INVERSE SEMIGROUPS ACTING ON GRAPHS

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There has been much work done recently on the action of semigroups on sets with some important applications to, for example, the theory and structure of semigroup amalgams. It seems natural to consider the actions of semigroups on sets 'with structure' and in particular on graphs and trees. The theory of group actions has proved a powerful tool in combinatorial group theory and it is reasonable to expect that useful techniques in semigroup theory may be obtained by trying to 'port' the Bass-Serre theory to a semigroup context. Given the importance of *transitivity* in the group case, we believe that this can only reasonably be achieved by restricting our attention to the class of inverse semigroups. However, it very soon becomes apparent that there are some fundamental differences with inverse semigroup actions and even such basic notions such as *free actions* have to be treated carefully. We make a start on this topic in this paper by first of all recasting some of Schein's work on representations by partial homomorphisms in terms of actions and then trying to 'mimic' some of the basic ideas from the group theory case. We hope to expand on this in a future paper [5].

1. Introduction

The Bass-Serre theory of group actions on graphs has proved a powerful tool for combinatorial group theorists and the aim of this paper is to consider whether there is any 'mileage' in trying to use these techniques in the context of semigroup theory as well. In particular we aim to develop the theory of (partial) actions of inverse semigroups on graphs and trees and highlight some of the connections with the case for group actions.

In section 2 we give a brief account of the classical theory for group actions on graphs and trees before introducing the basic concept of a partial inverse semigroup action in section 3. We include in this section a brief account of Schein's $\omega-$cosets and partial congruences together with a brief account of a concept of a free action. In section 4, $S-$Graphs are introduced and we present a few examples to illustrate the underlying concepts. We hope to extend this work in a future publication [5].

2. The Group Case

We 'set the scene' in this section by giving a very brief outline of the main players in the Bass-Serre theory of groups. The notation and terminology is mainly that of [2] and we refer the reader to that text for more details. Let G be a group. By a (left) $G-$set we mean a non-empty set, X, on which G acts by permutations, in the sense that there is a group homomorphism $\rho : G \to \mathrm{Sym}X$, where Sym X is the symmetric group on X. As usual, we denote $\rho(g)(x)$ by gx. If X and Y are $G-$sets then a function $f : X \to Y$ is called a $G-$map if for all $x \in X, g \in G, f(gx) = gf(x)$. Let X be a $G-$set. By the $G-$*stabilizer* of an element $x \in X$ we mean the set of elements of G that 'fix' x, i.e. $G_x = \{g \in G : gx = x\}$. It is easy to see that G_x is a subgroup of G and that for any $g \in G, G_{gx} \simeq gG_xg^{-1}$. A group G is said to act *freely* on X if $G_x = 1$ for all $x \in X$. The $G-$*orbit* of an element x is the set $Gx = \{gx : g \in G\}$ which is a $G-$subset of X and it is easy to prove that Gx is $G-$isomorphic to the $G-$set of cosets of G_x in G, denoted G/G_x. The *quotient set* for the $G-$set X is the set of $G-$orbits, $G\backslash X = \{Gx : x \in X\}$ which clearly has a natural map $X \to G\backslash X, x \mapsto Gx$. A $G-$*transversal in* X is a subset Y of X which contains exactly one element of each $G-$orbit of X. Hence the composite $Y \subseteq X \to G\backslash X$ is a bijection.

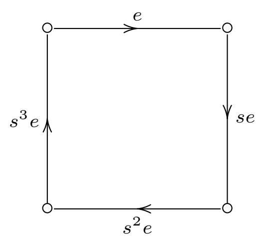
A $G-$graph (X, V, E, ι, τ) is a non-empty $G-$set X with disjoint non-empty $G-$subsets V and E such that $X = V \cup E$ and two $G-$maps $\iota, \tau : E \to V$. If Y is a $G-$subset of X then we write $VY = V \cap Y, EY = E \cap Y$. If Y is non-empty and both ιe and τe belong to VY for all e in Y then we say that Y is a $G-$subgraph of X.

By the *quotient graph*, $G\backslash X$, we mean the graph $(G\backslash X, G\backslash V, G\backslash E, \bar{\iota}, \bar{\tau})$ where $\bar{\iota}(Ge) = G\iota e, \bar{\tau}(Ge) = G\tau e$ for all $Ge \in G\backslash E$.

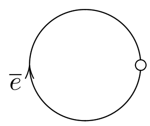
If $G\backslash X$ is connected then it can be shown (see [2]) that there exist subsets $Y_0 \subseteq Y \subseteq X$ such that Y is a $G-$transversal in X, Y_0 is a subtree of X with $VY_0 = VY$ and for each $e \in EY, \iota(e) \in VY$. In this case, Y is called a *fundamental transversal* in X.

The *Cayley graph* of G with respect to a subset T of G is the $G-$graph, $X(G,T)$, with vertex set $V = G$, edge set $E = G \times T$ and incidence function $\iota(g,t) = g, \tau(g,t) = gt$ for all $(g,t) \in E$.

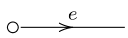
For example, consider the cyclic group $C_4 = \langle s : s^4 \rangle$ and $T = \{s\}$ then the Cayley graph can be represented as



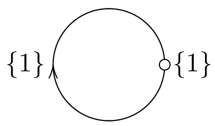
where $e = (1, s) \in G \times T$. The quotient graph is



and a corresponding fundamental $G-$transversal is



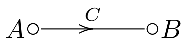
A *graph of groups* $(G(-), Y)$, is a connected graph $(Y, V, E, \overline{\iota}, \overline{\tau})$ together with a function $G(-)$ which assigns to each $v \in V$ a group $G(v)$ and to each edge $e \in E$ a subgroup $G(e)$ of $G(\overline{\iota}e)$ and a group monomorphism $t_e : G(e) \to G(\overline{\tau}e)$.
The standard example is given as follows. We start with a $G-$graph X such that $G\backslash X$ is connected and choose a fundamental transversal Y with subtree Y_0. For each edge e in EY, there are unique vertices $\overline{\iota}e, \overline{\tau}e$ in VY which belong to the same $G-$orbit as $\iota e, \tau e$ respectively. From the way that Y is defined, we see that in fact $\overline{\iota}e = \iota e$. By using the incidence functions $\overline{\iota}, \overline{\tau} : EY \to VY$ we can thus make Y into a graph.
For each e in EY, τe and $\overline{\tau}e$ belong to the same $G-$orbit and so there exists t_e in G such that $t_e\overline{\tau}e = \tau e$ - if $\overline{\tau}e = \tau e$ then we can take $t_e = 1$. Notice then that $G_{\tau e} = t_e G_{\overline{\tau}e} t_e^{-1}$. Now clearly, $G_e \subseteq G_{\iota e}, G_e \subseteq G_{\tau e}$ and so there is an embedding $G_e \to G_{\overline{\tau}e}$ given by $g \mapsto t_e^{-1} g t_e$.
Hence we have constructed a graph of groups *associated to* X.
For our previous example, we have $G_e = \{1\} = G_{\iota e}$ and the connecting element $t_e = s$.



