Simplified Decoding for Some Non–Coherent Codes over the Grassmannian

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Abstract—We consider multiple–antenna non–coherent communications over Rayleigh flat fading channels and single–antenna non–coherent systems with unknown channel phase. For both systems, we propose some sub–optimal simplified decodings for the class of unitary space–time codes obtained via the exponential map [1]. To this aim, we use the geometrical interpretation of these codes as sets of points over the Grassmannian manifold \( G_{T,M} \). We compare these codes with other propositions in the literature, for which a simplified decoder exists.

I. INTRODUCTION

The design of space–time codes for multiple–antenna channels with no channel state information at the receiver (CSIR) has received less attention than the case with full CSIR, despite the high information rates achievable by these systems [2], [3]. In fact, even if the parameters which control some useful performance indicators (e.g. the pairwise error probability) are well known [4], [5], the lack of appropriate mathematical tools to manipulate them constitutes one of the major difficulties code designers have to cope with. An approach to solve this problem is the so–called training–based strategy [6], [7]. Part of the frame period is dedicated to send a pilot signal so that a rough channel estimation is possible. Then, coding methods developed for the coherent space–time codes can be applied. Another approach, developed in this article, focuses on unitary codebooks, since they are optimal from both a rate and an error probability perspective at high SNR [3], [5]. In this case, a parameterization of the set of unitary matrices must be chosen [1], [8]. The parameterization allows us to control to some extent code parameters, but the map is highly non–linear, so that finding an effective decoding strategy is often difficult.

In this work we discuss some simplified decodings for unitary codes obtained via the exponential parameterization of the Grassmannian \( G_{T,M} \), the manifold of \( M \)–dimensional subspaces of \( C^T \) with \( T \geq 2M \), as in [1]. Based on a geometric interpretation of the coding procedure, we implement a threshold detector and some more complex decoding strategies based on Schnorr–Euchner and Pohst algorithms. We compare them with the Generalized Likelihood Ratio Test (GLRT) performances obtained by exhaustive search. Finally, in the case of one and two transmit and receiver antennas, we compare the performance of our codes with other propositions for which a simplified decoding exists [6]–[9].

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II. SYSTEM MODEL

Consider a block Rayleigh flat fading channel, with \( M \) antennas at the transmitter, \( N \) antennas at the receiver. The transmission is made by blocks of \( T \) symbol periods. Inside the block, the fading is assumed constant. The received signal, named also received super–symbol is

\[
Y_{T \times N} = X_{T \times M} H_{M \times N} + \sqrt{M/\rho} T \times N, \quad (1)
\]

where the subscripts indicate matrix dimensions. Channel entries and noise entries are i.i.d. circularly complex Gaussian variables with unit variance, \( \rho \) is the average SNR per receive antenna and per transmit super–symbol \( X \). \( X \in C \), a codebook of unitary \( T \times M \) matrices.

In the case \( M = N = 1 \), we use a channel model with unknown phase only

\[
y = x e^{j \varphi_h} + w / \sqrt{\rho T} \quad (2)
\]

to better discriminate among the performances of different receivers [9]. We suppose that the random phase \( \varphi_h \) is uniformly distributed in \([0, 2\pi)\).

At the receiver the decision is made according to the GLRT, which does not require the knowledge of channel statistics since it is defined as \( \hat{X} = \arg\max_{X \in C} \sup_{H \in H} p(Y|X,H) \) [5]. In our case, that is i.i.d. fadings and unitary codebook, the GLRT is equivalent to the Maximum Likelihood (ML) criterion [5], and it takes the form

\[
\hat{X} = \arg\max_{X \in C} \|Y^\dagger X\|^2_F = \arg\min_{X \in C} \|Y^\dagger X^\perp\|^2_F \quad (3)
\]

where \( X^\perp \) is the orthogonal complement of \( X \), so that \( [XX^\perp] \) is square unitary. \( \| \cdot \|_F \) is the Frobenius norm and \( (\cdot)^\dagger \) is the transpose conjugate. We assume \( N = M \) and \( 2M \leq T \), because it is the optimal choice of the transmit antenna number from an information theoretical perspective [3] and it permits to have full diversity [5].

