Independent subsets of powers of paths, and Fibonacci cubes

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Abstract
We provide a formula for the number of edges of the Hasse diagram of the independent subsets of the $h$-th power of a path ordered by inclusion. For $h = 1$ such a value is the number of edges of a Fibonacci cube. We show that, in general, the number of edges of the diagram is obtained by convolution of a Fibonacci-like sequence with itself.

Keywords: Independent subset, path, power of graph, Fibonacci cube.

1 Introduction

For a graph $G$ we denote by $V(G)$ the set of its vertices, and by $E(G)$ the set of its edges.

\textbf{Definition 1.1} For $n, h \geq 0$, the $h$-power of a path, denoted by $P_n^{(h)}$, is a graph with $n$ vertices $v_1, v_2, \ldots, v_n$ such that, for $1 \leq i, j \leq n$, $i \neq j$, $(v_i, v_j) \in E(P_n^{(h)})$ if and only if $|j - i| \leq h$.

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Thus, for instance, \( P_n^{(0)} \) is the graph made of \( n \) isolated nodes, and \( P_n^{(1)} \) is the path with \( n \) vertices.

**Definition 1.2** An independent subset of a graph \( G \) is a subset of \( V(G) \) not containing adjacent vertices.

**Notation.** (i) We denote by \( p_n^{(h)} \) the number of independent subsets of \( P_n^{(h)} \).
(ii) We denote by \( H_n^{(h)} \) the Hasse diagram of the poset of independent subsets of \( P_n^{(h)} \) ordered by inclusion, and by \( H_n^{(h)} \) the number of edges of \( H_n^{(h)} \).

In this work we evaluate \( p_n^{(h)} \) and \( H_n^{(h)} \). Our main result (Theorem 3.4) is that, for \( n, h \geq 0 \), the sequence \( H_n^{(h)} \) is obtained by convolving the sequence \( 1, \ldots, 1, p_0^{(h)}, p_1^{(h)}, p_2^{(h)}, \ldots \) with itself.

Clearly, \( H_n^{(0)} \) is the \( n \)-dimensional cube. Thus, on one hand, our work generalizes the known formula \( n2^n - 1 \) for the number of edges of the Boolean lattice with \( n \) atoms, obtained by the convolution of the sequence \( \{2^n\} \) with itself. From a different perspective, this work could be seen as yet another generalization of the notion of Fibonacci cube. Indeed, observe that every independent subset \( S \) of \( P_n^{(h)} \) can be represented by a binary string \( b_1b_2\cdots b_n \), where, for \( i = 1, \ldots, n \), \( b_i = 1 \) if and only if \( v_i \in S \). More specifically, each independent subset of \( P_n^{(h)} \) is associated with a binary string of length \( n \) such that the distance between any two 1’s of the string is greater than \( h \). For \( h = 1 \) the binary strings associated with independent subsets of \( P_n^{(h)} \) are Fibonacci strings of order \( n \), and the Hasse diagram of the set of all such strings ordered bitwise is a Fibonacci cube of order \( n \) (see [5,7]). Fibonacci cubes were introduced as an interconnection scheme for multicomputers in [3], and their combinatorial structure has been further investigated, e.g. in [6,7]. Several generalizations of the notion of Fibonacci cubes has been proposed (see, e.g., [4,5]). As far as we now, our generalization, described in terms of independent subsets of powers of paths ordered by inclusion, is a new one.

## 2 The independent subsets of powers of paths

We denote by \( p_{n,k}^{(h)} \) the number of independent \( k \)-subsets of \( P_n^{(h)} \).

**Lemma 2.1** For \( n, h, k \geq 0 \), \( p_{n,k}^{(h)} = \binom{n-hk+h}{k} \).

**Proof.** See [2, Theorem 1], and [1], where we establish a bijection between independent \( k \)-subset of \( P_n^{(h)} \) and \( k \)-subsets of a set with \( (n-hk+h) \) elements. \( \square \)
For $n, h \geq 0$, the number of all independent subsets of $P_n^{(h)}$ is

$$p_n^{(h)} = \sum_{k=0}^{\lceil n/(h+1) \rceil} p_{n,k}^{(h)} = \sum_{k=0}^{\lceil n/(h+1) \rceil} \binom{n-hk+h}{k}.$$ 

**Remark 2.2** Denote by $F_n$ the $n^{th}$ element of the Fibonacci sequence $F_1 = 1$, $F_2 = 1$, and $F_i = F_{i-1} + F_{i-2}$, for $i > 2$. Then, $p_n^{(1)} = F_{n+2}$.

**Lemma 2.3** For $n, h \geq 0$, $p_n^{(h)} = \begin{cases} n+1 & \text{if } n \leq h+1, \\ p_{n-1}^{(h)} + p_{n-h-1}^{(h)} & \text{if } n > h+1. \end{cases}$

**Proof.** See the first part of [2, Proof of Theorem 1], or [1].

### 3 The poset of independent subsets of powers of paths

Figure 1 shows a few Hasse diagrams $H_n^{(h)}$. Notice that, as mentioned in the introduction, for each $n$, $H_n^{(1)}$ is a Fibonacci cube.

