

A New $SU(2)$ Anomaly

Edward Witten

PCTS, October 3, 2018

A familiar anomaly says that in four spacetime dimensions, an $SU(2)$ gauge theory with a single multiplet of fermions that transform under $SU(2)$ in the spin $1/2$ representation is inconsistent (EW, 1982). In today's lecture - based on a new paper of the same name as the lecture with J. Wang and X.-G. Wen - I will describe a similar but more subtle anomaly for the case of a theory with a single multiplet of fermions in the spin $3/2$ representation of $SU(2)$. (For brevity I will call these fermions of isospin $3/2$.)

Before going on I want to mention one old and one new reference:

S. Coleman (1976, unpublished) was the first to point out that there was something strange about an $SU(2)$ gauge theory in four dimensions with a single multiplet of fermions of isospin $1/2$.

C. Cordova and T. Dumitrescu (arXiv:1806.09592) explored by a different method a problem that turns out to be related.

I will begin by reviewing the familiar anomaly for the case of isospin $1/2$. Let us make sure we agree on what the model is: in four dimensions, a left chirality spinor of isospin $1/2$ is a four-component object $\psi_{\alpha i}$, $\alpha = 1, 2$, $i = 1, 2$, where α is a Lorentz spinor index and i is an $SU(2)$ index. The hermitian adjoint is a right-handed spinor field $\tilde{\psi}_{\dot{\alpha} i}$ also of isospin $1/2$ ($\dot{\alpha}$ is a Lorentz spinor index of opposite chirality). Together they have eight hermitian components (eight real components at the classical level), which one can think of as the hermitian parts and i times the antihermitian parts of $\psi_{\alpha i}$. It is important to know that in Euclidean signature these fields are naturally real (not true for fermions in general). For example, $\psi_{\alpha i}$ is real in Euclidean signature because the $(1/2, 0)$ representation of $Spin(4)$ is pseudoreal, as is the isospin $1/2$ representation of $SU(2)$. (In general the tensor product of two pseudoreal representations is real.) Similarly $\tilde{\psi}_{\dot{\alpha} i}$ is real in Euclidean signature.

Because the Euclidean fermions are real, the Dirac operator

$$\not{D}_4 \Psi = \sum_{\mu=1}^4 \gamma^\mu (\partial_\mu + \sum_{a=1}^3 A_\mu^a t_a) \Psi, \quad \Psi = \begin{pmatrix} \psi \\ \tilde{\psi} \end{pmatrix}$$

is also real. Here γ_μ are gamma matrices obeying $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ and t_a are antihermitian $SU(2)$ generators obeying $[t_a, t_b] = \varepsilon_{abc} t_c$. The statement that \not{D}_4 is real is a little subtle. It would not be possible to pick 4×4 real gamma matrices, essentially because the spinor representations of $Spin(4)$ are pseudoreal rather than real, and similarly it is not possible to pick 2×2 real t_a 's, because the isospin $1/2$ representation of $SU(2)$ is likewise pseudoreal, not real. But there is no problem to pick 8×8 real gamma matrices, and a set of 8×8 real gamma matrices commutes with a set of three real t_a 's, letting us write a real 8×8 Dirac equation. Basically this reflects the fact that the tensor product of two pseudoreal representations is real.

Since the spinor representation of $\text{Spin}(1, 4)$ is pseudoreal and four-dimensional, its tensor product with the isospin $1/2$ representation of $\text{SU}(2)$ is an eight-dimensional real representation of $\text{Spin}(1, 4) \times \text{SU}(2)$, so classically it is possible to have a five-dimensional theory also with a single multiplet of fermions in the isospin $1/2$ representation of $\text{SU}(2)$. These fields are again real in Euclidean signature, for the same reason as before. So there is again a real Euclidean Dirac operator in five dimensions. Once we have four real 8×8 gamma matrices γ_μ that commute with the t_a , we can define a fifth one $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ that also commutes with the t_a . So now we can write a real five-dimensional Euclidean Dirac operator

$$\not{D}_5 = \sum_{\mu=1}^5 \gamma^\mu (\partial_\mu + \sum_{a=1}^3 A_\mu^a t_a)$$

that clearly is also real. (Note that my Dirac operators are real and skew-symmetric, that is, antihermitian. One can multiply by i to make \not{D}_4 and \not{D}_5 imaginary and hermitian.)

