y), \lambda_A(z)\} \leq \max\{\lambda(a), \lambda(x)\}$. Since $x, a \in L(\lambda_A; t)$, we have $\max\{\lambda_A(a), \lambda_A(x)\} \leq t$, and so $\max\{\lambda_A(y), \lambda_A(z)\} \leq t$. Thus $y \in L(\lambda_A; t)$ and $z \in L(\lambda_A; t)$. Hence $L(\lambda_A; t) \subseteq a + L(\lambda_A; t)$ and $L(\lambda_A; t) \subseteq L(\lambda_A; t) + a$.

Now, we show that $x \cdot y \subseteq L(\lambda_A; t)$ for every $x \in R$ and $y \in L(\lambda_A; t)$. Since A is a left intuitionistic fuzzy H_v -ideal of R , we have

$$\sup \{ \lambda_A(z) \mid z \in x \cdot y \} \leq \lambda_A(y) \leq t. \quad (3.16)$$

Therefore, for every $z \in x \cdot y$, we get $\lambda_A(z) \leq t$, which implies that $z \in L(\lambda_A; t)$, so $x \cdot y \subseteq L(\lambda_A; t)$. \square

THEOREM 3.9. *If $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy set of R such that all nonempty levels $U(\mu_A; t)$ and $L(\lambda_A; t)$ are H_v -ideals of R , then $A = (\mu_A, \lambda_A)$ is an intuitionistic fuzzy H_v -ideal of R .*

Proof. Assume that all nonempty levels $U(\mu_A; t)$ and $L(\lambda_A; t)$ are H_v -ideals of R . If $t_0 = \min\{\mu_A(x), \mu_A(y)\}$ and $t_1 = \max\{\lambda_A(x), \lambda_A(y)\}$ for $x, y \in R$, then $x, y \in U(\mu_A; t_0)$ and $x, y \in L(\lambda_A; t_1)$. So $x + y \subseteq U(\mu_A; t_0)$ and $x + y \subseteq L(\lambda_A; t_1)$. Therefore for all $z \in x + y$, we have $\mu_A(z) \geq t_0$ and $\lambda_A(z) \leq t_1$, that is,

$$\begin{aligned} \inf \{ \mu_A(z) \mid z \in x + y \} &\geq \min \{ \mu_A(x), \mu_A(y) \}, \\ \sup \{ \lambda_A(z) \mid z \in x + y \} &\leq \max \{ \lambda_A(x), \lambda_A(y) \}, \end{aligned} \quad (3.17)$$

which verifies the conditions (1) and (4) of Definition 3.1.

Now, if $t_2 = \min\{\mu_A(a), \mu_A(x)\}$ for $x, a \in R$, then $a, x \in U(\mu_A; t_2)$. So there exist $y_1, z_1 \in U(\mu_A; t_2)$ such that $x \in a + y_1$ and $x \in z_1 + a$. Also we have $t_2 \leq \min\{\mu_A(y_1), \mu_A(z_1)\}$. Therefore, Definition 3.1(2) is verified. If we put $t_3 = \max\{\lambda_A(a), \lambda_A(x)\}$, then $a, x \in L(\lambda_A; t_3)$. So there exist $y_2, z_2 \in L(\lambda_A; t_3)$ such that $x \in a + y_2$ and $x \in z_2 + a$, and we have $\max\{\lambda_A(y_2), \lambda_A(z_2)\} \leq t_3$, and so Definition 3.1(5) is verified.

