Non-Iterative Two-Dimensional Linear Discriminant Analysis

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Abstract

Linear discriminant analysis (LDA) is a well-known scheme for feature extraction and dimensionality reduction of labeled data in a vector space. Recently, LDA has been extended to two-dimensional LDA (2DLDA), which is an iterative algorithm for data in matrix representation. In this paper, we propose non-iterative algorithms for 2DLDA. Experimental results show that the non-iterative algorithms achieve competitive recognition rates with the iterative 2DLDA, while they are computationally more efficient than the iterative 2DLDA.

1. Introduction

Linear discriminant analysis (LDA) is a well-known scheme for feature extraction and dimensionality reduction of labeled data in a vector space. Therefore, if each datum is represented as matrix, then it must be transformed into a vector. However, this transformation from a matrix to a vector might cause the so-called singularity problem or the small sample size (S3) problem in which the scatter matrices become singular and thus the traditional LDA fails to work well. To address this problem, several extensions of LDA have been presented recently. Li and Yuan [2] and Xiong et al. [6] referred to 2DPCA [8] and presented two-dimensional linear discriminant analyses based on image between-class scatter matrix and image within-class scatter matrix. Their methods reduce the dimensionality of image matrices without converting the image matrices into vectors. Thus the singularity problem resulting from the high-dimensionality of vectors is artfully avoided. Liu et al. [3] also presented a similar method to theirs more than ten years ago. Yang et al. [7] and Song et al. [5] presented their variants. However, their methods reduce only the number of columns in each matrix and the number of rows is unchanged. Yang et al. [9] presented a two-stage algorithm and named it two-dimensional linear discriminant analysis (2DLDA), which reduces the number of columns first and then reduces the number of rows. Although Yang’s 2DLDA gives sequentially optimal transformation matrices, it is an order-dependent algorithm, i.e., we can consider the alternative algorithm, which reduces the number of rows first and then reduces the number of columns. Ye et al. [10] formulated the 2DLDA in an order-independent way and presented its iterative solution algorithm. Ye et al. also called their method 2DLDA. In their experiments, they showed that the accuracy curves of classification were stable with respect to the number of iterations. Therefore, they recommended stopping their algorithm at the first iteration. However, this simple algorithm is no less than the alternative algorithm of Yang’s 2DLDA. Bauckhage and Tsotsos [1] presented separable LDA toward the problem of binary (i.e., two-class) classification for visual object recognition.

In this paper, we propose a method for selecting the transformation matrices that give the higher value of the generalized Fisher criterion [7] from Yang’s 2DLDA and the simple algorithm of Ye’s 2DLDA. We also propose another non-iterative algorithm, namely the parallel algorithm, which treats rows and columns of matrices independently. Experimental results show that our non-iterative algorithms achieve competitive recognition rates with Ye’s 2DLDA, while our algorithms are computationally more efficient than Ye’s 2DLDA.

The organization of this paper is as follows: Section 2 summarizes the overview of Ye’s 2DLDA and exemplifies that the iterative algorithm of 2DLDA does not necessarily increase the value of the generalized Fisher criterion [7]. Section 3 proposes two non-iterative algorithms of 2DLDA, i.e., the selective and parallel algorithms. Section 4 presents the results of experiments. Section 5 offers our conclusion.

2. An Overview of 2DLDA

In this section, we give a brief overview of 2DLDA presented by Ye et al. [10].

Let $A_{i} \in \mathbb{R}^{r \times c}$, for $i = 1, \ldots, m$, be the $m$ images in a dataset, clustered into classes $\Pi_{1}, \ldots, \Pi_{n}$, where $\Pi_{j}$ has $n_{j}$ images. Let $M_{j} = \sum_{A_{i} \in \Pi_{j}} A_{i} / n_{j}$ be the mean of the

