Cardinality and Entropy for Bifuzzy Sets

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Summary

- **intuitionistic** fuzzy sets and **bifuzzy** sets
- **bipolar** space (τ, δ) for representation of bifuzzy sets
- **penta-valued** space (t, f, u, c, i) for representation of bifuzzy sets
- **new** metric **distance** for the interval [0,1]
- **entropy** measures for bifuzzy sets
- scalar **cardinality** measures for bifuzzy sets.

Intuitionistic Fuzzy Sets

Atanassov defined the intuitionistic fuzzy set as an extension of fuzzy set. An intuitionistic fuzzy set A in X is described by:

$\mu_A : X \to [0,1]$ **membership function**

$\nu_A : X \to [0,1]$ **non-membership function**

$\pi_A : X \to [0,1]$ **margin of hesitation**

These functions verify the properties:

$$\forall x \in X, \qquad \mu_A(x) + \nu_A(x) \le 1 \qquad (1)$$

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \qquad (2)$$

Bifuzzy Sets

A bifuzzy set A in X is defined by:

$\mu_A : X \rightarrow [0,1]$ the **membership** function

$\nu_A : X \rightarrow [0,1]$ the **non-membership** function

The two functions μ_A and ν_A are totally **independently**.

The ***union*** $A \bigcup B$:

$$\left(\mu_{A \bigcup B}, \nu_{A \bigcup B}\right) = \left(\mu_A \vee \mu_B, \nu_A \wedge \nu_B\right) \tag{1}$$

The ***intersection*** $A \bigcap B$:

$$\left(\mu_{A \bigcap B}, \nu_{A \bigcap B}\right) = \left(\mu_A \wedge \mu_B, \nu_A \vee \nu_B\right) \tag{2}$$

The ***complement*** A^c

$$\left(\mu_{A^c}, \nu_{A^c}\right) = \left(\nu_A, \mu_A\right) \tag{3}$$

The ***negation*** A^n

$$\left(\mu_{A^n}, \nu_{A^n}\right) = \left(1 - \mu_A, 1 - \nu_A\right) \tag{4}$$

The ***dual*** A^d

$$\left(\mu_{A^d}, \nu_{A^d}\right) = \left(1 - \nu_A, 1 - \mu_A\right) \tag{5}$$

The bipolar representation of bifuzzy sets (τ, δ)

$$\forall(\mu,\nu)\in[0,1]^2 \quad \rightarrow \quad (\tau,\delta)\in[-1,1]^2$$

net truth $$\tau=\mu\circ\bar{\nu}-\bar{\mu}\circ\nu \tag{1}$$

definedness $$\delta=\mu\circ\nu-\bar{\mu}\circ\bar{\nu} \tag{2}$$

where $\circ$ is a t-norm and $\bar{x}$ is the negation of x

$\circ$ is a **Frank t-norm**

$$\tau=\mu-\nu \tag{3}$$

$$\delta=\mu+\nu-1 \tag{4}$$

$$|\tau|+|\delta|\leq 1 \tag{5}$$

Penta-valued representation of bifuzzy sets

index of ***truth***: $t = \tau_+$ (1)

index of ***falsity***: $f = \tau_-$ (2)

index of ***contradiction***: $c = \delta_+$ (3)

index of ***undefinedness***: $u = \delta_-$ (4)

index of ***ambiguity***: $i = 1 - |\tau| - |\delta|$ (5)

$$t + f + u + c + i = 1 \quad (6)$$

$$t \cdot f = u \cdot c = 0 \quad (7)$$

$$\tau = \mu - \nu \tag{1}$$

$$\delta = \mu - \overline{\nu} = \nu - \overline{\mu} \tag{2}$$

$$x \ominus y = \frac{x-y}{1+|x-y|+|x+y-1|} \tag{3}$$

$$\tau^* = \mu \ominus \nu \tag{4}$$

$$\delta^* = \mu \ominus \bar{\nu} = \nu \ominus \bar{\mu} \tag{5}$$

New metric distance on the interval [0,1]

$$d(x,y) = |x - y| \quad (1)$$

$$D(x,y) = |x \ominus y| \quad (2)$$

$$D(x,y) = \frac{2|x-y|}{1+|x-y|+|x+y-1|} \quad (3)$$

- $D(x,y) = 0 \Leftrightarrow x = y$ (4)
- $D(x,y) = D(y,x)$ (5)
- $D(x,y) + D(y,z) \geq D(x,z)$ (6)

New similarity on the interval [0,1]

$$d(x, y) = |x - y| \tag{1}$$

$$s(x, y) = 1 - |x - y| \tag{2}$$

$$S(x, y) = 1 - D(x, y) \tag{3}$$

$$S(x, y) = \frac{1 - |x - y| + |x + y - 1|}{1 + |x - y| + |x + y - 1|} \tag{4}$$