III. CODING PROCEDURE

A. The exponential parameterization

Each codeword \( X \) of \( C \) can be seen as a point over \( G_{T,M} \), i.e. it represents the subspace generated by its columns. We adopt the exponential parameterization of the Grassmannian presented in [1]: \( \{X = \exp(\alpha B)X_{ref}, B \in B\} \). \( B \) is a full–rate codebook designed for coherent systems, formed by \( M \times (T - M) \) complex matrices. \( X_{ref} = \exp(0_{T,M}) = I_{T,M} = \)}
\[ [I_M \ 0_{T-M}]^\dagger \] is called the reference subspace. The factor \( \alpha > 0 \) controls the distribution of the principal angles between \( X \) and \( X_{ref} \). The distribution of the principal angles among different codewords in \( C \) is imposed by the choice of the code \( B \) itself. Let
\[
[X X^\perp] = \exp \left\{ [0_M \ B \ 0_{T-M}] \right\}.
\]
(4)

Let \( B = U \Lambda V^\dagger \) be the thin singular value decomposition (SVD) \([10]\) of \( B \), where \( U \) is \( M \times M \) and unitary, \( V \) is \((T-M) \times M \) and has orthonormal columns, \( \Lambda \) is \( M \times M \) and diagonal. The Cosine-Sine (CS) decomposition \([10]\) of (4) yields
\[
X = \begin{bmatrix}
UCU^\dagger & \end{bmatrix}, \quad X^\perp = \begin{bmatrix}
USV^\dagger & VCV^\dagger + V^\perp (V^\perp)^\dagger
\end{bmatrix}
\]
(5)

where \( C = \cos \Lambda \) and \( S = \sin \Lambda \). Hence \( \Lambda \) is the matrix of the principal angles between \( X \) and \( X_{ref} \). Multiplying by \( \alpha \) modifies only the singular values matrix of \( aB \) which becomes \( \alpha \Lambda \). It is possible to inverse the map. Let \( X^\dagger = [X_1^\dagger \ X_2^\dagger] \), where \( X_1 \) is a positive defined Hermitian \( M \times M \) matrix, \( X_2 \) is \((T-M) \times M \) and apply the thin SVD to \( -X_2 = V_2 S_2 U_2^\dagger \).

If all the singular values in \( S \) are distinct, \( U_2 = U \) \( D \) is equal to \( U \) up to a right diagonal factor \( D = \text{diag}(d_1, \ldots, d_M) \) with \( |d_m| = 1 \) for all \( m \) \([10]\). \( V_2 \) is uniquely determined by \( U_2 \) and \( D \). As long as \( U_2 \) and \( V_2 \) comes from the same matrix \( X_2, B = U_2 \sin^{-1}(S)V_2^\dagger \) can be recovered, since the diagonal factor annihilates. Since \( C^2 + S^2 = I_M \), the columns of \( X \) are orthogonal. Condition \( 0 < \alpha \lambda \leq \pi / 2 \) on the generic singular value of \( \alpha B \) prevents the folding up of the constellation over the Grassmannian \([1]\).

**B. The One Transmit Antenna Case**

If \( M = 1 \), matrices \( X \) and \( B \) become respectively the \( T \times 1 \) vector \( x \) and the \( 1 \times (T-1) \) vector \( b \). Map (4) simplifies in
\[
x = \begin{bmatrix}
\cos \lambda & \\
-\sin \lambda & b
\end{bmatrix}, \quad \lambda = \|b\| < \pi / 2,
\]
(6)

where \( \lambda \) represents the principal angle between the subspace spanned by \( x \) and the reference subspace \( x_{ref} = e_1 = [1 \ 0 \ldots 0]^\dagger \). It is possible to find an explicit inverse map from \( G_{T,1} \) to the set of vectors \( b \in C^{T-1} \) whose norm \( \|b\| = \lambda \) satisfies \( \lambda < \pi / 2 \) (see \([11]\)). Let \( z = [z_1^\dagger, \ldots, z_T^\dagger] \in G_{T,1} \) be of the form (6), i.e. \( z_1 > 0 \). Then the corresponding \( b_{2}^\dagger \) is
\[
b_{2}^\dagger = -\frac{\rho_z}{\sin \rho_z} [z_2 \ldots z_T]^\dagger, \quad \rho_z = \arccos z_1.
\]
(7)

For codewords with small principal angle (\( \lambda \rightarrow 0 \)), map (6) can be linearized (\( x^\dagger \simeq [1 - b] \)).