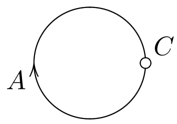
Let $(G(-), Y)$ be a graph of groups. Choose a spanning subtree Y_0 of Y. It follows that $VY_0 = VY$. The *fundamental group* $\pi(G(-), Y, Y_0)$ is the group with generating set $\{t_e : e \in E\} \cup \bigcup_{v \in V} G(v)$ and relations : the relations for $G(v)$, for each $v \in VY$; $t_e^{-1} g t_e = t_e(g)$ for all e in EY; $t_e = 1$ for all $e \in EY_0$.

From the fundamental group, $G = \pi(G(-), Y, Y_0)$,we can construct a *standard* $G-$graph as follows: Let T be the $G-$set generated by Y and relations saying that for each y in Y, $G(y)$ stabilizes y. Then T has $G-$subsets $VT = GV$ and $ET = GE$. Define $\iota, \tau : ET \to VT$ by $\iota(ge) = g\bar{\iota}(e), \tau(ge) = gt_e\overline{\tau}(e)$. Then T is a $G-$graph with fundamental transversal Y.
It is straightforward to verify that the graph of groups associated to this $G-$graph is isomorphic to the original graph of groups. Conversely, we can show that given a group, G, acting on a *tree* we can form the graph of groups and the fundamental group, π, is then isomorphic to G and the standard graph is isomorphic to the original $G-$tree. See [2,§3.4 & Theorem 4.1] for details.

The two classic examples of fundamental groups arise from the following two graphs of groups:



and



In the former case, the fundamental group is the amalgamated free product $A *_C B$ while in the later case it is the HNN-extension $A *_C t$.

3. Partial Actions of Inverse Semigroups

3.1. *Introduction*

Throughout the rest of this paper, S will denote an inverse semigroup. We refer the reader to [3] or [4] for basic concepts relating to semigroups and inverse semigroups. By a (left) partial $S-$act, X, we mean a partial action of S on the set X such that $(st)x$ exists if and only if $s(tx)$ exists and then

$$(st)x = s(tx).$$

In addition, we require that whenever $sx = sy$ then $x = y$. The *domain* of an element $s \in S$ is the set

$$D_s^X = \{x \in X : sx \in X\}.$$

We shall denote D_s^X as simply D_s when the context is clear. Notice then that $x \in D_{st}$ if and only if $x \in D_t$ and $tx \in D_s$. Right partial $S-$acts are defined dually. By a *partial biact* X we mean a set which is both a left partial $S-$act and a right partial $S-$act and which satisfies the additional condition that for all $s, t \in S, x \in X$, if sx and xt exist then $s(xt)$ and $(sx)t$ exist and $s(xt) = (sx)t$. To signify that X is an $S-$biact we shall sometimes denote it by ${}_SX_S$.
Note that if X is a left $S-$act then it is also a right $S-$act with multiplication given by $xs = s^{-1}x$. However, it may not be an $S-$biact.

Example 3.1. Let S be an inverse semigroup and let $s \in S$. Define $D_s = s^{-1}sS$ and for $x \in D_s$ define $s \cdot x = sx$. Then $(S, \cdot)$ is a left partial $S-$act. This is the partial act induced by the Preston-Wagner representation of S. In a similar way, S is also a right partial $S-$act and in fact a partial $S-$biact.

Example 3.2. Let S be an inverse semigroup and X a non-empty set. Define a left partial action of S on X by $sx = x$ for all $x \in X, s \in S$. This is called the *trivial left partial action of* S *on* X. There is clearly a similar construction for trivial right partial actions.

Notice that if X is a left partial $S-$act, then we can consider it a partial $S-$biact with the trivial right partial action.

Lemma 3.1. *Let X be a partial $S-$act and let $s \in S, e \in E(S), x, y \in X$.*

(1) If $sx = y$ then $x = s^{-1}y$;
(2) if $x \in D_s$ then $s^{-1}sx = x$;
(3) if $x \in D_e$ then $ex = x$.

Proof.

(1) $sx = ss^{-1}sx = ss^{-1}y$ and so $x = s^{-1}y$.
(2) $sx = ss^{-1}sx$ and so $x = s^{-1}sx$.
(3) $e(ex) = ex$ and so $ex = x$. □

The set of all elements of S that act on x will be denoted $D^x = \{s \in S : x \in D_s\}$. In addition we shall use the notation $D^X = \bigcup_{x \in X} D^x$.

Let H be a subset of an inverse semigroup S. Denote by $H\omega$ the set $\{s \in S : s \geq h, \text{for some } h \in H\}$. This is called the *closure* of H and we say that H is *closed* if $H\omega = H$. Notice that $H \subseteq H\omega$ and that $(H\omega)\omega = H\omega$.

Lemma 3.2. *Let S be an inverse semigroup and X a left partial $S-$act. Let $x \in X$.*

(1) D^x is a union of $\mathcal{L}-$classes;
(2) D^x is a closed subset of S;
(3) D^X is a union of $\mathcal{D}-$classes;
(4) D^X is closed under the taking of inverses.

Proof.

(1) If $s \in D^x$ and $t \,\mathcal{L}\, s$ then $t^{-1}t = s^{-1}s$ and so $t \in D^x$.
(2) If $s \in D^x$ and $t \geq s$ then there exists $e \in E(S)$ with $s = et$ and so $t \in D^x$.
(3) If $s \in D^x$ and $s \,\mathcal{D}\, t$ then $s \,\mathcal{L}\, u \,\mathcal{R}\, t$ for some $u \in S$. Hence $u \in D^x$ by part (1) and $uu^{-1} = tt^{-1}$ and so $t \in D^{t^{-1}ux}$.
(4) If $s \in D^x$ then $s^{-1} \in D^{sx}$. □

A element x of X is said to be *effective* if $D^x \neq \emptyset$. A partial $S-$act X is *effective* if all its elements are effective. A partial $S-$act is *transitive* if for all $x, y \in X$, there exists $s \in S$ with $y = sx$.

