Random matrix theory applied to acoustic backscattering and imaging in complex media

Alexandre Aubry\textsuperscript{(1,2)}, Arnaud Derode\textsuperscript{(2)}

(1) John Pendry's group, Imperial College London, United Kingdom

(2) Mathias Fink, Institut Langevin “Ondes et Images”, ESPCI ParisTech, Paris, France
Introduction

Wave propagation in random media is a multidisciplinary subject (optics, solid state physics, microwaves, acoustics, seismology, etc.)

Experimental flexibility of ultrasound

- Multi-element array
- Time-resolved measurement of the amplitude and phase of the wave field

Numerous applications

- Imaging (adaptive focusing)
- Time reversal
- Spatial and temporal evolution of multiple scattering intensity (coherent backscattering)

Interest of a matrix approach

- Acquisition of the inter-element matrix $K$.
- This matrix contains all the information available on the medium under investigation.
**Introduction**

**Matrix K in « simple » media**  

D.O.R.T method: Singular value decomposition of $K$

$$K = U \Lambda^t V^*$$  
$\Lambda$: Diagonal matrix containing the $N$ singular values ($\lambda_1 > \lambda_2 > ... > \lambda_N$)  
$U, V$: Unitary matrices whose columns are the singular vectors

Simple media: one eigenstate $\Leftrightarrow$ one scatterer

Each eigenvector $V_i$ back-propagates respectively towards each scatterer of the medium

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**Matrix K in complex media?**

- Fundamental interest: link with random matrix theory
- Interest for detection and imaging (separation between single and multiple scattering)
- Interest for the study of multiple scattering (characterization)
Introduction

One important parameter: the scattering mean free path $l_e$

Single scattering ($t \ll l_e/c$)
- Classical imaging techniques (ultrasound imaging, D.O.R.T method)

Multiple scattering ($t \gg l_e/c$)
- Nightmare for imaging
- Statistical approach, measurements of transport parameters ($l_e$, $D$...)

Which frequency, which medium?

→ 2 MHz à 5 MHz, $\lambda \sim 0.5$ mm

Random scattering sample of steel rods ($l_e \sim 10$ mm)
Agar-agar gel ($l_e \sim 1000$ mm)
Human trabecular bone ($l_e \sim 20$ mm)
Human soft tissues ($l_e \sim 100$ mm)
Outline

Propagation operator in random media

➢ Link with random matrix theory
➢ Different statistical properties between single and multiple scattering

Separation of the single and multiple scattering contributions

➢ Target detection in strongly scattering media
➢ Multiple scattering in weakly scattering media (breast tissues)

Imaging of diffusive media with multiple scattering

➢ Local measurements of the diffusion constant
➢ Application to human trabecular bone imaging
Propagation matrix in random media
**Experimental procedure: Time frequency analysis**

1/ Acquisition of the inter-element matrix

\[
H(t) = [h_{ij}(t)]
\]

2/ Short-time Fourier analysis:

\[
H(t) = [h_{ij}(t)]
\]

\[
w(t) = 1 \text{ pour } t \in [-5,5 \mu s]
\]

\[
w(t) = 0 \text{ elsewhere}
\]

\[
k_{ij}(T,t) = h_{ij}(T - t)w(t)
\]

\[
K(T,t)
\]

Numerical Fourier Transform

\[
K(T,f)
\]

Keep the temporal resolution provided by ultrasonic measurements.

The length \( \delta t \) of time window is chosen so that signals associated with the same scattering path arrive in the same time-window.
Propagation matrix in random media

Statistical properties of $\mathbf{K}$ in the multiple scattering regime

Experiment in a highly scattering medium

Random scattering sample of steel rods ($l_e \sim 10$ mm)

Matrix $\mathbf{K}$ ($32 \times 32$)

Real part of matrix $\mathbf{K}$, $T=120\mu s$, $f=3.1$ MHz

Random distribution of scatterers

$\mathbf{K}(T,f) =$ random matrix

$\text{SVD } \mathbf{K} = \mathbf{U} \Lambda^t \mathbf{V}^*$

No longer one-to-one equivalence between eigenstates and scatterers of the medium

Statistical approach

Distribution of singular values = Quarter circle law

$\rho(\lambda) = \frac{1}{\pi} \sqrt{4 - \lambda^2}$

V. Marcenko and L. Pastur, Math. USSR-Sbornik 1, 457, 1967

Single scattering

Occurrence of a deterministic coherence along the antidiagonals of $K$ in the single scattering regime

Multiple scattering

Random feature in the multiple scattering regime

Propagating matrix in random media

Deterministic coherence in the single scattering regime

Paraxial approximation, point-like reflectors.