Entropy variants for intuitionistic fuzzy sets

- Bustince – Burillo entropy

$$e_{BB} = \pi \qquad (1)$$

- Zeng – Li entropy

$$e_{ZL} = 1 - |\mu - \nu| \qquad (2)$$

- Grzegorzewski – Mrówka **vector** entropy:

$$\vec{e}_{GM} = \left(1 - |\mu - \nu| \quad ; \quad \pi \right) \qquad (3)$$

- Szmidt – Kacprzyk entropy:

$$e_{SK} = \frac{1 - |\mu - \nu| + \pi}{1 + |\mu - \nu| + \pi} \qquad (4)$$

Entropy = component $_1$ + component $_2$

Bustince – Burillo $$? = \pi + ? \quad (1)$$

Zeng – Li $$1-|\mu-\nu| = ? + ? \quad (2)$$

Grzegorzewski – Mrówka $$1-|\mu-\nu| = \pi + ? \quad (3)$$

Szmidt – Kacprzyk $$\frac{1-|\mu-\nu|+\pi}{1+|\mu-\nu|+\pi} = ? + ? \quad (4)$$

Bifuzzy entropy in the space (t, f, u, c, i)

Entropy = **similarity** measure between the **vector** $\mathrm{v} = (t, f, u, c, i)$ and **its complement** $\mathrm{v}^{c} = (f, t, u, c, i)$

$$e = S_B(\mathrm{v}, \mathrm{v}^c) \tag{1}$$

the **Bhattacharyya similarity**:

$$S_B(\mathrm{v}, \mathrm{v}^c) = \sqrt{t \cdot f} + \sqrt{f \cdot t} + \sqrt{u \cdot u} + \sqrt{c \cdot c} + \sqrt{i \cdot i} \tag{2}$$

$$t \cdot f = 0$$

$$e = u + c + i \tag{3}$$

Bifuzzy entropy components

$$e = u + c + i$$

- the first one is connected with the **missing information** that results in a ”gap” between the membership function μ and the non-membership function ν.

- the second is connected with the **supplementary information** that results in a ”glut” between the membership function μ and the non-membership function ν.

- the third is connected with the **lack of distinction** between the vectors $x=(\mu,\nu)$ and $a=(0.5,0.5)$

Bifuzzy entropy in the space (μ, ν)

Entropy = similarity measure between **membership** function and **non-membership** function values.

$$e = s(\mu,\nu) = 1 - |\mu - \nu| \qquad (1)$$

$$e^* = S(\mu,\nu) = \frac{1 - |\mu - \nu| + |\mu + \nu - 1|}{1 + |\mu - \nu| + |\mu + \nu - 1|} \qquad (2)$$

Bifuzzy entropy components in the space (μ, ν) (I)

Using the classic difference $(x-y)$ one obtains:

- entropy

$$e = 1 - |\mu - \nu| \tag{1}$$

- undefinedness

$$u = (1 - \mu - \nu)_{+} \tag{2}$$

- contradiction

$$c = (\mu + \nu - 1)_{+} \tag{3}$$

- ambiguity

$$i = 1 - |\mu - \nu| - |\mu + \nu - 1| \tag{4}$$

Bifuzzy entropy components in the space (μ, ν) (II)

Using the difference $(x \ominus y)$ one obtains:

$$e^{*} = \frac{1-|\mu-\nu|+|\mu+\nu-1|}{1+|\mu-\nu|+|\mu+\nu-1|} \quad (1)$$

- undefinedness

$$u = \frac{2\cdot(1-\mu-\nu)_{+}}{1+|\mu-\nu|+|\mu+\nu-1|} \quad (2)$$

- contradiction

$$c = \frac{2\cdot(\mu+\nu-1)_{+}}{1+|\mu-\nu|+|\mu+\nu-1|} \quad (3)$$

- ambiguity

$$i = \frac{1-|\mu-\nu|-|\mu+\nu-1|}{1+|\mu-\nu|+|\mu+\nu-1|} \quad (4)$$

Bifuzzy entropy properties (I)

In the space (μ, ν) the entropy function $e(\mu, \nu)$ verifies these five conditions:

1) $e(1{,}0) = e(0{,}1) = 0$

2) $e(\mu, \nu) = 1 \;\Leftrightarrow\; \mu = \nu$

3) if $[\nu_1, \mu_1] \subset [\nu_2, \mu_2]$ then $e(\mu_1, \nu_1) > e(\mu_2, \nu_2)$

4) if $[\mu_1, \nu_1] \subset [\mu_2, \nu_2]$ then $e(\mu_1, \nu_1) > e(\mu_2, \nu_2)$

5) $e(\mu, \nu) = e(\nu, \mu) = e(1-\nu, 1-\mu) = e(1-\mu, 1-\nu)$

These properties represent an extension of properties considered by **De Luca** and **Termini** for fuzzy sets and by **Szmidt** and **Kacprzyk** for intuitionistic fuzzy sets.