Example 3.3. Let S and T be (disjoint) inverse semigroups and consider the $0-$direct union $R = S \,\dot{\cup}\, T \,\dot{\cup}\, \{0\}$ which is of course also an inverse semigroup. Then S is a left partial $R-$act if we define $r \cdot s = rs$ whenever $s = r^{-1}rs$. Notice that in this case, we must have $D^s \subseteq S$. Notice also that S is an effective left partial $R-$act since for all $s \in S, \{s^{-1}\} \in D^s$. In a similar way, we can consider T as an effective left partial $R-$act and for all $t \in T, D^t \subseteq T$. Now $S \times T$ is a left partial $R-$act with the induced action $r \cdot (s,t) = (r \cdot s, r \cdot t)$ and it then follows that for all $(s,t) \in S \times T, D^{(s,t)} = \emptyset$.

If X is a left partial $S-$act and Y is a subset of X then we shall say that Y is a partial $S-$*subact* of X if for all $s \in S, y \in D_s^X \cap Y \Rightarrow sy \in Y$. Notice that this makes Y a left partial $S-$act with the partial action that induced from X and $D_s^Y = D_s^X \cap Y$ for all $s \in S$.

Example 3.4. Let U be an inverse subsemigroup of an inverse semigroup S. Then S is a partial $U-$biact with action defined by $u \cdot s = us$ for

$s = u^{-1}us$ and $s \cdot u = su$ for $s = suu^{-1}$. If follows that U (with partial action given by Example 3.1) is then a $U-$subact of S.

Let $x \in X$ and define the $S-$*orbit* of x as $S^1x = \{sx : s \in S\} \cup \{x\}$. We use S^1 instead of S to take account of those elements x which are not effective. Notice that S^1x is a left partial $S-$subact of X (the *partial subact generated by* x) and that the action is such that, for all $tx \in S^1x$ and all $s \in S$, $tx \in D_s^{S^1x}$ if and only if $x \in D_{st}^X$ and in which case $s(tx) = (st)x$. Then we have

Lemma 3.3. *For all $x \in X$ the $S-$orbit S^1x, is a transitive left partial $S-$act. If x is effective then so is S^1x.*
Conversely, if a left partial $S-$act is effective and transitive then it has only one $S-$orbit.

Proof. If $y = s_1x$ and $z = s_2x$ then put $t = s_1s_2^{-1}$ to get $y = tz$.
Suppose that x is effective. Then let $sx \in S^1x$ and notice that $ss^{-1}(sx) = sx \in S^1x$ and so S^1x is effective.
The converse is easy. □

Notice that $S^1x = S^1y$ if and only if $y \in S^1x$.

Example 3.5. Consider S as a left partial $S-$act with partial action given as in Example 3.1. Then for all $s \in S, S^1s = L_s$ the $\mathcal{L}-$class containing s.

This is easy to establish : if $t \in S^1s$ then $t = us$ and $s = u^{-1}us$ and so $u^{-1}t = u^{-1}us = s$ and hence $t \in L_s$. Conversely, if $t \in L_s$ then $t^{-1}t = s^{-1}s$ and so $t = tt^{-1}t = (ts^{-1})s$ with $(ts^{-1})^{-1}(ts^{-1})s = st^{-1}ts^{-1}s = ss^{-1}s = s$ as required.

Let X and Y be two partial $S-$acts. A function $f : X \to Y$ is called an $S-$*map* if for all $s \in S$, $x \in D_s^X$ *if and only if* $f(x) \in D_s^Y$ and then $f(sx) = sf(x)$.
For example, if Y is an $S-$subact of an left partial $S-$act X, then the inclusion map $\iota : Y \to X$ is an $S-$map.
The use of "if and only if" as opposed to "only if" in the definition of $S-$map may seem unnecessary but note that both conditions are needed in the proof of Theorem 3.2 below.

Lemma 3.4. *Let S be an inverse semigroup and let $s, t \in S$ then the following are equivalent:*

(1) $L_s \cong L_t$ *as left partial* $S-$*acts,*
(2) for all $u \in S$, $s \in D_u^{L_s}$ *if and only if* $t \in D_u^{L_t}$,
(3) $s\,\mathcal{D}\,t$ *in* S.

Proof.
(1) $\Rightarrow$ (2). Suppose that $L_s \cong L_t$. Then there is a left $S-$ isomorphism $\phi : L_s \to L_t$ which maps s to t' say. But $L_{t'} = L_t$ and so we may as well assume that $t' = t$. Now since ϕ is an $S-$map then for all $u \in S$, $s \in D_u^{L_s}$ if and only if $t \in D_u^{L_t}$.
(2) $\Rightarrow$ (3). Since $s \in D_{s^{-1}}^{L_s}$ then $t \in D_{s^{-1}}^{L_t}$ and so $s^{-1}t \in L_t$. In other words $t = ss^{-1}t$ and therefore $t^{-1} = t^{-1}ss^{-1}$. In a similar way, $s^{-1} = s^{-1}tt^{-1}$. Now $t\,\mathcal{L}\,s^{-1}t$ and so $tt^{-1}\,\mathcal{L}\,s^{-1}tt^{-1} = s^{-1}$. But $t\,\mathcal{R}\,tt^{-1}$ and so $t\,\mathcal{D}\,s^{-1}\,\mathcal{D}\,s$ as required.
(3) $\Rightarrow$ (1). Let $(s, t) \in \mathcal{D}$. If $s = t$ then clearly $L_s \cong L_t$ and so we can assume that $s \neq t$. Then there exists $u \in S$ such that $\rho_u : L_s \to L_t, \rho_{u^{-1}} : L_t \to L_s$ given by $\rho_u(x) = xu, \rho_{u^{-1}}(y) = yu^{-1}$ are mutually inverse $\mathcal{R}-$class preserving bijections. This clearly means that ρ_u and $\rho_{u^{-1}}$ are left $S-$maps and so $L_s \cong L_t$ as required. □

Notice that the superscript in the expression "$s \in D_u^{L_s}$" is superfluous and we have simply inserted it the previous lemma for the sake of clarity. We shall in future denote $D_u^{L_s}$ as simply D_u.