**C. Metrics**

The exponential map has a clear geometrical interpretation. The \( T \times M \) matrix \( \Delta = -X_{ref}^\perp B^\dagger \) represents the direction of \( X \in G_{T,M} \) in the space tangent at \( X_{ref} \), denoted by \( T_x \) \([12]\). In the following, with a slight abuse, we say that \( B \) “lies on (tangent space) \( T_x \), to refer to this interpretation. A canonical inner product is defined in tangent space \( T_x \) as
\[
\langle \Delta_1, \Delta_2 \rangle = \text{tr}(\Delta_1^\dagger \Delta_2) = 2 \text{Re}(\text{tr}(B_i B_i^\dagger))
\]
(8)

where \( \Delta_i \) is the direction of \( X_i \) and Re(\cdot) is the real part. From the canonical metric (8) a canonical distance is derived
\[
d_{can}(X_1, X_2) = ||\Delta_1 - \Delta_2||_F^2 = 2||B_1 - B_2||^2_F.
\]
(9)

Different distances \( d(X_1, X_2) \) can be defined over the manifold, all depending on the principal angles \([12]\). A simple link between these distances and (9) seems hard to find.

**IV. DECODING**

The continuous maximization problem
\[
Y_c = \arg \max_{x \in G_{T,M}} \|Y^\dagger X\|_{F}^2
\]
(10)

has a well known solution: \( Y_c \) is the eigenspace spanned by the first \( M \) greatest eigenvalues of \( YY^\dagger \). Supposing that the non-zero eigenvalues of \( YY^\dagger \) are all distinct, in our case, the solution is unique when \( N \geq M \), since \( Y \) in (1) can not have rank greater than \( N \).

The maximization in (3) is made over a discrete set contained in \( G_{T,M} \). In this case the problem has always a solution, since it is a maximum over a discrete set, but a reduced complexity algorithm is hard to find.

![Fig. 1. (a) The noise (dotted sphere) uniformly perturbs the direction of x. (b) The level set \( L_r \) is given by \( s \in S^T \) so that \( \psi(s) = \cos \theta_r \).](image)
symmetry (see Fig. 1 (a)). The GLRT function in this case is \( \psi(x) = |x^T y|/|y| = |x^T y_c| = \cos(\lambda_{x,y_c}) \), where \( \lambda_{x,y_c} \) is the principal angle between \( x \) and \( y_c \). The level set \( L_r \) of \( \psi(x) \) over \( \mathbb{S}^T \) corresponding to the value \( \cos \theta_r \) is formed by the vectors \( x \in \mathbb{S}^T \) with the same principal angle \( \theta_r \) with \( y_c \). \( L_r \) is a complex sphere \( \mathbb{S}^{T-1} \) centered at \( o = \cos \theta_ry_c \) and of radius \( \sin \theta_r \). The parameter \( \theta_r \) can be linked to the noise variance in order to have a certain probability to find the vector \( x \) inside the part of \( \mathbb{S}^T \) bounded by \( L_r \).

We would like to use an efficient closest point search algorithm [13] in \( \mathbb{S}^T \), but this is not possible, because the exponential map does not transform a lattice in tangent space \( \mathbb{T}_x \) into a lattice in \( \mathbb{S}^T \). Conversely, we can use this algorithm in \( \mathbb{T}_x \), since \( B \) is carved from a lattice. However, due to the map non-linearity, the detection rule in \( \mathbb{T}_x \), which corresponds to the GLRT in the Grassmannian, does not coincide with the smallest Euclidean distance except for points close to the reference subspace (see Sec. III-B and the linearization of the exponential map). Hence, our idea is to map back to \( \mathbb{T}_x \) the level set \( L_r \), and obtain its preimage \( L_r^{-1} \). As a matter of fact, the codeword of \( B \) corresponding to the sent codeword will be inside the subset of \( \mathbb{T}_x \) bounded by \( L_r^{-1} \) with high probability. Our proposition is to list all codewords inside this subset by using the Pohst algorithm [13]. Then, GLRT metrics of these “candidate” codewords are calculated and the codeword with maximum metric is selected. This decoding strategy will be called local GLRT, because the GLRT is performed only on a list of codewords.