Number of scatterers contained in the isochronous volume

Reflectivity of the $s^{th}$ scatterer

Position of the $i^{th}$ element

Position of the $s^{th}$ scatterer

$A_{s}$

$\exp\left(\frac{2jkR}{R}\sum_{s=1}^{N} A_{s}\right) \exp\left(jk \frac{|x_{i} - x_{s}|^{2}}{2R}\right) \exp\left(jk \frac{|x_{j} - X_{s}|^{2}}{2R}\right)$

### Propagation matrix in random media

Deterministic coherence in the single scattering regime

\[
\begin{align*}
    k_{ij}(T, f) & = \frac{\exp(2jkr)}{R} \sum_{s=1}^{N_s} A_s \exp \left( jk \frac{[x_i - X_s]^2}{2R} \right) \exp \left( jk \frac{[x_j - X_s]^2}{2R} \right) \\
    k_{ij}(T, f) & = \frac{\exp(2jkr)}{R} \exp \left( jk \frac{[x_i - x_j]^2}{4R} \right) \sum_{s=1}^{N_s} A_s \exp \left( jk \frac{[x_i + x_j - \sqrt{2}X_s]^2}{4R} \right)
\end{align*}
\]

Deterministic term (independent from the distribution of scatterers)

Random term

Along the antidiagonals of \( K \), the term \((x_i + x_j)\) is constant. Whatever the realization of disorder, there is a deterministic phase relation between coefficients \( k_{i-m,i+m} \) of \( K \) located on the same antidiagonal:

\[
\beta_m = \frac{k_{i-m,i+m}}{k_{ii}} = \exp \left( \frac{[mp]^2}{R} \right)
\]

\( p \) is the array pitch

Propagating matrix in random media

Statistical properties of $\mathbf{K}$ in the single scattering regime

Matrix $\mathbf{K}$ in the single scattering regime:

$$
\mathbf{K} = \begin{bmatrix}
  a_{i+1, j-1} \exp\left[j k \frac{(i-j)^2}{4R}\right]
\end{bmatrix}
$$

Coefficients $a_p$ are random complex variables

Random Hankel matrix

Matrix whose antidiagonals are constant

$$
\mathbf{K} = \begin{bmatrix}
  a_{i+1, j-1}
\end{bmatrix}
$$


Detection and Imaging of a target embedded in a diffusive medium
Detection and imaging in a random medium

Experimental configuration

TARGET: steel hollow cylinder φ=15 mm
L=20 mm

Array of transducers

Random scattering sample made of steel rods (l_e=7.7±0.3 mm)

The target echo has to come through six mean free paths of the diffusive medium. It undergoes a decreasing of a factor $e^{-6} \approx 1/400$.

Classical imaging techniques fail in detecting and imaging the target placed behind the diffusive slab due to multiple scattering.

Idea: Take advantage of the deterministic coherence of single scattered signals

Building of a smart radar/sonar which separates single scattered echoes from the multiple scattering background

Detection and imaging in a random medium

Extract the single scattered echoes with our smart radar

\[ T = 94.5 \mu s \text{ (expected time-of-flight for the target) } \quad f = 2.7 \text{ MHz} \]

Need to combine the smart radar with an imaging technique

**The D.O.R.T method**  

Detection and imaging in a random medium

The D.O.R.T method applied to the “single scattering” matrix $K^s$

Example for the time of flight expected for the target ($f=2.7$ MHz)

Measured matrix $K$

DORT method applied to $K$

"Single Scattering" matrix $K^s$

DORT method applied to $K^s$

Detection and imaging in a random medium

The D.O.R.T method applied to the “single scattering matrix” $K^F$

Our approach allows to:
- Strongly diminish the noisy effect of multiple scattering
- Smooth aberration effects and provide an image of the target of high quality.