An important detail is that a single multiplet of fermions of isospin $1/2$ cannot have a bare mass: a Lorentz-invariant and gauge-invariant bilinear would be

$$\varepsilon^{\alpha\beta} \varepsilon^{ij} \psi_{\alpha i} \psi_{\beta j} + \text{h.c.},$$

and vanishes by fermi statistics. So an anomaly is conceivable. On the other hand, a *pair* of such multiplets, say ψ and χ , could have a bare mass

$$\varepsilon^{\alpha\beta} \varepsilon^{ij} \psi_{\alpha i} \chi_{\beta j} + \text{h.c.},$$

and so cannot contribute to any anomaly. So a possible anomaly will be only a mod 2 effect, that is it will depend on the number of isospin $1/2$ multiplets mod 2. Similarly, an odd number of multiplets of any half-integer isospin j might have an anomaly. For integer j , a single multiplet can have a bare mass and an anomaly is not possible.

A systematic explanation of the anomaly involves the fact that $\pi_4(\text{SU}(2)) = \mathbb{Z}_2$, and the relation of this to the mod 2 index in five dimensions. However, there is an easier way to see that there is an anomaly. On a four-sphere S^4 , let A be an $\text{SU}(2)$ gauge field of instanton number 1. According to the Atiyah-Singer index theorem, a multiplet of Weyl fermions of isospin $1/2$ has a single zero-mode in an instanton field, call it $\beta_{\alpha i}$. Having a single zero-mode means that in an instanton field, the elementary fermion field $\psi_{\alpha i}(x)$ has an expectation value,

$$\langle \psi_{\alpha i}(x) \rangle \sim \beta_{\alpha i}.$$

This expectation value obviously violates the symmetry $(-1)^F$ that acts as -1 on fermions and as $+1$ on bosons, so that is an anomaly.

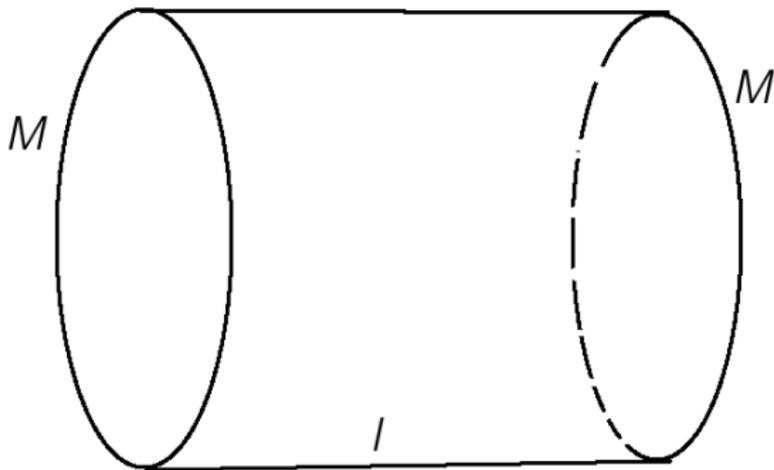
In this theory, $(-1)^F$ can be viewed as a gauge transformation: the gauge transformation by the central element $-1 \in SU(2)$. So the anomaly can be viewed as a breakdown of $SU(2)$ gauge invariance.

To determine if a fermion multiplet of isospin j contributes to this anomaly, we just count zero modes in an instanton field. From the index theorem, the number of such zero-modes is $(2/3)j(j+1)(2j+1)$. This is odd precisely if j is of the form $2n+1/2$, so fermion fields in those representations of $SU(2)$ contribute to the anomaly. A theory free of this anomaly has an overall even number of fermions multiplets of isospin of the form $2n+1/2$.

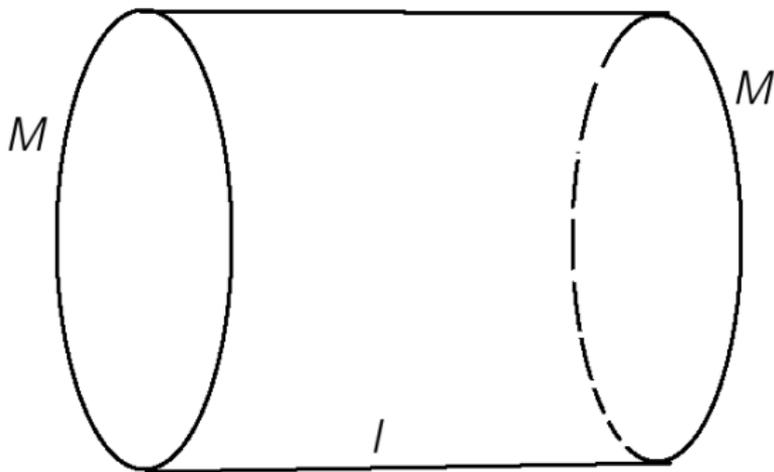
In general, fermion anomalies in four dimensions are related to a topological invariant in five dimensions. In the present case, the five-dimensional topological invariant that is relevant is the *mod 2 index*. Briefly, such an invariant exists for every theory of fermions because the fermion action $\int d^D x \sqrt{g} \bar{\Psi} (\not{D} + \dots) \Psi$ is antisymmetric, by fermi statistics.