Notice that the orbits of an effective partial $S-$act X, partition X. In this case the *quotient set* of X is defined as the set $S \setminus X = \{S^1 x : x \in X\}$.

Let X be a left partial $S-$act and ρ an equivalence on X. If ρ has the additional property that whenever $s \in S, (x, y) \in \rho$ are such that $x \in D_s$ if and only if $y \in D_s$, then $(sx, sy) \in \rho$ whenever $x \in D_s$, then we shall call ρ a *left* $S-$*congruence* on X. In other words, whenever $(x, y) \in \rho$ then either both $x, y \in D_s$ or neither are, and when both are then $(sx, sy) \in \rho$. *Right* $S-$*congruences* are defined dually. It is then easy to check that the quotient X/ρ is a left partial $S-$act with action given by $s(x\rho) = (sx)\rho$ whenever $x \in D_s$.

Example 3.6. Let X be a partial $S-$biact and define a relation $\mathcal{L}^X$ on X by $\mathcal{L}^X = \{(x, y) : S^1 x = S^1 y\}$. Then it is easy to show that $\mathcal{L}^X$ is a right $S-$congruence on X and so the quotient $X/\mathcal{L}^X \cong S \setminus X$ is a right partial $S-$act. If X is only a left partial $S-$act then $\mathcal{L}^X$ is an equivalence on X.

It is clear (see Example 3.5) that if $X = S$ with partial action given by the Preston-Wagner action, then $\mathcal{L}^X$ is Green's $\mathcal{L}-$relation.

Dually, we define $\mathcal{R}^X = \{(x, y) : xS^1 = yS^1\}$ and note that $\mathcal{R}^X$ is a left $S-$congruence on X and that $\mathcal{R}^S = \mathcal{R}$, Green's $\mathcal{R}-$relation.

The intersection of the equivalences $\mathcal{L}^X$ and $\mathcal{R}^X$ will be denoted $\mathcal{H}^X$ and the join $\mathcal{L}^X \vee \mathcal{R}^X$ by $\mathcal{D}^X$.

Denote the $\mathcal{R}^X-$class of x by R_x^X. Similar meaning is attached to L_x^X, H_x^X, D_x^X. The following results are then easy to prove.

Theorem 3.1. *Let S be an inverse semigroup and X a partial $S-$biact. Then*

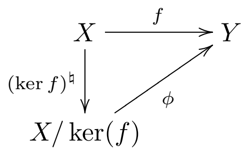
(1) $\mathcal{L}^X \circ \mathcal{R}^X = \mathcal{R}^X \circ \mathcal{L}^X = \mathcal{D}^X$.
(2) If $(x, y) \in \mathcal{L}^X$ then there exists $s \in S^1$ such that $x = sy$ and the functions $\lambda_s : R_y^X \to R_x^X, \lambda_{s^{-1}} : R_x^X \to R_y^X$ defined by $\lambda_s(z) = sz, \lambda_{s^{-1}}(z) = s^{-1}z$ are mutually inverse $\mathcal{L}^X-$class preserving bijections.

It is clear that the intersection of two left $S-$congruences is again a left $S-$congruence. Let σ be a relation on a left partial $S-$act X and suppose there exists a left $S-$congruence ρ with $\sigma \subseteq \rho$. Then the smallest left $S-$congruence containing σ will be referred to as the left $S-$congruence *generated by* σ and denoted $\sigma^\sharp$. It is clearly equal to the intersection of all left $S-$congruences which contain σ. However, it is important to realise that not every relation generates an $S-$congruence.
For example, let S be an inverse semigroup which is *not* bisimple. Let $s, t \in S$ be such that $(s, t) \notin \mathcal{D}$ and let $\sigma = \{(s, t)\}$. Then the left $S-$congruence generated by σ on the left partial $S-$act S does not exist. If it did then there would exist a left $S-$congruence on S, ρ say, such that $(s, t) \in \sigma \subseteq \rho$. But then for all $u \in S, s \in D_u$ if and only if $t \in D_u$ which is impossible by Lemma 3.4.

Also, if ρ is an $S-$congruence on X then the natural map $\rho^\natural : X \to X/\rho$ is an $S-$map.

Theorem 3.2. *If $f : X \to Y$ is an $S-$map then* $\ker(f)$ *is an $S-$congruence on X.*
Moreover there exists a unique $S-$monomorphism $\phi : X/\ker(f) \to Y$ such

that



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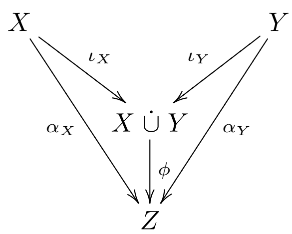
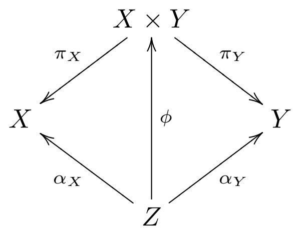
Proof. It is clear that $\ker(f)$ is an equivalence and if $(x, x') \in \ker(f), s \in S$ then if $x \in D_s^X$, it follows that $f(x') = f(x) \in D_s^Y$ since f is an $S-$map and so $x' \in D_s^X$. It is then clear that $(sx, sx') \in \ker(f)$.
Define $\phi : X/\ker(f) \to Y$ by $\phi(x\ker(f)) = f(x)$. That this is well-defined and one to one and that the diagram commutes follows in the usual way. Moreover, $x\ker(f) \in D_s^{X/\ker(f)}$ if and only if $x \in D_s$ if and only if $f(x) \in D_s^Y$ and then $sf(x) = f(sx)$ which means that $\phi(s(x\ker(f))) = s\phi(x\ker(f))$. It is clear that ϕ is unique. □

Let X and Y be left partial $S-$acts and consider the disjoint union $X \dot\cup Y$. This is clearly a left partial $S-$act and it is easy to check that it satisfies the properties of a *coproduct* of the acts X and Y. In more detail, we have $S-$maps $\iota_X : X \to X \dot\cup Y$ and $\iota_Y : Y \to X \dot\cup Y$ and if there is a left partial $S-$act Z and $S-$maps $\alpha_X : X \to Z, \alpha_Y : Y \to Z$ then we can define an $S-$map $\phi : X \dot\cup Y \to Z$ by $\phi(x) = \alpha_X(x), \phi(y) = \alpha_Y(y)$ which makes the diagram



commute.
In a similar way, we can define an obvious left $S-$action on the cartesian product $X \times Y$ by $s(x, y) = (sx, sy)$ whenever both $x, y \in D_s$. This makes $X \times Y$ into a *product* with projections $\pi_X : X \times Y \to X, \pi_Y : X \times Y \to Y$ such that given any left partial $S-$act Z and maps $\alpha_X : Z \to X, \alpha_Y : Z \to Y$ there is a unique $S-$map $\phi : Z \to X \times Y$ (given by $z \mapsto (\alpha_X(x), \alpha_Y(y))$)

such that the diagram



commutes.