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Iterative 2DLDA

The above algorithm is summarized as follows:

\begin{equation}
D_w = \text{tr} \left( \sum_{j=1}^{n} \sum_{A_i \in \Omega_j} L^T(A_i - M_j)RR^T(A_i - M_j)^T L \right),
\end{equation}

\begin{equation}
D_b = \text{tr} \left( \sum_{j=1}^{n} n_j L^T(M_j - M)RR^T(M_j - M)^T L \right),
\end{equation}

where \( \text{tr} \) denotes the matrix trace. The optimal transformations \( L \) and \( R \) would maximize \( f(L, R) = D_w/D_b \). Due to the difficulty of computing the optimal \( L \) and \( R \) simultaneously, Ye et al. [10] derived the following iterative algorithm. Initially, we fix \( R = (I_{l_0}, 0)^T \). Then we can compute the optimal \( L \). Next, we fix the \( L \) and compute the optimal \( R \). This procedure is repeated a certain number of times. The above algorithm is summarized as follows:

**Iterative 2DLDA**

**Step 1:** Initialize \( R_0 = (I_{l_0}, 0)^T \) and \( k = 1. k \) indicates the number of iterations.

**Step 2:** Compute \( S_w^R = \sum_{j=1}^{n} \sum_{A_i \in \Omega_j} (A_i - M_j)R_{k-1}R_{k-1}^T(A_i - M_j)^T \) and \( S_b^R = \sum_{j=1}^{n} n_j(M_j - M)R_{k-1}R_{k-1}^T(M_j - M)^T \).

**Step 3:** Compute the first \( l_1 \) eigenvectors \( \phi_{1}^R, ..., \phi_{l_1}^R \) of \( (S_w^R)^{-1}S_b^R \) and form \( L_k = [\phi_{1}^R, ..., \phi_{l_1}^R] \).

**Step 4:** Compute \( S_w^L = \sum_{j=1}^{n} \sum_{A_i \in \Omega_j} (A_i - M_j)L_kL_k^T(A_i - M_j)^T \) and \( S_b^L = \sum_{j=1}^{n} n_j(M_j - M)^T L_kL_k^T(M_j - M) \).

**Step 5:** Compute the first \( l_2 \) eigenvectors \( \phi_{1}^L, ..., \phi_{l_2}^L \) of \( (S_w^L)^{-1}S_b^L \) and form \( R_k = [\phi_{1}^L, ..., \phi_{l_2}^L] \).

**Step 6:** Increase \( k \) by 1. If \( k < k_{\text{max}} \) then go to Step 2, otherwise output \( L = L_{k_{\text{max}}} \) and \( R = R_{k_{\text{max}}} \), where \( k_{\text{max}} \) is the maximal number of iterations.

However, this iterative algorithm does not necessarily increase \( f(L, R) \) monotonically. An example with the ORL face image database [4] is shown in Fig. 1, where the horizontal axis is the number of iterations and the vertical axis is \( f(L, R) \). We set \( l_1 = l_2 = 10 \) in this example. This non-monotonicity corroborates the perturbation in the accuracy curve of 2DLDA reported in [10]. Due to the non-monotonicity, it is hard to determine appropriate criteria for stopping the iteration. Therefore, Ye et al. [10] adopted a simple algorithm that outputs the result of the first iteration.

3. Non-Iterative 2DLDA

In this section, we propose two non-iterative algorithms for 2DLDA, i.e., the selective and parallel algorithms.

3.1 Selective Algorithm

The first iteration of 2DLDA computes \( L \) and \( R \) in turn with the initialization \( R = (I_{l_0}, 0)^T \). Alternatively, we can consider another algorithm that computes \( R \) and \( L \) in turn with the initialization \( L = (I_{l_0}, 0)^T \). This alternative algorithm is equivalent to another 2DLDA presented by Yang et al. [9]. In this subsection, we propose an algorithm unifying them. That is, we select \( L \) and \( R \) which give larger \( f(L, R) \). The selective algorithm is as follows:

**Selective Algorithm**

**Step 1:** Initialize \( R = (I_{l_0}, 0)^T \) and compute \( L \) and \( R \) in turn. Let \( L^{(1)} \) and \( R^{(1)} \) be the computed \( L \) and \( R \).