![Fig. 1. Some $H_n^{(h)}$.](image)

Since in $H_n^{(h)}$ each non-empty independent $k$-subset covers exactly $k$ independent $(k-1)$-subsets, we can write

$$H_n^{(h)} = \sum_{k=1}^{\lceil n/(h+1) \rceil} k p_{n,k}^{(h)} = \sum_{k=1}^{\lceil n/(h+1) \rceil} k \binom{n-hk+h}{k}.$$ \hspace{1cm} (1)

Let now $T_{n,h}^{(i)}$ be the number of independent $k$-subsets of $P_n^{(h)}$ containing the vertex $v_i$, and let, for $h, k \geq 0$, $n \in \mathbb{Z}$, $\overline{p}_{n,k}^{(h)} = \begin{cases} p_{0,k}^{(h)} & \text{if } n < 0, \\ p_{n,k}^{(h)} & \text{if } n \geq 0. \end{cases}$

**Lemma 3.1** For $n, h, k \geq 0$, and $1 \leq i \leq n$,

$$T_{k,i}^{(n,h)} = \sum_{r=0}^{k-1} \overline{p}_{i-h-1,r}^{(h)} \overline{p}_{n-i-h,k-1-r}^{(h)}.$$ 

**Proof.** No independent subset of $P_n^{(h)}$ containing $v_i$ contains any of the elements $v_{i-h}, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{i+h}$. Let $r$ and $s$ be non-negative integers whose
sum is \( k - 1 \). Each independent \( k \)-subset of \( P_n^{(h)} \) containing \( v_i \) can be obtained by adding \( v_i \) to a \((k - 1)\)-subset \( R \cup S \) such that

(a) \( R \subseteq \{ v_1, \ldots, v_{i-h-1} \} \) is an independent \( r \)-subset of \( P_n^{(h)} \),

(b) \( S \subseteq \{ v_{i+h+1}, \ldots, v_n \} \) is an independent \( s \)-subset of \( P_n^{(h)} \).

Vice versa, one can obtain each of this pairs of subsets by removing \( v_i \) from an independent \( k \)-subset of \( P_n^{(h)} \) containing \( v_i \). Thus, \( T_{k,i}^{(n,h)} \) is obtained by counting independently the subsets of type (a) and (b). Noting that the subsets of type (b) are in bijection with the independent \( s \)-subsets of \( P_{n-i-h}^{(h)} \), the lemma is proved.

In order to obtain our main result, we prepare a lemma.

**Lemma 3.2** For positive \( n \),

\[
H_n^{(h)} = \sum_{k=1}^{\lfloor n/(h+1) \rfloor} \sum_{i=1}^{n} T_{k,i}^{(n,h)}.
\]

**Proof.** The inner sum counts the number of \( k \)-subsets exactly \( k \) times, one for each element of the subset. That is, \( \sum_{i=1}^{n} T_{k,i}^{(n,h)} = k F_{n,k}^{(h)} \). The lemma follows directly from Equation (1). □

Next we introduce a family of Fibonacci-like sequences.

**Definition 3.3** For \( h \geq 0 \), and \( n \geq 1 \), the \( h \)-Fibonacci sequence \( F_n^{(h)} = \{ F_n^{(h)} \}_{n \geq 1} \) is the sequence whose elements are

\[
F_n^{(h)} = \begin{cases} 1 & \text{if } n \leq h + 1, \\ F_{n-1}^{(h)} + F_{n-h-1}^{(h)} & \text{if } n > h + 1. \end{cases}
\]

From Lemma 2.3, and setting for \( h \geq 0 \), and \( n \in \mathbb{Z} \), \( p_n^{(h)} = \begin{cases} p_0^{(h)} & \text{if } n < 0, \\ p_n^{(h)} & \text{if } n \geq 0, \end{cases} \)

we have that,

\[
F_i^{(h)} = p_{i-h-1}^{(h)}, \quad \text{for each } i \geq 1. \tag{2}
\]

Thus, we can write \( F^{(h)} = 1, \ldots, 1, p_0^{(h)} , p_1^{(h)} , p_2^{(h)} , \ldots. \)

In the following, we use the discrete convolution operation \( * \), as follows.

\[
(F^{(h)} * F^{(h)}) (n) = \sum_{i=1}^{n} F_i^{(h)} F_{n-i+1}^{(h)}. \tag{3}
\]
Theorem 3.4 For $n, h \geq 0$, the following holds.

$$H_n^{(h)} = (\mathcal{F}^{(h)} \ast \mathcal{F}^{(h)}) (n).$$

Proof. The sum $\sum_{k=1}^{[n/(h+1)]} T^{(n,h)}_{k,i}$ counts the number of independent subsets of $P_n^{(h)}$ containing $v_i$. We can also obtain such a value by counting the independent subsets of both $\{v_1, \ldots, v_{i-h-1}\}$, and $\{v_{i+h+1}, \ldots, v_n\}$. Thus, we have:

$$\sum_{k=1}^{[n/(h+1)]} T^{(n,h)}_{k,i} = \bar{p}_{i-h-1}^{(h)} \bar{p}_{n-h-i}^{(h)}.$$

Using Lemma 3.2 we can write

$$H_n^{(h)} = \sum_{k=1}^{[n/(h+1)]} \sum_{i=1}^{n} T_{k,i}^{(n,h)} = \sum_{i=1}^{n} \sum_{k=1}^{[n/(h+1)]} T_{k,i}^{(n,h)} = \sum_{i=1}^{n} \bar{p}_{i-h-1}^{(h)} \bar{p}_{n-h-i}^{(h)}.$$

By Equation (2) we have $\sum_{i=1}^{n} \bar{p}_{i-h-1}^{(h)} \bar{p}_{n-h-i}^{(h)} = \sum_{i=1}^{n} F_{i}^{(h)} F_{n-i+1}^{(h)}$. By (3), the theorem is proved. \hfill \Box

Further properties of coefficients $H_n^{(h)}$, and $p_n^{(h)}$ are discussed in [1]. Moreover, in [1] we investigate the case of powers of cycles, and its connection with Lucas cubes.

References