Notice that if X is a left partial $S-$act and Y is a partial right $S-$act then $X \times Y$ is a partial $S-$biact with obvious action $s(x, y) = (sx, y), (x, y)t = (x, yt)$.

We shall say that a left partial $S-$act X is *complete* if it satisfies the property that for all $s \in S$ and for all $x, y \in X$, $x \in D_s$ if and only if $y \in D_s$. This means that an element $s \in S$ either acts on all elements of X or on none. Notice that X is complete if and only if the universal equivalence $\nabla_X = X \times X$ is an $S-$congruence on X.
If X is a complete partial $S-$subact of a left partial $S-$act Y, then we can define a left $S-$congruence $\sigma_X = \nabla_X \cup 1_Y$ on Y. We shall denote the quotient Y/σ_X as Y/X and refer to it as the *Rees quotient* of Y by X.

3.2. *Stabilisers and $\omega-$cosets*

Lemma 3.5. *Let H be an inverse subsemigroup of an inverse semigroup S. Then H is closed in S if and only if H is unitary in S.*

Proof. Suppose that H is closed in S and that $hs = h'$ for some $h, h' \in H, s \in S$. Then $h^{-1}hs = h^{-1}h'$ and so $s \geq h^{-1}h' \in H$ and therefore $s \in H$ and H is left unitary in S. Right unitary follows in a similar manner.
Conversely, if H is unitary in S and if $s \geq h$ for some $h \in H, s \in S$ then $hh^{-1} = sh^{-1}$ and so $s \in H$ as required. □

Let T and R be inverse subsemigroups of an inverse semigroup S and suppose that R is closed. If $T \subseteq R$ then $T\omega \subseteq R$. To see this notice that if $h \in T\omega$ then $h \geq t$ for $t \in T \subseteq R$. Hence $h \in R\omega = R$.

We briefly review Schein's theory of partial congruences (see [1] or [3] for more details).

Let $T \subseteq S$ be sets and suppose that ρ is an equivalence on T. Then we say that ρ is a *partial equivalence* on S with domain T. It is easy to establish that ρ is a partial equivalence on S if and only if it is symmetric and transitive. If now T is an inverse subsemigroup of an inverse semigroup S and if ρ is left compatible with the multiplication on S (in the sense that for all $s \in S, (u, v) \in \rho$ either $su, sv \in T$ or $su, sv \in S \setminus T$ and $(su, sv) \in \rho$ in the former case) then ρ is called a *left partial congruence* on S and the set T/ρ of $\rho-$classes will often be denoted by S/ρ.

Theorem 3.3. ([1,Theorem 7.10]) *Let H be a closed inverse subsemigroup of an inverse semigroup S. Define*

$$\pi_H = \{(s, t) \in S \times S : s^{-1}t \in H\}.$$

Then π_H is a left partial congruence on S and the domain of π_H is the set $D_H = \{s \in S : s^{-1}s \in H\}$.
The (partial) equivalence classes are the sets $(sH)\omega$ for $s \in D_H$. The set $(sH)\omega$ is the equivalence class that contains s and in particular H is one of the π_H-classes.

The sets $(sH)\omega$, for $s \in D_H$, are called the *left $\omega-$cosets* of H in S. The set of all left $\omega-$cosets is denoted by S/H. Notice that if $(sH)\omega$ is a left $\omega-$coset then $s^{-1}s \in H$.

Proposition 3.1. ([3, Proposition 5.8.3]) *Let H be an inverse subsemigroup of an inverse semigroup S, and let $(aH)\omega, (bH)\omega$ be left $\omega-$cosets of H. Then the following statements are equivalent:*

(1) $(aH)\omega = (bH)\omega$;
(2) $b^{-1}a \in H\omega$;
(3) $a \in (bH)\omega$;
(4) $b \in (aH)\omega$.

Lemma 3.6. ([1,Lemma 7.11]) *With H and S as in Theorem 3.3,*

(1) precisely one left $\omega-$coset, namely H, contains idempotents,
(2) each left $\omega-$coset is closed,
(3) π_H is left cancellative i.e. $((xa)H)\omega = ((xb)H)\omega$ implies that $(aH)\omega = (bH)\omega$.

Let X be a partial $S-$act and let $x \in X$. Denote by S_x the set of elements of S that fix x, i.e. $S_x = \{s \in S : sx = x\}$. We shall call S_x the $S-$*stabiliser* of x. Notice that $s^{-1}t \in S_x$ if and only if $sx = tx$.

For example, if X is a trivial partial $S-$act, then $S_x = S$ for all $x \in X$, while if S is a group (considered as a partial $S-$act) then $S_s = \{1\}$ for all $s \in S$.

Theorem 3.4. (cf. [1,Lemma 7.18]) *For all $x \in X$, S_x is either empty or a closed inverse subsemigroup of S.*

Proof. Assume that $S_x \neq \emptyset$. If $s, t \in S_x$ then $x = sx = s(tx) = (st)x$ and so S_x is a subsemigroup. Also $sx = x$ implies that $x = s^{-1}x$ and so S_x is an inverse subsemigroup of S. Let $h \geq s$ with $s \in S_x$. Then $s^{-1}h = s^{-1}s$ and so $s^{-1}sx = s^{-1}hx$ which means that $sx = hx$ and so $h \in S_x$. □

From Lemma 3.6 we can easily deduce the following important result.