Now, we verify the conditions (3) and (6). Let $t_4 = \mu_A(y)$ and $t_5 = \lambda_A(y)$ for some $x, y \in R$. Then $y \in U(\mu_A; t_4)$, $y \in L(\lambda_A; t_5)$. Since $U(\mu_A; t_4)$ and $L(\lambda_A; t_5)$ are H_v -ideals of R , then $x \cdot y \subseteq U(\mu_A; t_4)$ and $x \cdot y \subseteq L(\lambda_A; t_5)$. Therefore for every $z \in x \cdot y$, we have $z \in U(\mu_A; t_4)$ and $z \in L(\lambda_A; t_5)$ which imply that $\mu_A(z) \geq t_4$ and $\lambda_A(z) \leq t_5$. Hence

$$\begin{aligned} \inf \{ \mu_A(z) \mid z \in x \cdot y \} &\geq t_4 = \mu_A(y), \\ \sup \{ \lambda_A(z) \mid z \in x \cdot y \} &\leq t_5 = \lambda_A(y). \end{aligned} \quad (3.18)$$

This completes the proof. \square

COROLLARY 3.10. *Let I be a left H_v -ideal of an H_v -ring R . If fuzzy sets μ and λ are defined on R by*

$$\mu(x) = \begin{cases} \alpha_0 & \text{if } x \in I, \\ \alpha_1 & \text{if } x \in R \setminus I, \end{cases} \quad \lambda(x) = \begin{cases} \beta_0 & \text{if } x \in I, \\ \beta_1 & \text{if } x \in R \setminus I, \end{cases} \quad (3.19)$$

where $0 \leq \alpha_1 < \alpha_0$, $0 \leq \beta_0 < \beta_1$, and $\alpha_i + \beta_i \leq 1$ for $i = 0, 1$, then $A = (\mu, \lambda)$ is an intuitionistic fuzzy H_v -ideal of R and $U(\mu; \alpha_0) = I = L(\lambda; \beta_0)$.

COROLLARY 3.11. *Let χ_s be the characteristic function of a left H_v -ideal I of R . Then $A = (\chi_I, \chi_I^c)$ is a left intuitionistic fuzzy H_v -ideal of R .*

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THEOREM 3.12. *If $A = (\mu_A, \lambda_A)$ is a left intuitionistic fuzzy H_v -ideal of R , then for all $x \in R$,*

$$\begin{aligned}\mu_A(x) &= \sup \{ \alpha \in [0, 1] \mid x \in U(\mu_A; \alpha) \}, \\ \lambda_A(x) &= \inf \{ \alpha \in [0, 1] \mid x \in L(\lambda_A; \alpha) \}.\end{aligned}\tag{3.20}$$

Proof. Let $\delta = \sup \{ \alpha \in [0, 1] \mid x \in U(\mu_A; \alpha) \}$ and let $\varepsilon > 0$ be given. Then $\delta - \varepsilon < \alpha$ for some $\alpha \in [0, 1]$ such that $x \in U(\mu_A; \alpha)$. This means that $\delta - \varepsilon < \mu_A(x)$ so that $\delta \leq \mu_A(x)$ since ε is arbitrary.

We now show that $\mu_A(x) \leq \delta$. If $\mu_A(x) = \beta$, then $x \in U(\mu_A; \beta)$, and so

$$\beta \in \{ \alpha \in [0, 1] \mid x \in U(\mu_A; \alpha) \}.\tag{3.21}$$

Hence

$$\mu_A(x) = \beta \leq \sup \{ \alpha \in [0, 1] \mid x \in U(\mu_A; \alpha) \} = \delta.\tag{3.22}$$

Therefore

$$\mu_A(x) = \delta = \sup \{ \alpha \in [0, 1] \mid x \in U(\mu_A; \alpha) \}.\tag{3.23}$$

Now let $\eta = \inf \{ \alpha \in [0, 1] \mid x \in L(\lambda_A; \alpha) \}$. Then

$$\inf \{ \alpha \in [0, 1] \mid x \in L(\lambda_A; \alpha) \} < \eta + \varepsilon\tag{3.24}$$

for any $\varepsilon > 0$, and so $\alpha < \eta + \varepsilon$ for some $\alpha \in [0, 1]$ with $x \in L(\lambda_A; \alpha)$. Since $\lambda_A(x) \leq \alpha$ and ε is arbitrary, it follows that $\lambda_A(x) \leq \eta$.