**Step 2:** Initialize \( L = (I_{l_0}, 0)^T \) and compute \( L \) and \( R \) in turn. Let \( L^{(2)} \) and \( R^{(2)} \) be the computed \( L \) and \( R \).

**Step 3:** If \( f(L^{(1)}, R^{(1)}) \geq f(L^{(2)}, R^{(2)}) \) then output \( L = L^{(1)} \) and \( R = R^{(1)} \), otherwise output \( L = L^{(2)} \) and \( R = R^{(2)} \).

3.2 Parallel Algorithm

In this subsection, we propose another non-iterative algorithm, which computes \( L \) and \( R \) independently. Let us define the row-row within-class and between-class scatter matrices as follows:

\begin{equation}
S_w^R = \sum_{j=1}^{n} \sum_{A_i \in \Omega_j} (A_i - M_j)(A_i - M_j)^T,
\end{equation}

\begin{equation}
S_b^R = \sum_{j=1}^{n} n_j(M_j - M)(M_j - M)^T.
\end{equation}
The optimal left side transformation matrix $L$ would maximize $\text{tr}(L^T S_w^L L) / \text{tr}(L^T S_w^U L)$. This optimization problem is equivalent to the following constrained optimization problem:

$$\max_L \quad \text{tr}(L^T S_w^L L)$$
subj.to $L^T S_w^U L = I_l,$  \hspace{1cm} (5)

where $I_l$ is the identity matrix of size $l$. Let $S_w^U = U \Lambda U^T$ be the eigen-decomposition of $S_w^U$, where $\Lambda$ is a diagonal matrix whose diagonal elements are eigenvalues of $S_w$ and $U$ is an orthonormal matrix whose columns are the corresponding eigenvectors. Substitution of $\tilde{L} = \Lambda^{1/2} U^T L$ into (5) gives

$$\max_L \quad \text{tr}(\tilde{L}^T \Lambda^{-1/2} U^T S_w^U \Lambda^{-1/2} \tilde{L})$$
subj.to $\tilde{L}^T \tilde{L} = I_l,$  \hspace{1cm} (6)

Let $\tilde{\phi}_1, \ldots, \tilde{\phi}_l$ be the first $l$ eigenvectors of $\Lambda^{-1/2} U^T S_w^U \Lambda^{-1/2}$. Then the optimal solution of (6) is expressed as $\tilde{L} = [\tilde{\phi}_1, \ldots, \tilde{\phi}_l]$. Consequently we can calculate the optimal solution of (5) as $L = U \Lambda^{-1/2} \tilde{L}$.

Alternatively, we define the column-column within-class and between-class scatter matrices as follows:

$$S_w = \sum_{j=1}^n \sum_{A \in \Omega_j} (A_i - M_j)^T (A_i - M_j),$$  \hspace{1cm} (7)

$$S_c = \sum_{j=1}^n n_j (M_j - M)^T (M_j - M).$$  \hspace{1cm} (8)

The optimal right side transformation matrix $R$ would maximize $\text{tr}(R^T S_c^R R) / \text{tr}(R^T S_w^R R)$. This optimization problem is equivalent to the following constrained optimization problem:

$$\max_R \quad \text{tr}(R^T S_c^R R)$$
subj.to $R^T S_w^R R = I_{l_2},$  \hspace{1cm} (9)

where $I_{l_2}$ is the identity matrix of size $l_2$. Let $S_c^R = V \Sigma V^T$ be the eigen-decomposition of $S_c^R$, where $\Sigma$ is a diagonal matrix whose diagonal elements are eigenvalues of $S_c^R$ and $V$ is an orthonormal matrix whose columns are the corresponding eigenvectors. Substitution of $\tilde{R} = \Sigma^{1/2} V^T R$ into (9) gives

$$\max_{\tilde{R}} \quad \text{tr}(\tilde{R}^T \Sigma^{-1/2} V^T S_c^R V \Sigma^{-1/2} \tilde{R})$$
subj.to $\tilde{R}^T \tilde{R} = I_{l_2},$  \hspace{1cm} (10)

Let $\tilde{\psi}_1, \ldots, \tilde{\psi}_{l_2}$ be the first $l_2$ eigenvectors of $\Sigma^{-1/2} V^T S_c^R V \Sigma^{-1/2}$. Then the optimal solution of (10) is expressed as $\tilde{R} = [\tilde{\psi}_1, \ldots, \tilde{\psi}_{l_2}]$. Consequently we can calculate the optimal solution of (9) as $R = V \Sigma^{-1/2} \tilde{R}$.