Due to lack of space, we rapidly sketch how to bound \( L_r^{-1} \) with a family of spheres. To calculate \( L_r^{-1} \), we express \( L_r \) in the orthonormal basis \( \{z, f_1, \ldots, f_{T-1}\} \), where \( z = y_c e^{-i \arg(y_c, 1)} \), \( f_1 = (e_1 - \cos \lambda_z, z)/\sin \lambda_c \), and the other vectors complete the basis of \( \mathbb{C}^T \). Then

\[
L_r = \{e^{i\xi} (\cos \theta_z z + e^{i\xi_1} \sin \theta_z (\sin \theta_1 f_1 + \cos \theta_1 v)) : \\
\xi_r, \xi_1 \in [0, 2\pi), \theta_1 \in [0, \pi/2), |v| = 1\}
\]

where \( v \) belongs to a \((T-2)\)-dimensional sphere in the space spanned by \( f_2, \ldots, f_{T-1} \). The parameter \( e^{i\xi_1} \) is used to impose form (6) to the points of \( L_r \), in order to invert the map and to obtain \( L_r^{-1} \). It can be shown that, if \( 0 \leq \lambda_c \leq \max(\theta_r, \pi/2 - \theta_r) \) (condition verified with probability one when noise variance and \( \theta_r \) go to zero), \( L_r^{-1} \) is contained inside the closed surface bounded by the set of spheres in \( \mathbb{C}^{T-2} \) of constant radius \( \mathcal{R} = (\lambda_c + \theta_r) \sin \theta_r / \sin(\lambda_c + \theta_r) \) and centered at a set of points \( b_o = b_o z \) with \( z = [z_2^* \ldots z_T^*] \), and \( b_o \in \mathbb{C} \) depends on \( \xi_1, \theta_1, \theta_0 \) and \( \lambda_c \).

In order to simplify the decoder, the set of spheres is approximated by three spheres (hence our algorithm is suboptimal). To summarize, once \( \theta_r < \pi/4 \) is fixed, the proposed decoding algorithm is as follows: 1) calculate \( z, b_o \) and \( \lambda_c \). 2) If \( \lambda_c < \pi/5 \), the non-linearity effect can be neglected: find the closest codeword to \( b_o \) in \( \mathbb{T}_x \) and stop. 3) If \( \pi/5 \leq \lambda_c \leq \pi/2 - \theta_r \), perform three sphere searches of radius \( R \) and centers \( b_{1,2,3} = o e^{i\varphi_{1,2,3}} z \), with \( \varphi_{1,2,3} = 0, \pm \arcsin(\tan \theta_r \tan \lambda_c)/2 \), \( o = \rho_0 \cos \theta_r \sin \lambda_c / \sin \rho_0 \), \( \rho_0 = \arccos(\cos \theta_r \cos \lambda_c) \). 4) If \( \pi/2 - \theta_r < \lambda_c < \pi/2 \), perform three sphere searches with the same parameters as in case 3) except \( \varphi_{1,2,3} = 0, \pm \arccos(1 - (R/o)^2)/2 \). 5) Calculate GLRT metrics on the list coming from step 3) or 4) and select the codeword corresponding to the maximum.

The computational complexity of the proposed algorithm is roughly \( O(8(T - 1)^3) \), due to the Pohst algorithm. About \( L_{av,T} \) cells of memory are needed to store the candidate codewords, where \( L_{av} \) is the average length of the list. \( L_{av} \) can be controlled, since the lattice is known and \( \theta_r \) can be chosen to set properly the radius \( R \) of the sphere (if \( \theta_r \to 0 \), then \( R \to 0 \)) according to the noise level.

In the case \( N > 1 \), the same decoder can be used, preceded by an algorithm which exploits the different received vector to provide a better estimation \( y_c \) of \( x \).