To prove that $\lambda_A(x) \geq \eta$, let $\lambda_A(x) = \zeta$. Then $x \in L(\lambda_A; \zeta)$, and thus $\zeta \in \{ \alpha \in [0, 1] \mid x \in L(\lambda_A; \alpha) \}$. Hence

$$\inf \{ \alpha \in [0, 1] \mid x \in L(\lambda_A; \alpha) \} \leq \zeta,\tag{3.25}$$

that is, $\eta \leq \zeta = \lambda_A(x)$. Consequently

$$\lambda_A(x) = \eta = \inf \{ \alpha \in [0, 1] \mid x \in L(\lambda_A; \alpha) \},\tag{3.26}$$

which completes the proof. \square

4. Relations

Let $\alpha \in [0, 1]$ be fixed and let $\text{IF}(R)$ be the family of all intuitionistic fuzzy left H_v -ideals of an H_v -ring R . For any $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ from $\text{IF}(R)$, we define two binary relations \mathfrak{U}^α and \mathfrak{L}^α on $\text{IF}(R)$ as follows:

$$\begin{aligned}(A, B) \in \mathfrak{U}^\alpha &\iff U(\mu_A; \alpha) = U(\mu_B; \alpha), \\ (A, B) \in \mathfrak{L}^\alpha &\iff L(\lambda_A; \alpha) = L(\lambda_B; \alpha).\end{aligned}\tag{4.1}$$

These two relations \mathfrak{U}^α and \mathfrak{L}^α are equivalence relations. Hence $\text{IF}(R)$ can be divided into

the equivalence classes of \mathfrak{U}^α and \mathfrak{L}^α , denoted by $[A]_{\mathfrak{U}^\alpha}$ and $[A]_{\mathfrak{L}^\alpha}$ for any $A = (\mu_A, \lambda_A) \in \text{IF}(R)$, respectively. The corresponding quotient sets will be denoted as $\text{IF}(R)/\mathfrak{U}^\alpha$ and $\text{IF}(R)/\mathfrak{L}^\alpha$, respectively.

For the family $LI(R)$ of all left H_ν -ideals of R , we define two maps U_α and L_α from $\text{IF}(R)$ to $LI(R) \cup \{\emptyset\}$ putting

$$U_\alpha(A) = U(\mu_A; \alpha), \quad L_\alpha(A) = L(\lambda_A; \alpha) \quad (4.2)$$

for each $A = (\mu_A, \lambda_A) \in \text{IF}(R)$.

It is not difficult to see that these maps are well defined.

LEMMA 4.1. *For any $\alpha \in (0, 1)$, the maps U_α and L_α are surjective.*

Proof. Let $\mathbf{0}$ and $\mathbf{1}$ be fuzzy sets on R defined by $\mathbf{0}(x) = 0$ and $\mathbf{1}(x) = 1$ for all $x \in R$. Then $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1}) \in \text{IF}(R)$ and $U_\alpha(\mathbf{0}_\sim) = L_\alpha(\mathbf{0}_\sim) = \emptyset$ for any $\alpha \in (0, 1)$. Moreover, for any $K \in LI(R)$, we have $I_\sim = (\chi_K, \chi_K^c) \in \text{IF}(R)$, $U_\alpha(I_\sim) = U(\chi_K; \alpha) = K$, and $L_\alpha(I_\sim) = L(\chi_K^c; \alpha) = K$. Hence U_α and L_α are surjective. \square

THEOREM 4.2. *For any $\alpha \in (0, 1)$, the sets $\text{IF}(R)/\mathfrak{U}^\alpha$ and $\text{IF}(R)/\mathfrak{L}^\alpha$ are equipotent to $LI(R) \cup \{\emptyset\}$.*