The above procedure is summarized as follows:

**Parallel Algorithm**

**Step A1:** Compute $S_w^U$ and $S_c^R$.
**Step A2:** Compute the eigen-decomposition $S_w^U = U \Lambda U^T$.
**Step A3:** Compute the first $l_1$ eigenvectors $\phi_1, \ldots, \phi_{l_1}$ of $\Lambda^{-1/2} U^T S_w^U \Lambda^{-1/2}$ and form $\tilde{L} = [\phi_1, \ldots, \phi_{l_1}]$.
**Step A4:** Compute $L = U \Lambda^{-1/2} \tilde{L}$.
**Step B1:** Compute $S_c^R$ and $S_w^U$.
**Step B2:** Compute the eigen-decomposition $S_c^R = V \Sigma V^T$.
**Step B3:** Compute the first $l_2$ eigenvectors $\psi_1, \ldots, \psi_{l_2}$ of $\Sigma^{-1/2} V^T S_c^R V \Sigma^{-1/2}$ and form $\tilde{R} = [\psi_1, \ldots, \psi_{l_2}]$.
**Step B4:** Compute $R = V \Sigma^{-1/2} \tilde{R}$.

Since the above algorithm computes $L$ and $R$ independently, we can interchange Step A1,...,A4 and Step B1,...,B4. Finally, we transform $A_i$ into $B_i = L^T A_i R$ for $i = 1, \ldots, m$.

**3.3 Classification**

Let $A$ be a test image to be classified. Then we classify $A$ into the $j^*$th class selected by the nearest-neighbor rule:

$$j^* = \arg \min_{j \in \{1, \ldots, n\}} \left( \min_{B_i \in H_j} \| B_i - B \|_F^2 \right),$$  \hspace{1cm} (11)

where $B = L^T A R$ and $\| \cdot \|_F$ denotes the Frobenius norm.

**4. Experimental Results**

In this section, we experimentally evaluate the performance of iterative and non-iterative 2DLDA on the ORL face image database. The ORL database [4] contains face images of 40 persons. Each person has 10 different images. That is, the total number of the images is 400. The size of each image is $112 \times 92$, i.e., $r = 112$ and $c = 92$. Some example images are shown in Fig. 2. We select five images per person for training, and the remaining five images for testing. So the sizes of training and testing data are both 200, i.e., $m = 200$. We make $l_1$ and $l_2$ the same size $p$ for simplifying our experiments. Figure 3 shows the result of face recognition. The horizontal axis denotes $p$ and the vertical axis denotes the recognition rate. We fixed the number of iterations of the iterative 2DLDA for 10. The recognition rate of the selective algorithm denoted by broken line is higher than that of the iterative 2DLDA denoted by solid line on the whole. The recognition rate of the parallel algorithm denoted by dotted line is a little lower than that of the iterative 2DLDA in the wide range of $p$.

We performed our experiments using Matlab on a Pentium IV 3.4GHz machine with 2GB RAM. Figure 4 shows the CPU time for training. This figure shows that the selective algorithm denoted by broken line is computationally
more efficient than the iterative 2DLDA denoted by solid line and the parallel algorithm denoted by dotted line is the fastest.

5. Conclusion

In this paper, we presented two non-iterative algorithms for 2DLDA, i.e., the selective and parallel algorithms. It is experimentally shown that our algorithms achieve competitive recognition rates with the iterative 2DLDA, while the CPU times of our algorithms are much shorter than that of the iterative 2DLDA.

2DLDA can be regarded as a sort of bilinear discriminant analysis. Extensions of 2DLDA to multilinear discriminant analyses are future works.

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References