Theorem 3.5. (cf. [1,Lemma 7.19]) *If H is a closed inverse subsemigroup of an inverse semigroup S then S/H is a left partial $S-$act with action given by $s \cdot X = (sX)\omega$ whenever $X, sX \in S/H$. Moreover, it is easy to establish that $S_{H\omega} = H$*

Lemma 3.7. ([1,Lemma 7.26]) *Let K be a closed inverse subsemigroup of an inverse semigroup S. Let $s^{-1}s \in K$ so that $(sK)\omega$ is a left $\omega-$coset of K. Then $sKs^{-1} \subseteq S_{(sK)\omega}$.*

If H and K are two closed inverse subsemigroups of S then we say that H and K are *conjugate* if $S/H \cong S/K$ (as partial $S-$acts).

Theorem 3.6. ([1,Theorem 7.27]) *Let H and K be closed inverse subsemigroups of an inverse semigroup S. Then H and K are conjugate if and only if there is an element $s \in S$ such that*

$$s^{-1}Hs \subseteq K \text{ and } sKs^{-1} \subseteq H.$$

Moreover, any such element s necessarily satisfies $ss^{-1} \in H, s^{-1}s \in K$.

Theorem 3.7. *Let H and K be closed inverse subsemigroups of an inverse semigroup S. Then H and K are conjugate if and only if there is an element $s \in S$ such that*

$$(s^{-1}Hs)\omega = K \text{ and } (sKs^{-1})\omega = H.$$

Proof. From Theorem 3.6, if H and K are conjugate, then there is an element $s \in S$ such that

$$s^{-1}Hs \subseteq K \text{ and } sKs^{-1} \subseteq H.$$

Now it is clear that $(s^{-1}HS)\omega \subseteq K$ so let $k \in K$ and let $l = sks^{-1} \in H$. Now put $m = s^{-1}ls = s^{-1}sks^{-1}s \in s^{-1}Hs$ and notice that $m \leq k$ and so $k \in (s^{-1}Hs)\omega$ as required. □

Notice that if $ss^{-1} \in H$ then $s^{-1}Hs$ is an inverse subsemigroup of H. To see this note that $s^{-1}hss^{-1}h^{-1}ss^{-1}hs = s^{-1}h(ss^{-1})(h^{-1}ss^{-1}h)s = s^{-1}h(h^{-1}ss^{-1}h)(ss^{-1})s = s^{-1}ss^{-1}hh^{-1}hs = s^{-1}hs$.

Suppose that H is a closed inverse subsemigroup of S and that there exists $s \in S$ such that $ss^{-1} \in H$ is the identity of H. The map $\vartheta : H \rightarrow s^{-1}Hs$ given by $\vartheta(t) = s^{-1}ts$ is an embedding, since $\vartheta(tr) = s^{-1}trs = s^{-1}tss^{-1}rs = \vartheta(t)\vartheta(r)$ and if $s^{-1}ts = s^{-1}rs$ then $t = r$.

Conversely, if $ss^{-1} \in H$ is such that $\vartheta : H \rightarrow s^{-1}Hs$ given by $\vartheta(t) = s^{-1}ts$ is an embedding, then $\vartheta(tss^{-1}) = s^{-1}tss^{-1}s = s^{-1}ts = \vartheta(t)$ and so $tss^{-1} = t$. Similarly, $ss^{-1}t = t$ and so ss^{-1} is the identity of H.

In fact,

Theorem 3.8. *Let S be an inverse semigroup, $s \in S$ and suppose that H is a closed inverse subsemigroup with $ss^{-1} \in H$. Let $K = (s^{-1}Hs)\omega$. Then the following are equivalent*

(1) $H \rightarrow s^{-1}Hs$ *given by* $h \mapsto s^{-1}hs$ *is an embedding of inverse semigroups,*
(2) there exists an inverse semigroup embedding $\phi' : H \rightarrow s^{-1}Hs$ *with* $\phi'(ss^{-1}) = s^{-1}s$,
(3) there exists an inverse semigroup embedding $\phi' : H \rightarrow s^{-1}Hs$ *and* ss^{-1} *is the identity of* H,
(4) ss^{-1} *is the identity of* H,
(5) $H = sKs^{-1}$,
(6) $H \subseteq (ss^{-1})S(ss^{-1})$,

Proof. (1) ⇒ (2) This is clear.
(2) ⇒ (3) Let $t \in H$ and suppose that $\phi'(t) = s^{-1}t's$ for some $t' \in H$. Then we have $\phi'(tss^{-1}) = \phi'(t)\phi'(ss^{-1}) = s^{-1}t'ss^{-1}s = s^{-1}t's = \phi'(t)$ and so $tss^{-1} = t$. In a similar way, $ss^{-1}t = t$ and so ss^{-1} is the identity of H.
(3) ⇒ (4) Obvious.

(4) $\Rightarrow$ (5) For any $h \in H, h = ss^{-1}hss^{-1} \in sKs^{-1}$. Conversely, let $k \in K$ so that $k \geq s^{-1}hs$ for some $h \in H$. So $ke = s^{-1}hs$ for some $e \in E(S)$. Hence $skes^{-1} = ss^{-1}hss^{-1} = h$ and so $h = (sks^{-1})(ses^{-1})$. Hence $sks^{-1} \geq h$ and so $sKs^{-1} \subseteq H$ since H is closed.
(5) $\Rightarrow$ (6) Clear.
(6) $\Rightarrow$ (1) Let $\theta : H \to s^{-1}Ss$ be given by $\theta(h) = s^{-1}hs$. From the remarks preceding this theorem, it is clear that θ is a morphism. If $\theta(h) = \theta(h')$ then $s^{-1}ss^{-1}tss^{-1}s = s^{-1}ss^{-1}t'ss^{-1}s$ and hence $h = ss^{-1}tss^{-1} = ss^{-1}t'ss^{-1} = h'$ as required. □

Note that under the conditions of the preceding theorem, $E(s^{-1}Hs) = s^{-1}E(H)s$. Moreover, we will see in Example 4.1, that $s^{-1}s$ need not be the identity in K.