Proof. Let $\alpha \in (0, 1)$. Putting $U_\alpha^*([A]_{\mathfrak{U}^\alpha}) = U_\alpha(A)$ and $L_\alpha^*([A]_{\mathfrak{L}^\alpha}) = L_\alpha(A)$ for any $A = (\mu_A, \lambda_A) \in \text{IF}(R)$, we obtain two maps:

$$U_\alpha^* : \text{IF}(R)/\mathfrak{U}^\alpha \longrightarrow LI(R) \cup \{\emptyset\}, \quad L_\alpha^* : \text{IF}(R)/\mathfrak{L}^\alpha \longrightarrow LI(R) \cup \{\emptyset\}. \quad (4.3)$$

If $U(\mu_A; \alpha) = U(\mu_B; \alpha)$ and $L(\lambda_A; \alpha) = L(\lambda_B; \alpha)$ for some $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ from $\text{IF}(R)$, then $(A, B) \in \mathfrak{U}^\alpha$ and $(A, B) \in \mathfrak{L}^\alpha$, whence $[A]_{\mathfrak{U}^\alpha} = [B]_{\mathfrak{U}^\alpha}$ and $[A]_{\mathfrak{L}^\alpha} = [B]_{\mathfrak{L}^\alpha}$, which means that U_α^* and L_α^* are injective.

To show that the maps U_α^* and L_α^* are surjective, let $K \in LI(R)$. Then for $I_\sim = (\chi_K, \chi_K^c) \in \text{IF}(R)$, we have $U_\alpha^*([I_\sim]_{\mathfrak{U}^\alpha}) = U(\chi_K; \alpha) = K$ and $L_\alpha^*([I_\sim]_{\mathfrak{L}^\alpha}) = L(\chi_K^c; \alpha) = K$. Also $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1}) \in \text{IF}(R)$. Moreover, $U_\alpha^*([\mathbf{0}_\sim]_{\mathfrak{U}^\alpha}) = U(\mathbf{0}; \alpha) = \emptyset$ and $L_\alpha^*([\mathbf{0}_\sim]_{\mathfrak{L}^\alpha}) = L(\mathbf{1}; \alpha) = \emptyset$. Hence U_α^* and L_α^* are surjective. \square

Now for any $\alpha \in [0, 1]$, we have the new relation \mathfrak{R}^α on $\text{IF}(R)$ putting

$$(A, B) \in \mathfrak{R}^\alpha \iff U(\mu_A; \alpha) \cap L(\lambda_A; \alpha) = U(\mu_B; \alpha) \cap L(\lambda_B; \alpha), \quad (4.4)$$

where $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$. Obviously, \mathfrak{R}^α is an equivalence relation.

LEMMA 4.3. *The map $I_\alpha : \text{IF}(R) \rightarrow LI(R) \cup \{\emptyset\}$ defined by*

$$I_\alpha(A) = U(\mu_A; \alpha) \cap L(\lambda_A; \alpha), \quad (4.5)$$

where $A = (\mu_A, \lambda_A)$, is surjective for any $\alpha \in (0, 1)$.

Proof. Indeed, if $\alpha \in (0, 1)$ is fixed, then for $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1}) \in \text{IF}(R)$, we have

$$I_\alpha(\mathbf{0}_\sim) = U(\mathbf{0}; \alpha) \cap L(\mathbf{1}; \alpha) = \emptyset, \quad (4.6)$$

and for any $K \in LI(R)$, there exists $I_\sim = (\chi_K, \chi_K^c) \in IF(R)$ such that $I_\alpha(I_\sim) = U(\chi_K; \alpha) \cap L(\chi_K^c; \alpha) = K$. \square

THEOREM 4.4. *For any $\alpha \in (0, 1)$, the quotient set $IF(R)/\mathfrak{R}^\alpha$ is equipotent to $LI(R) \cup \{\emptyset\}$.*