B. Simplified decoding, case \( M > 1 \)

Codes over \( G_{T,M} \) have no apparent structure, since the exponential map is non-linear. Moreover, in this case the analytical study of Sec. IV-A cannot be pursued. Again, our idea is to take the decision on the tangent space at \( x_{ref} = I_{T,M} \), exploiting the properties of code \( B \). The detector chooses the codeword whose direction in \( \mathbb{T}_x \) is closest to the direction \( \Delta_x = -X_{ref} b_\| \), corresponding to the solution of (10). To this aim, a maximization of the cosine between pairs of directions \( \langle \Delta_x, \Delta \rangle / (||\Delta_x|| F ||\Delta|| F) \) is performed. From (9) it is equivalent to

\[
\min_{B \in \mathbb{B}} \frac{||B - \mathbf{B}_0||_F}{||\mathbf{B}_0||_F}.
\]

We give an algorithm to calculate \( \mathbf{B}_z \), which yields the exact sent matrix when no additive white Gaussian noise is present in the channel. The received codeword can always be written as

\[
\mathbf{Y} = \mathbf{Z} \mathbf{H}_c, \quad \text{with } \mathbf{Z}^\dagger \mathbf{Z} = \mathbf{I}_M, \mathbf{H}_c \in \mathbb{C}^{M \times N}
\]

where \( \mathbf{Z} \) has the form of Equation (5) and \( \mathbf{H}_c \) is in general a noisy estimation of the channel.

1) Let \( \mathbf{Y}_{c1} \) be the upper \( M \times M \) sub-matrix of \( \mathbf{Y}_c \), given by (10). Calculate the SVD \( \mathbf{Y}_{c1} = \mathbf{U}_1 \mathbf{C}_1 \mathbf{V}_1^\dagger \).

2) \( \mathbf{Z} \) in form of (5) is \( \mathbf{Z} = \mathbf{Y}_c \mathbf{V}_1 \mathbf{U}_1^\dagger \). Inverse the exponential map to obtain \( \mathbf{B}_z \), as described in Sec. III-A. Finally, \( \mathbf{H}_c \) (a rough channel estimation) can be obtained from (12), i.e. \( \mathbf{H}_c = \mathbf{Z}^\dagger \mathbf{Y} \).

Then, an efficient method to implement (11) is the Schnorr–Euchner algorithm [13] as long as \( ||B||_F \) is constant for all the codewords in \( B \). If this is not the case, the same algorithm can be used for each code subset with constant \( ||B||_F \). To improve the performance, the GLRT metric can be computed on the list of codewords (local GLRT). Since the code lattice is known a priori, the radius of the sphere can be controlled. Then, the Pohst algorithm efficiently calculates lists with stable lengths, roughly independent of the SNR [14].
C. Mismatched Decoding

Another decoder for the case \( M > 1 \) is introduced. This heuristic mismatched distance takes into account the effect of the channel:

\[
d_{\text{heu}}(Z, X) = \| H^T (B_2 - B) \|_F . \tag{13}
\]

Rule (13) can be justified as follows. If all the codewords have small principal angles with respect to \( X_{ref} \), in a first order approximation, the exponential map (5) becomes \( X = [I_M - B X] \), since \( \cos(A) \approx I_M \) and \( \sin(A) \approx A \). Then, the system approximatively reduces to the training-based systems as in [7]. The derivation of (13) is trivial. For our codebooks, the approximation is good only for codewords close to \( X_{ref} \) (in the sense of small principal angles). On the other hand, when codewords are far away from the reference space, the mismatched metric does not take into account the non-linearity of the exponential map. Thus, a performance degradation is expected. The computational complexity of this decoder can be roughly estimated as \( O(8((T - M) M^3)) + O(L_u T N M) \), where \( L_u \) is the average length of candidate codewords lists. The first term is due to the Pohst algorithm, while the second one is due to the GLRT metric calculations of the codewords in the list.

V. EXAMPLES: \( G_{T,1} \) AND \( G_{4,2} \)

In the \( G_{T,1} \) case, with \( N = 1 \), we use a uncoded unit-energy (16–QAM) \( T^{-1} \) constellation, with \( T = 3 \) and \( T = 6 \). The factor \( \alpha = \pi \sqrt{10}/(2(3\sqrt{2}(T - 1) + 1)) \) is justified in [11]. Spectral efficiency \( \eta \) is \( 4(1 - 1/T) \) bits/s/Hz.