What I explained in my original paper on this subject is that the anomaly in this four-dimensional theory is given, in general, by the mod 2 index in five dimensions. Concretely, let M be a Riemannian four-manifold, with metric g , and with some background gauge field A . We consider the fermion path integral in the presence of background fields g, A . Let φ be a gauge transformation and/or diffeomorphism of the background fields. To decide if the fermion path integral is φ -invariant, we construct a five-manifold known as the mapping torus.

If φ is a *symmetry* of the bosonic background – as in the example with $\varphi = (-1)^F$ – then the construction is particularly simple. One just takes the five-manifold $M \times I$ where I is the unit interval $0 \leq t \leq 1$, and glues together the two ends using the symmetry φ :



If φ is not a symmetry of g, A , then one lets g, A be t -dependent so as to interpolate from the original g, A at $t = 0$ to g^φ, A^φ at $t = 1$ and then glues as before:



Anyway we use φ to construct a five-manifold that we might call $M \times S^1$ along with metric and gauge field g, A . Then the general claim is that the anomaly under φ in the original fermion path integral on M , for a fermion field of isospin j , is the mod 2 index of the isospin j Dirac operator on $M \times S^1$.

Let us see what this claim means for our example with $\varphi = (-1)^F$. Since $(-1)^F$ acts trivially on g and A , the five-manifold is a simple product $M \times S^1$, and the gauge field on $M \times S^1$ is a “pullback” from M . The $(-1)^F$ means that fermions are periodic in going around the S^1 . In this example, the mod 2 index is easily computed. A fermion zero-mode on $M \times S^1$ must have zero momentum along the S^1 , so it comes from a fermion zero-mode on M . So the number of fermion zero modes on $M \times S^1$ is the number of zero-modes on M . In the example we started with, there was a single zero-mode on M , and therefore the mod 2 index on $M \times S^1$ is also 1. That is how one sees, in this particular example, that the anomaly can be described by the mod 2 index in five dimensions. The general argument uses the relation between the mod 2 index in five dimensions and spectral flow of a family of Dirac operators in four dimensions.

In general, the mod 2 index in five dimensions is hard to calculate. There is a mod 2 version of the Atiyah-Singer index theorem, but the information that it gives is somewhat abstract. However, if we have a gauge transformation plus diffeomorphism φ that is a symmetry of the bosonic background g, A , the anomaly is easy to calculate. We will first view the matter in four-dimensional terms and then reinterpret it in terms of the mod 2 anomaly in five dimensions.

Let us think in general about how a symmetry φ of the bosonic background might act on fermion zero-modes on M . The fermion field $\Psi_{\alpha i}$ on M is naturally real, as we discussed at the beginning, and this real structure is invariant under diffeomorphisms and gauge transformations. Diffeomorphisms and gauge transformations also preserve the L^2 norm. So φ acts as a real orthogonal transformation on a finite-dimensional space of zero-modes. The anomaly is just the transformation of the zero-mode measure under φ ; in other words, it is $\det \varphi$ where φ is viewed as a linear transformation of the space of zero-modes. In general a real orthogonal transformation can have complex eigenvalues that come in pairs $\exp(\pm i\theta)$, and it can have real eigenvalues 1 or -1 that are not necessarily paired. The paired eigenvalues $\exp(\pm i\theta)$ do not contribute to $\det \varphi$. If m_1 and m_{-1} are the number of zero-modes on which φ acts as 1 or -1 , then $\det \varphi = (-1)^{m_{-1}}$, and from a four-dimensional point of view, the anomaly under any diffeomorphism and gauge transformation that is a symmetry of the bosonic background is

$$(-1)^{m_{-1}}.$$

Note that the total number of zero-modes, mod 2, is $m_1 + m_{-1}$, since a pair of complex conjugate eigenvalues does not contribute. So the condition for no anomaly under $(-1)^F$ is that $m_1 + m_{-1}$ is even, or in other words $m_1 = m_{-1} \pmod{2}$.