Multiple scattering generates a *speckle* image without any direct relation with the reflectivity of the medium

A perfect image of the target is obtained as if the highly scattering slab had been removed

Multiple scattering in weakly scattering media
Multiple scattering in weakly scattering media

Ultrasound imaging

Human soft tissues are known for being weakly scattering, $I_e > cT$

Multiple scattering is usually neglected for imaging purposes

Idea: extract the multiple scattering contribution hidden by a predominant single scattering contribution

→ Test of the Born approximation

→ Characterization of weakly scattering media (measurement of transport parameters)

Multiple scattering in weakly scattering media

The proof by the coherent backscattering peak

Spatial intensity profile as a function of the distance between the source and receiver

Multiple scattering of ultrasound is far from being negligible in human soft tissues around 4.3 MHz.

Experimental test for the Born approximation

Multiple scattering in weakly scattering media

Characterization of weakly scattering media

\[ \frac{1}{\ell_{\text{ext}}} = \frac{1}{\ell_e} + \frac{1}{\ell_a} \]

- \( \ell_e \): scattering mean free path (scattering losses)
- \( \ell_a \): absorption mean free path (absorption losses)

Time evolution of single and multiple scattering intensities

The separation of single and multiple scattering allows to distinguish between scattering and absorption losses by measuring \( \ell_e \) and \( \ell_a \) independently (here \( \ell_a = 50 \text{ mm} \) and \( \ell_e = 2000 \text{ mm} \)).

Imaging of diffusive media
Imaging of diffusive media

Local measurements of $D$: Use of collimated beams

Random media can be inhomogeneous in disorder (e.g. trabecular bone)

- Local measurements of $D$ are needed (near-field)
- Use of collimated beams

1/ Acquisition of the inter-element matrix $H$

- N-element array of transducers
  
- $h_{ij}(t) =$ impulse response between $i$ and $j$

2/ Gaussian beamforming applied to matrix $H$

- Creation of a virtual array of sources/receivers located in the near-field of the bone

Imaging of diffusive media

Measurement of the diffusion constant $D$ in the near-field

Random scattering sample of steel rods ($l_e \sim 10$ mm)

Spatial intensity profile at a given time $T$

![Graph showing spatial intensity profile with labels $w_0$, $\propto \sqrt{Dt}$, $X_E - X_R [\text{mm}]$, and $\text{Normalized intensity}$.

- Coherent backscattering peak
- Incoherent background: Diffusive halo

The information about the diffusion constant is contained in the diffusive halo.

The separation of coherent and incoherent intensities is needed to obtain a measurement of the diffusion constant $D$.

Imaging of diffusive media

Building of a fictive antireciprocal medium

Measured matrix $H \Rightarrow$ Fictive matrix $H^A$:

- $i < j \Rightarrow h_{ij}^A = h_{ij}$
- $i = j \Rightarrow h_{ii}^A = 0$
- $i > j \Rightarrow h_{ij}^A = -h_{ij}$

$H^A$, Antisymmetric matrix $h_{ij}^A = -h_{ji}^A$

Reciprocal paths are exactly out of phase (artificial antireciprocity)

Destructive interference between reciprocal paths

Measured matrix $H$  

\[ I = I_{inc} + I_{coh} \]

Virtual matrix $H^A$  

\[ I^A = I_{inc} - I_{coh} \]

Antireciprocal medium:

An anticone is obtained instead of the classical coherent backscattering cone!

Imaging of diffusive media

Building of a fictive antireciprocal medium

Once the incoherent intensity is isolated, local measurements of the diffusion constant is possible. Multiple scattered waves can be used to image random media.

Application to the imaging of a slice of trabecular bone

Imaging of diffusive media

Experimental results

frequency range $[2.5 ; 3.5]$ MHz

$\frac{\rho}{\rho_{\text{max}}}$

The highest values of $D$ correspond to the areas where the bone is less dense, hence the weaker scattering.

Image obtained with local measurements of $D$ shows more contrast than the image provided by density measurements.

Conclusion & perspectives
Conclusion

Propagation operator in random media

- Statistical behaviour of singular values (RMT)
- Deterministic coherence of the single scattering contribution

Separation single scattering / multiple scattering

- Detection and imaging embedded or hidden behind a highly scattering slab
- Study of multiple scattering in weakly scattering media: Application to human soft tissues
- Imaging with local measurements of the diffusion constant: Application to human trabecular bone imaging

Perspectives

This work can be extended to all fields of wave physics for which multi-element array technology is available and allows time-resolved measurements of the amplitude and the phase of the wave-field

- Application to real random media: detection of defects in coarse grain austenitic steels (nuclear industry), landmine detection
- Imaging of human soft tissues with multiple scattering?
Thank you for your attention!