In the \( G_{4,2} \) case, let \( M = N = 2 \) and \( T = 4 \), \( s = [s_1^* s_2^* s_3^* s_4^*]^T \). Symbols \( s_i \) are drawn from a 4–QAM and an 8–QAM, so that the spectral efficiency \( \eta \) is respectively 2 bits/s/Hz and 3 bits/s/Hz. Two codes are compared. Code \( B_1 \) is proposed in [15], gives the non–coherent code \( C_1 \) (see also [11]), and its codewords are given by

\[
B = \frac{1}{\sqrt{2}} \begin{bmatrix}
1 & \phi_2 & \theta(s_3 + \phi s_4) & \phi (s_3 + \phi s_4) \\
\phi_2 & \theta(s_3 - \phi s_4) & s_1 - \phi s_2
\end{bmatrix}
\]

with \( \psi^2 = \phi = e^{\pi/4} \). It holds \( \| B \|_F = \| \text{vec}(B) \| = \| \Phi_1 \| = \| s \| \), because \( \Phi_1 \) is unitary [15], and where \( \text{vec}(B) \) is a vector built with the columns of \( B \).

Code \( B_2 \), presented in [16], gives the non–coherent code \( C_2 \), and its codewords are given by

\[
B = \frac{1}{\sqrt{5}} \begin{bmatrix}
\phi_r (s_1 + r s_2) & \phi_r (s_3 + r s_4) \\
\phi_r (s_3 - r s_4) & \phi_r (s_1 + r s_2)
\end{bmatrix}
\]

where \( r = (1 + \sqrt{5})/2, \bar{r} = (1 - \sqrt{5})/2, \phi_r = 1 + i(1 - r) \) and \( \phi_2 = 1 + i(1 - \bar{r}) \). As in the previous case \( \| B \|_F = \| \Phi_2 \| = \| s \| \) is constant for M–PSK alphabets, because \( \Phi_2 \) is unitary.

For the 4–QAM case, \( \alpha \) has been chosen to minimize the super–symbol error probability at 15 dB SNR. For \( C_1 \) we chose \( \alpha = 0.566 \) and for \( C_2 \) we set \( \alpha = 0.6 \). In the 8–QAM case, we maximize the minimum product distance \( \prod_{m=1}^{\overline{M}} \sin^2 \theta_m \) where \( \theta_1, \ldots, \theta_M \) are the principal angles between any two codewords [1]. For \( C_1 \) we chose \( \alpha = 1.267 \), while for \( C_2 \) \( \alpha = 1.400 \).
block decoding algorithm [9], which works only with PSK\(^{T-1}\) constellations.

In the case \(M = 2\), with ML decoding (Fig. 3), code \(\mathcal{C}_2\) slightly outperforms code \(\mathcal{C}_1\). This is due to a better distances distribution of the principal angles between pairs of codewords. The change of the curve slope (loss of diversity) for the threshold detector is due to the fact that the metric (11) does not take into account the channel effect, and it does not correspond to the GLRT rule in \(G_{T,M}\).

The performance of the proposed threshold decoder slightly outperforms the behavior of the decoders in [8, fig. 2] and [17] below 18 dB SNR, while the trend changes at higher SNR. Our detector is very simple, it inverts the map, it rotates the symbol and makes two comparisons. The super-symbol error probability can be decreased performing a local GLRT over a list of points around \(b_r\). In Fig. 3, by using lists of average length 12 an improvement of about 4 dB can be appreciated, but the diversity is not recovered.

At low SNR, the training–based codes of [6], [7], [18] with sphere decoder have comparable performance to the proposed local GLRT decoder, where the list of points is calculated using the mismatched metric (13) (see Fig. 3 and Fig. 4). Our proposed decoder does not maintain full diversity, as explained in Sec. IV–B. However, even if training–based non–unitary codes are far better than unitary ones [18], our proposed unitary codes, which do not follow a genuine training–based approach, are still comparable with the best training–based non–unitary codes at spectral efficiencies of 2 and 3 bits/s/Hz (with ML decoding for every SNR, with mismatched decoding only at low SNR).

VII. CONCLUSIONS

In this paper, we presented some simplified decoding strategies for unitary space–time codes obtained via the exponential map from coherent codes. In the one transmit antenna case, we were able to analyze the shape of the decision regions induced by the GLRT in the tangent space. Hence, we succeeded in coping with the strong non–linearity introduced by the non–linear map and a sub–optimal decoder based on the Pohst algorithm has been proposed. In the multiple–antenna case, the analytical approach was not feasible. We proposed other simplified decodings which, nevertheless, do not achieve full diversity. Finally, we made comparisons with other codes proposed in the literature.

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