Now let us explain the five-dimensional interpretation of the anomaly. In doing this, for simplicity, I will assume that there is no anomaly under $(-1)^F$ so that $m_1 = m_{-1}$. (If there is an anomaly in $(-1)^F$, then one of φ and $\varphi(-1)^F$ has an anomaly and one does not. It is subtle to explain which is which.) From a five-dimensional point of view, the anomaly is the mod 2 index of the Dirac operator on $M \times S^1$. A zero-mode on $M \times S^1$ is just a φ -invariant mode on M . So the mod 2 index is m_1 (or m_{-1}), and again this illustrates that the anomaly in four dimensions is the mod 2 index in five dimensions. As I explained before, this is true in general though one needs a more subtle argument if φ is not a symmetry of the classical background.

This completes our review of the old $SU(2)$ anomaly. Now I want to navigate towards the new one. Consider an $SU(2)$ gauge theory in which fermions are in representations of half-integer isospin and bosons in representations of integer isospin. This means that the fields of the theory are not in arbitrary representations of $\text{Spin}(4) \times SU(2)$ but in representations of the quotient $(\text{Spin}(4) \times SU(2))/\mathbb{Z}_2$, where \mathbb{Z}_2 is the product of a 2π rotation in $\text{Spin}(4)$ and the element $-1 \in SU(2)$. Such a theory can be formulated on a four-manifold M without a spin structure, with only a weaker structure that we will call a spin- $SU(2)$ structure. A spin- $SU(2)$ structure is a connection with structure group $(\text{Spin}(4) \times SU(2))/\mathbb{Z}_2$ with the property that if you forget $SU(2)$ and project this group to $\text{Spin}(4)/\mathbb{Z}_2 = SO(4)$, you get the Riemannian connection (for some Riemannian metric on M). A spin- $SU(2)$ structure gives the information one needs to define parallel transport of fermions of half-integer isospin or bosons of integer isospin.

A spin-SU(2) structure has an analog for U(1) that is perhaps more familiar. Consider a U(1) gauge theory in which fermions have odd electric charge and bosons have even electric charge. (For example, if there were no neutrons, and nuclei were bound states of protons only, then ordinary matter could be described by such a theory.) This condition means that all fields are in representations of

$$\text{Spin}_c(4) = (\text{Spin}(4) \times \text{U}(1))/\mathbb{Z}_2$$

rather than more general representations of $\text{Spin}(4) \times \text{U}(1)$. A theory with this property can be formulated on a manifold with a spin_c structure, which is a $\text{Spin}_c(4)$ connection that reduces to the Riemannian connection if one forgets U(1) and projects $\text{Spin}_c(4)$ to $\text{Spin}(4)/\mathbb{Z}_2 = \text{SO}(4)$.

The difference between spin and spin_c structures is most striking on a four-manifold that does not admit a spin structure. This means that parallel transport of neutral fermions cannot be defined, because the Riemannian connection, if taken in the spinor representation of $\text{Spin}(4)$, does not satisfy Dirac quantization. We have to compensate for this by coupling the fermions to a $U(1)$ gauge field that also violates Dirac quantization, by a compensating amount. This makes a spin_c structure.

Concretely, the “U(1) gauge field” in a Spin_c theory does not obey conventional Dirac flux quantization, which would say that the flux through any two-manifold C is an integer multiple of 2π . Rather

$$\int_C \frac{F}{2\pi} = \frac{1}{2} \int_C w_2(M) \pmod{\mathbb{Z}}.$$

Here $w_2(M)$ is a mod 2 cohomology class known as the second Stiefel-Whitney class of M .