Proof. Let $I_\alpha^* : IF(R)/\mathfrak{R}^\alpha \rightarrow LI(R) \cup \{\emptyset\}$, where $\alpha \in (0, 1)$, be defined by the formula

$$I_\alpha^*([A]_{\mathfrak{R}^\alpha}) = I_\alpha(A) \quad \text{for each } [A]_{\mathfrak{R}^\alpha} \in IF(R)/\mathfrak{R}^\alpha. \quad (4.7)$$

If $I_\alpha^*([A]_{\mathfrak{R}^\alpha}) = I_\alpha^*([B]_{\mathfrak{R}^\alpha})$ for some $[A]_{\mathfrak{R}^\alpha}, [B]_{\mathfrak{R}^\alpha} \in IF(R)/\mathfrak{R}^\alpha$, then

$$U(\mu_A; \alpha) \cap L(\lambda_A; \alpha) = U(\mu_B; \alpha) \cap L(\lambda_B; \alpha), \quad (4.8)$$

which implies that $(A, B) \in \mathfrak{R}^\alpha$ and, in the consequence, $[A]_{\mathfrak{R}^\alpha} = [B]_{\mathfrak{R}^\alpha}$. Thus I_α^* is injective.

It is also onto because $I_\alpha^*(\mathbf{0}_\sim) = I_\alpha(\mathbf{0}_\sim) = \emptyset$ for $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1}) \in IF(R)$, and $I_\alpha^*(I_\sim) = I_\alpha(K) = K$ for $K \in LI(R)$ and $I_\sim = (\chi_K, \chi_K^c) \in IF(R)$. \square

The relation γ^* is the smallest equivalence relation on R such that the quotient R/γ^* is a ring. γ^* is called the fundamental equivalence relation on R and R/γ^* is called the fundamental ring, see [16].

According to [16], if \mathcal{U} denotes the set of all finite polynomials of elements of R over \mathbb{N} , then a relation γ can be defined on R as follows:

$$x\gamma y \quad \text{iff } \{x, y\} \subseteq u \text{ for some } u \in \mathcal{U}. \quad (4.9)$$

According to [16], the transitive closure of γ is the fundamental relation γ^* , that is, $a\gamma^*b$ if and only if there exist $x_1, \dots, x_{m+1} \in R$; $u_1, \dots, u_m \in \mathcal{U}$ with $x_1 = a$, $x_{m+1} = b$ such that

$$\{x_i, x_{i+1}\} \subseteq u_i, \quad i = 1, \dots, m. \quad (4.10)$$

Suppose that $\gamma^*(a)$ is the equivalence class containing $a \in R$. Then both the sum \oplus and the product \odot on R/γ^* are defined as follows:

$$\begin{aligned} \gamma^*(a) \oplus \gamma^*(b) &= \gamma^*(c) \quad \forall c \in \gamma^*(a) + \gamma^*(b), \\ \gamma^*(a) \odot \gamma^*(b) &= \gamma^*(d) \quad \forall d \in \gamma^*(a) \cdot \gamma^*(b). \end{aligned} \quad (4.11)$$

We denote the unit of the group $(R/\gamma^*, \oplus)$ by ω_R .

Defintion 4.5. Let R be an H_v -ring and μ a fuzzy subset of R . The fuzzy subset μ_{γ^*} on R/γ^* is defined as follows:

$$\begin{aligned} \mu_{\gamma^*} : R/\gamma^* &\longrightarrow [0, 1], \\ \mu_{\gamma^*}(\gamma^*(x)) &= \sup \{\mu(a) \mid a \in \gamma^*(x)\}. \end{aligned} \quad (4.12)$$

THEOREM 4.6. *Let R be an H_v -ring and $A = (\mu_A, \lambda_A)$ a left intuitionistic fuzzy H_v -ideal of R . Then $A/\gamma^* = (\mu_{\gamma^*}, \lambda_{\gamma^*})$ is a left intuitionistic fuzzy ideal of the fundamental ring R/γ^* .*

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