Theorem 3.9. *Let X be an effective left partial $S-$act and let $x \in X$. Then $S^1x \cong S/S_x$.*

Proof. Consider the map $\alpha : S^1x \to S/S_x$ given by $\alpha(sx) = (sS_x)\omega$. Notice that if $sx = tx$ then $s^{-1}t \in S_x$ and so $ss^{-1}t \in sS_x$, which means that $t \in (sS_x)\omega$. Similarly, $s \in (tS_x)\omega$ and so α is well-defined. It is clear that α is an $S-$map that is onto. Suppose then that $(sS_x)\omega = (tS_x)\omega$. Then $s^{-1}t \in S_x$ and so $sx = tx$ as required. □

Lemma 3.8. *Let S be an inverse semigroup and X a left partial $S-$act. Let $s \in S$ and $x \in D_s$. Then sS_xs^{-1} is an inverse subsemigroup of S.*

Proof. This follows from Theorem 3.4 and the remarks preceding Theorem 3.8. □

Theorem 3.10. *Let S be an inverse semigroup and X a partial $S-$act. Let $s \in S$ and $x \in D_s$. Then S_x and S_{sx} are conjugate.*

Proof. Since $S^1x = Ssx$ the result follows from Theorem 3.9. In fact, we have that $(sS_xs^{-1})\omega = S_{sx}$. □

3.3. $\mathcal{L}-$*classes and free partial acts*

Let X be a left partial $S-$act, $x \in X$ and consider the $S-$orbit S^1x. In general, not every element of S will act on x so recall that we put $D^x = \{s \in S : x \in D_s\}$. Then we have

Lemma 3.9. *For every $x \in X$, $D^x = \{s \in S : s^{-1}s \in S_x\} = \bigcup_{s \in D^x} L_s$.*

Proof. If $s^{-1}s \in S_x$ then $x \in D_s$ and so $s \in D^x$. Conversely, if $x \in D_s$ then $s^{-1}s \in S_x$ from Lemma 3.1. The last equation follows from Lemma 3.2. □

Let X be a left partial $S-$act and let $x \in D_s$. Notice then that if $y = sx$ then $s^{-1}sx = s^{-1}y = x$ so that $s^{-1}s \in S_x$ and $S_x \cap E(S) \neq \emptyset$. Following the definition for group actions, we shall say that an inverse semigroup S acts *freely* on a partial $S-$act X if for all $x \in X$ there exists $e \in E(S)$ such that $S_x = e\omega (= \{e\}\omega)$. Notice that if $S_x = e\omega$ then $S_{sx} = (ses^{-1})\omega$ for all $sx \in S^1x$. To see this note that if $t \in S_{sx}$ then $tsx = sx$ and so $s^{-1}ts \in S_x$. Hence $s^{-1}ts \geq e$ and so $t \geq ses^{-1}$.

Lemma 3.10. *Let S be an inverse semigroup, $s \in S$ and consider S as a left partial $S-$act. Then $S_s = \{s^{-1}s\}\omega$.*

Proof. If $t \in S_s$ then $s = t^{-1}ts = t^{-1}s$ and so $ss^{-1} = t^{-1}ss^{-1}$ and so $t^{-1} \in (ss^{-1})\omega$ and hence $t \in (ss^{-1})\omega$. Moreover, if $t \in (ss^{-1})\omega$ then $t^{-1} \in (ss^{-1})\omega$ and so $ss^{-1} = t^{-1}ss^{-1}$. But then $s = t^{-1}s$ and so $t \in D_s$ and $ts = tt^{-1}s = s$. Hence $S_s = (ss^{-1})\omega$. □

Theorem 3.11. *The partial $S-$act X is free, effective and transitive if and only if it is isomorphic to an orbit of the left partial $S-$act S (i.e. an $\mathcal{L}-$class of S).*

Proof. Let $s \in S$ and consider S as a left partial $S-$act. By Lemma 3.10 we see that the $S-$orbit S^1s is a free partial $S-$act.
Conversely, let X be a free, effective and transitive partial $S-$act. Then X has only 1 orbit and so choose any $x \in X$ so that $X = S^1x$. By Theorem 3.9 and Lemma 3.10 $X \cong S/S_x = S/e\omega = S/S_e \cong S^1e$ for some $e \in E(S)$. □

Notice that each $\mathcal{L}-$class of S is then a free left partial $S-$subact of S. Moreover, If $(s,t) \in \mathcal{D}$ then $L_s \cong L_t$. Hence each $\mathcal{D}-$class contains, up to isomorphism, a unique free left partial $S-$subact. So in general, there may be more than one non-isomorphic free left partial $S-$act of rank 1.

The following is clear

Lemma 3.11. *Coproducts of free partial $S-$acts are free. Hence S is a free partial $S-$act.*

Theorem 3.12. *An inverse semigroup S is $E-$unitary if and only if for all free left partial $S-$acts X and all $x \in X$, $S_x \subseteq E(S)$.*

Proof. Suppose S is $E-$unitary and that X is a free left partial $S-$act. Then if $x \in X$ and $s \in S_x$ it follows since X is free that there exists $e \in E(S)$ such that $s \in e\omega$ and so there exists $f \in E(S)$ with $sf = e$. Hence $s \in E(S)$ as required.
Conversely, suppose that $sf = e$ for some $s \in S, e, f \in E(S)$. Then clearly $se = e$ and so $e = se = ss^{-1}se = ses^{-1}s = es^{-1}s = s^{-1}se$. Hence in the left partial $S-$act L_e we see that $e \in D_s$ and so $s \in S_e$. Consequently $s \in E(S)$ and S is $E-$unitary as required. □

3.4. *Generators and relations*

When dealing with actions over a group G, it is easy to establish that every $G-$set X is a quotient of a free $G-$set. In particular we can construct the (free) $G-$set $G \times X$ with action defined by $g(h, x) = (gh, x)$ and construct a $G-$map $G \times X \to X$ with $(g, x) \mapsto gx$ which is clearly onto. A similar construction for inverse semigroups and partial actions however fails. While we can construct a left partial $S-$act $S \times X$ with action induced by the left $S-$action of S on S (which is free being a coproduct of $|X|$ copies of S), the corresponding map, $S \times X \to X$ given by $(s, x) \mapsto sx$ is not in general an $S-$map. To see this notice that for $t \in S$, $(s, x) \in D_t^{S\times X}$ if and only if $s \in D_t^S$ (in other words $s = t^{-1}ts$ in S) whereas $sx \in D_t^X$ if and only if $x \in D_{ts}^X$. As an example, let X be any set with the trivial $S-$action, $sx = x$ for all $s \in S, x \in X$.