For an example of a four-manifold with no spin structure, we can take $\mathbb{C}\mathbb{P}^2$, which can be described by three complex coordinates z_1, z_2, z_3 , with $\sum_{i=1}^3 |z_i|^2 = 1$, and an equivalence relation $z_i \cong \exp(i\alpha)z_i$ for real α . $\mathbb{C}\mathbb{P}^1$ can be embedded in $\mathbb{C}\mathbb{P}^2$ by setting, say, $z_3 = 0$. We have

$$\int_{\mathbb{C}\mathbb{P}^1} w_2 = 1$$

and therefore a spin_c connection A on $\mathbb{C}\mathbb{P}^2$ will have half-integral flux $f = n + 1/2$ for some integer n :

$$\int_{\mathbb{C}\mathbb{P}^1} \frac{F}{2\pi} = f = \frac{1}{2} + n.$$

A spin_c connection on A that is also invariant under the $SU(3)$ symmetries of $\mathbb{C}\mathbb{P}^2$ is uniquely determined by the half-integer f . An index theorem shows that the number of zero modes of a charge 1 fermion coupled to this spin_c connection is

$$J_f = \frac{4f^2 - 1}{8} = \frac{n(n+1)}{2}.$$

Thus $J_{1/2} = 0$, $J_{3/2} = 1$.

Now let us construct a $\text{spin-SU}(2)$ structure on $\mathbb{C}\mathbb{P}^2$. We can do this by just embedding $U(1)$ in $SU(2)$:

$$A \rightarrow \hat{A} = \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}.$$

If A defines a spin_c structure then \hat{A} defines a $\text{spin-SU}(2)$ structure.

The number of zero modes of a fermion of isospin j interacting with the spin-SU(2) structure \hat{A} is easily computed. For simplicity let us consider the basic case that A has flux $1/2$. An isospin $j = k + 1/2$ fermion has components of electric charge $q = 2j, 2j - 2, \dots, -2j$ (all of which are odd integers) and a fermion of charge q "sees" flux $\hat{f} = fq = q/2$. So we just sum $J_{\hat{f}}$ over $\hat{f} = j, j - 1, \dots, -j$ and learn that the total number of zero-modes (if $j = k + 1/2$) is

$$\mathfrak{I}_j = \frac{1}{3}k(k+1)(k+2).$$

This number is always even, so there is no anomaly in $(-1)^F$. For example

$$\mathfrak{I}_{1/2} = 0, \quad \mathfrak{I}_{3/2} = 2.$$

Though there is no anomaly in $(-1)^F$, it is not hard to find a combined diffeomorphism plus gauge transformation that does have an anomaly. Let φ be the symmetry of \mathbb{CP}^2 that acts by $z_i \rightarrow \bar{z}_i$. It is not a symmetry of any spin_c connection on \mathbb{CP}^2 because (since it reverses the orientation of \mathbb{CP}^1) it changes the sign of the flux. But we can combine the diffeomorphism φ with an $SU(2)$ gauge transformation to make a symmetry of the $\text{spin-SU}(2)$ connection \hat{A} . We just introduce the gauge transformation

$$W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

which (with the way we embedded $U(1)$ in $SU(2)$) also changes the sign of the flux. Thus $\hat{\varphi} = \varphi \cdot W$ is a combined diffeomorphism and $SU(2)$ gauge transformation that does leave fixed the $\text{spin-SU}(2)$ connection \hat{A} . This means that it will be easy to determine whether $\hat{\varphi}$ has an anomaly. We just have to look at how it acts on fermion zero modes.

First let us consider a fermion field of isospin $1/2$. Since we had $\mathfrak{J}_{1/2} = 0$, in this case there are no zero-modes, and therefore there is no anomaly under $\hat{\varphi}$. Now take a fermion field ψ of isospin $3/2$. Now there are two zero-modes. One is a zero mode of the component of ψ of charge $3/2$, and one is a mode of the component of charge $-3/2$. (This follows from the formula for J_f .) These two modes are exchanged by W and therefore by $\hat{\varphi}$. One linear combination of the two modes is even under $\hat{\varphi}$ and one is odd. Therefore in the two-dimensional space of zero-modes, $\hat{\varphi}$ has one eigenvalue 1 and one eigenvalue -1 , and its determinant is -1 . So the path integral measure is odd under $\hat{\varphi}$ and this diffeomorphism plus gauge transformation has an anomaly.