Bifuzzy entropy in the space (τ, δ)

Entropy = the ratio between the distance to the **nearest crisp element** and the distance to the **farthest crisp element.**

$$\mathrm{v} = (\tau, \delta) \qquad\qquad e = \frac{d(\mathrm{v}, \mathrm{v}_{near})}{d(\mathrm{v}, \mathrm{v}_{far})} \qquad (1)$$

$$e = \frac{\min(|1-\tau| + |\delta|\ , |1+\tau| + |\delta|)}{\min(|1-\tau| + |\delta|, |1+\tau| + |\delta|)} \qquad (2)$$

$$e = \frac{1 - |\tau| + |\delta|}{1 + |\tau| + |\delta|} \qquad (3)$$

Bifuzzy entropy components in the space (τ, δ)

- entropy

$$e = \frac{1 - |\tau| + |\delta|}{1 + |\tau| + |\delta|} \qquad (1)$$

- undefinedness

$$u = \frac{2 \cdot \delta_-}{1 + |\tau| + |\delta|} \qquad (2)$$

- contradiction

$$c = \frac{2 \cdot \delta_+}{1 + |\tau| + |\delta|} \qquad (3)$$

- ambiguity

$$i = \frac{1 - |\tau| - |\delta|}{1 + |\tau| + |\delta|} \qquad (4)$$

Bifuzzy entropy properties (II)

In the space (τ, δ) the entropy function $e(\tau, \delta)$ verifies these five conditions:

1) $e(1,0) = e(-1,0) = 0$

2) $e(0, \delta) = 1$

3) if $|\tau_1| > |\tau_2|$ then $e(\tau_1, \delta) < e(\tau_2, \delta)$

4) if $|\delta_1| > |\delta_2|$ then $e(\tau, \delta_1) \geq e(\tau, \delta_2)$

5) $e(\tau, \delta) = e(-\tau, \delta) = e(\tau, -\delta) = e(-\tau, -\delta)$

Intuitionistic fuzzy entropy variants (I)

Bustince - Burillo entropy $e_{BB} = \pi$

Zeng - Li entropy $e_{ZL} = 1 - |\mu - \nu|$

Grzegorzewski - Mrówka **vector** entropy: $\vec{e}_{GM} = \left(1 - |\mu - \nu|; \pi \right)$

$$1 - |\mu - \nu| = \pi + 2 \cdot \min(\mu, \nu) \qquad (1)$$

Intuitionistic fuzzy entropy variants (II)

Szmidt – Kacprzyk entropy:

$$e_{SK} = \frac{1 - |\mu - \nu| + \pi}{1 + |\mu - \nu| + \pi} \qquad (1)$$

Entropy = undefinedness + ambiguity

$$e_{SK} = \frac{\pi}{1 - \min(\mu, \nu)} + \frac{\min(\mu, \nu)}{1 - \min(\mu, \nu)} \qquad (2)$$

Cardinality in the space (t, f, c, u, i)

$\mathrm{v} = (t, f, c, u, i)$

$$t + u + c + i + f = 1 \qquad (1)$$

$$\left(t + \frac{u + c + i}{2}\right) + \left(\frac{u + c + i}{2} + f\right) = 1 \qquad (2)$$

$$n(\mathrm{v}) = t + \frac{u + c + i}{2} \qquad (3)$$

the scalar **cardinality** for a **bifuzzy** set A:

$$card(A) = \sum_{x \in X} n(x) \qquad (4)$$

Bifuzzy cardinality functions

- Using the **classic** difference $(x - y)$ one obtains:

$$n = \frac{1}{2} + \frac{\mu - \nu}{2} \tag{1}$$

- Using the **generalized** difference $(x \ominus y)$ one obtains:

$$n^* = \frac{1}{2} + \frac{\mu - \nu}{1 + |\mu - \nu| + |\mu + \nu - 1|} \tag{2}$$

Bifuzzy cardinality function properties

In the space (μ,ν) the cardinality function $n(\mu,\nu)$ verifies these five conditions:

1) $n(1,0)=1,\ n(0,1)=0$

2) if $\mu_1<\mu_2$ then $n(\mu_1,\nu)<n(\mu_2,\nu)$

3) if $\nu_1>\nu_2$ then $n(\mu,\nu_1)<n(\mu,\nu_2)$

4) $n(\mu,\nu)+n(\nu,\mu)=1$

5) $n(\mu,\nu)=n(1-\nu,1-\mu)$

Scalar cardinality for intuitionistic fuzzy sets

$$n = \mu + \frac{\pi}{2} \tag{1}$$

$$n^{*} = \frac{1}{2} + \frac{\mu - \nu}{1 + |\mu - \nu| + \pi} \tag{2}$$

Scalar cardinality for fuzzy sets

$$n = \mu \tag{1}$$

$$n^* = \frac{1}{2} + \frac{2\mu - 1}{1 + |2\mu - 1|} \tag{2}$$

Thank you for your attention !