We can give the anomaly a five-dimensional interpretation in a way that I described earlier. We glue together the two ends of $\mathbb{C}\mathbb{P}^2 \times I$



to make the mapping torus $\mathbb{C}\mathbb{P}^2 \rtimes S^1$, which now comes with a spin-SU(2) structure. A fermion zero-mode on $\mathbb{C}\mathbb{P}^2 \rtimes S^1$ is a $\hat{\varphi}$ -invariant zero-mode on $\mathbb{C}\mathbb{P}^2$. Since there is precisely one such mode, the mod 2 index on $\mathbb{C}\mathbb{P}^2 \rtimes S^1$ is 1, and thus the anomaly in the fermion path integral on $\mathbb{C}\mathbb{P}^2$ is measured by a mod 2 index on $\mathbb{C}\mathbb{P}^2 \rtimes S^1$. (Of course, earlier, we gave a general explanation of why this is always so.) Using formulas that I gave earlier, one can generalize this to any half-integral isospin j , and one finds that the anomaly receives a contribution from fermion multiplets of isospin $4r + 3/2$ ($r \in \mathbb{Z}$) and not other values.

We considered two examples and found two anomalies. Are there any more? The answer is “no,” in the following sense. First of all, the mod 2 index \mathcal{I}_j in five dimensions for a fermion of any isospin j is a “cobordism invariant,” meaning that it vanishes on any five-manifold Y with spin or spin-SU(2) structure if Y is the boundary of a six-manifold Z over which the spin or spin-SU(2) structure of Y extends. (This cobordism invariance is a special case of a more general story. Fermion anomalies in D dimensions are always governed by an η -invariant in $D + 1$ dimensions. When perturbative anomalies cancel, the η -invariant is a cobordism invariant. When fermions are real in Euclidean signature - as in the cases we are considering today - η reduces to a mod 2 index.)

For five-manifolds with spin structure and an $SU(2)$ bundle, there is only a single \mathbb{Z}_2 -valued cobordism invariant, which one can take to be $\mathcal{I}_{1/2}$, the mod 2 index for the $j = 1/2$ representation. So the anomaly of any $SU(2)$ theory in $D = 4$ (defined on a spin manifold) is either 0 or $\mathcal{I}_{1/2}$. In particular, the mod 2 index \mathcal{I}_j for any j is either 0 or equal to $\mathcal{I}_{1/2}$. So by computing for the example of $S^4 \times S^1$ (with instanton number 1 on S^4) that we started with, we learn that $\mathcal{I}_j = \mathcal{I}_{1/2}$ if j is of the form $2n + 1/2$, and otherwise $\mathcal{I}_j = 0$. So this is the complete story for anomalies in $SU(2)$ gauge theory on a spin four-manifold. In particular, a $j = 3/2$ fermion on a spin manifold has no anomaly, though it has an anomaly on a spin- $SU(2)$ manifold.

On a five-manifold with $\text{spin-SU}(2)$ structure, there are two \mathbb{Z}_2 -valued invariants, which we can take to be $\mathcal{I}_{1/2}$ and $\mathcal{I}_{3/2}$. Hence \mathcal{I}_j for any j is a linear combination of these. The coefficients can be computed by checking our two examples $S^4 \times S^1$ and $\mathbb{C}\mathbb{P}^2 \times S^1$. One finds $\mathcal{I}_{2n+1/2} = \mathcal{I}_{1/2}$, $\mathcal{I}_{4r+3/2} = \mathcal{I}_{3/2}$, $\mathcal{I}_{4r+7/2} = 0$. So our two examples detect all anomalies on $\text{spin-SU}(2)$ manifolds.

There is another way to look at the spin-SU(2) anomaly which shows that it is never an anomaly in gauge-invariance only; it is always an anomaly in a combined gauge transformation plus diffeomorphism. Another cobordism invariant in five dimensions, which can be defined without a spin-SU(2) structure and still exists if there is one, is

$$J = \int_Y w_2 w_3.$$

So it must be that J is a linear combination of $\mathcal{I}_{1/2}$ and $\mathcal{I}_{3/2}$. We can determine the coefficients by checking the examples $S^4 \times S^1$ and $\mathbb{C}P^2 \times S^1$ and we learn

$$\mathcal{I}_{3/2} = J.$$

Now to test invariance of the four-dimensional fermion path integral under a gauge transformation (not accompanied by any diffeomorphism) we would have to compute a mod 2 index on a five-manifold $Y = M \times S^1$ (with some $SU(2)$ bundle on Y that depends on the gauge transformation we are trying to test). But J vanishes for any $M \times S^1$, and therefore the relation $\mathcal{I}_{3/2} = J$ implies that $\mathcal{I}_{3/2} = 0$ in any such example. So an isospin $3/2$ (or $4r + 3/2$) fermion field has no anomaly just under gauge-invariance.

A theory that has the old $SU(2)$ anomaly is simply inconsistent; it does not make sense to ask about its dynamics. That is not so at all for a theory with the new $SU(2)$ anomaly, because the anomaly is never an anomaly just in gauge-invariance. In Minkowski spacetime, a theory with the new anomaly is perfectly consistent and it makes sense to ask about its dynamics. (This is not necessarily an easy question to answer. The minimal example with a single multiplet of isospin $3/2$ is asymptotically free and flows potentially to strong coupling in the IR, so it is difficult to know how it behaves.)

One thing we can do with $SU(2)$ gauge theory is to add a Higgs field and spontaneously break some of the gauge symmetry. In a spin- $SU(2)$ theory, the Higgs field has to have integer isospin. For example, we can add a bosonic field $\vec{\phi}$ of isospin 1. Its expectation value can reduce $SU(2)$ to $U(1)$, or more exactly it will reduce spin- $SU(2)$ to spin_c . However, Higgsing of a spin- $SU(2)$ theory that carries the anomaly produces a $U(1)$ (or more precisely spin_c) theory with unusual properties.

Let us first consider a spin-SU(2) theory that *doesn't* have any anomaly. For example, to avoid trouble, we can include *two* fermion multiplets of isospin 1/2. After symmetry breaking to U(1), this theory has fermions of electric charge 1 and magnetic charge 0; we will say the charges are (1,0). This theory also has 't Hooft-Polyakov monopoles, which classically have spin 0. As first shown by Jackiw and Rebbi (1976), the monopoles can have unusual quantum numbers under global symmetries, because of fermion zero-modes. But the fermion zero-modes have angular momentum 0 and do not affect the spin of the monopoles. The monopoles are thus spinless bosons, of charges (0,1). We can also make a dyon of charges (1,1) by bringing together a charged fermion and a monopole. Taking account of the angular momentum in the electromagnetic field, the charge (1,1) dyon is again a boson, like the monopole.

However, suppose we can find a U(1) theory in which the (0,1) monopole is a fermion, just like the (1,0) electric charge. Taking account again of the angular momentum in the field, a (1,1) composite is now a fermion. This would be “all-fermion electrodynamics,” which is known to have a subtle anomaly (Wang, Potter, and Senthil, 2014). It is known that the anomaly is related to the five-dimensional cobordism invariant $J = \int w_2 w_3$ (Kravec, McGreevy, and Swingle, 2014).

Since our $SU(2)$ theory with the isospin $3/2$ fermion has the same anomaly as all-fermion electrodynamics, it is natural to guess that if we Higgs this theory down to $U(1)$, what we will get is indeed all-fermion electrodynamics. All we really have to do to prove this is to show that the 't Hooft-Polyakov monopole in this theory is a fermion. For this, we just have to look at the zero-modes of an isospin $3/2$ fermion in the field of the 't Hooft-Polyakov monopole. These zero-modes are governed by the Callias index theorem (1978).

There are four fermion zero-modes, three that transform with angular momentum 1 and a fourth with angular momentum 0. We follow the logic of Jackiw and Rebbi. Upon quantization, the fermion zero-modes turn into operators that obey the anticommutation relations of gamma matrices. Three of them are operators $\vec{\gamma}$ with angular momentum 1; they behave as Pauli sigma matrices, acting on 2 states of angular momentum 1/2. The fourth gamma matrix γ_0 forces a doubling of the spectrum, but because it has no angular momentum, it does not affect the fact that the states have spin 1/2. Therefore, the monopole is a spin 1/2 fermion and the Higgsing of an $SU(2)$ theory with the new anomaly has given us an ultraviolet completion of all-fermion electrodynamics.

Although a theory with a single fermion multiplet of $j = 3/2$ on a spin-SU(2) manifold is inconsistent, it does provide a “boundary state” for a five-dimensional gapped (or topological) theory in which the partition function on a five-manifold without boundary is $(-1)^{\int_Y w_2 w_3}$. This statement is a discrete version of “anomaly inflow” (Callan and Harvey, 1985). A careful argument is a little too long to explain now, so I refer to the paper on which today’s talk is based. The intuitive idea is just that since the spin-SU(2) theory with the $j = 3/2$ multiplet has the $w_2 w_3$ anomaly, it provides a suitable boundary state for a theory whose bulk partition function is $(-1)^{\int_Y w_2 w_3}$. Under Higgsing to spin_c , this boundary state reduces to one that is already known (Kravec, McGreevy, and Swingle, 2014).