We wish to give a useful meaning to the notation $\langle x|R\rangle$ where x is a symbol and R is a set of equations of the form $sx = tx$ for $s, t \in S$.
First notice that this equation is equivalent to $t^{-1}s \in S_x$ and that S_x is a closed inverse subsemigroup of S. Suppose then that $R = \{E_i : i \in I\}$ where E_i is the equation $s_ix = t_ix$ for $s_i, t_i \in S$. Let $P = \{t_i^{-1}s_i, s_i^{-1}t_i : i \in I\} \subseteq S$, let K be the smallest inverse subsemigroup of S containing P and let $H = K\omega$. Then $\langle x|R\rangle$ shall denote the left partial $S-$act S/H. It is clear that in practice we do not need to include both elements $t_i^{-1}s_i$ and $s_i^{-1}t_i$ but it disposes of the need for a "well-defined" argument. Notice also that H is the smallest closed inverse subsemigroup of S containing P.
If a partial $S-$act contains x and satisfies an equation $s_ix = t_ix$ then there are a number of other consequences for the partial act. In particular note that $s_i^{-1}s_i, t_i^{-1}t_i \in S_x$. Also, if the element s_i can be factorised as

$s_i = p_i q_i$ then $x \in D_{q_i}$ and so $q_i^{-1} q_i \in S_x$ and if there exists $r_i \in S$ with $x \in D_{r_i s_i} \cap D_{r_i t_i}$ then $r_i s_i x = r_i t_i x$ and so $(r_i t_i)^{-1}(r_i s_i) \in S_x$.
With the notation above we have:

Lemma 3.12.

(1) If $s \in H$ and $s = se$ for $e \in E(S)$ then $e \in H$.
(2) For all $i \in I$, $s_i^{-1} s_i, t_i^{-1} t_i \in H$.
(3) If for some $i \in I$, $s_i = p_i q_i$ then $q_i^{-1} q_i \in H$.
(4) If for some $i \in I$, there exists $r_i \in S$ with $s_i, t_i \in D_{r_i}^S$ then $(r_i t_i)^{-1}(r_i s_i) \in H$.
(5) For each $i \in I$, $(s_i H)\omega = (t_i H)\omega$.

Proof.

(1) This follows since H is closed in S.
(2) Since $t_i^{-1} s_i = t_i^{-1} s_i s_i^{-1} s_i$ and $t_i^{-1} s_i \in H$ then $s_i^{-1} s_i \in H$.
(3) Notice that $t_i^{-1} s_i = t_i^{-1} s_i q_i^{-1} q_i$ and so the result follows from (1).
(4) Notice that $r_i t_i \in L_{t_i}$ and so $t_i^{-1} r_i^{-1} r_i t_i = t_i^{-1} t_i$. Hence $t_i^{-1} r_i^{-1} r_i s_i = t_i^{-1} r_i^{-1} r_i t_i t_i^{-1} s_i = t_i^{-1} s_i \in H$.
(5) Notice that $(s_i H)\omega = (t_i H)\omega$ if and only if $t_i^{-1} s_i \in H$. □

Given any transitive, effective left partial $S-$act X we can construct a presentation associated with X in an obvious way. Let $X = S^1 y$ for some $y \in X$ and consider the presentation $\langle x | sx = x, s \in S_y \rangle$.

Theorem 3.13. *Let $X = S^1 y$ be an effective, transitive left partial $S-$act. Then the left partial $S-$act associated with the presentation $\langle x | sx = x, s \in S_y \rangle$ is isomorphic to X.*

Proof. This follows almost immediately from Theorem 3.9. □

4. Graphs

4.1. *S−graphs*

We define $S-$graphs for an inverse semigroup S in a similar way as we do for groups. Specifically, an $S-$graph (X, V, E, ι, τ) is a non-empty $S-$set X with disjoint non-empty $S-$subsets V and E such that $X = V \cup E$ and two maps $\iota, \tau : E \to V$ with the property that for all $e \in E, s \in S$ if $s \in D^e$ then $s \in D^{\iota e} \cap D^{\tau e}$ and $s\iota e = \iota(se), s\tau e = \tau(se)$.

If we have the stronger property that ι and τ are $S-$maps then we shall refer to X as a *complete* $S-$*graph.* Notice that in this case, since ι and τ are $S-$maps then for any edge $e \in E$ and any $s \in S$, $\iota e \in D_s$ if and only if $\tau e \in D_s$. Hence it follows that for all $s \in S$, either s acts on all vertices in a connected component of X or it acts on none. It also follows that if Y is a connected subgraph of a complete $S-$graph X and $s \in S$ acts on an element (and hence all) of Y then sY is a connected subgraph of X.

A *path* p in X is a finite sequence

$$v_0, e_1^{\epsilon_1}, v_1, \ldots, v_{n-1}, e_n^{\epsilon_n}, v_n$$

where $n \geq 0, v_i \in VX$ for each $1 \leq i \leq n$, $e_i^{\epsilon_i} \in EX^{\pm 1}, \iota(e_i^{\epsilon_i}) = v_{i-1}, \tau(e_i^{\epsilon_i}) = v_i$ for each $1 \leq i \leq n$.
We extend the incidence functions to paths by defining $\iota(p) = v_0, \tau(p) = v_n$ and define the *length* of p to be n. The inverse path p^{-1} is defined to be the path $v_n, e_n^{-\epsilon_n}, v_{n-1}, \ldots, v_1, e_1^{-\epsilon_1}, v_0$. If for each $1 \leq i \leq n-1, e_{i+1}^{\epsilon_{i+1}} \neq e_i^{-\epsilon_i}$ then p is said to be *reduced*.
Let $u, v \in V$ and consider $P = \{p : p \text{ is a reduced path with } \iota(p) = u, \tau(p) = v\}$. Any path $p \in P$ of minimal length is called a *geodesic* from u to v. A path p of length $n \geq 1$ is called a *closed path* or *cycle* if $\iota(p) = \tau(p)$. A connected $S-$graph with no cycles will be called an $S-$tree. It is clear that there is a unique path connecting any two vertices in an $S-$tree and so this will be a geodesic.
If (X, V, E, ι, τ) is a tree and $W \subseteq V$ then by the *subtree generated by* W we mean the subgraph of X consisting of all edges and vertices which occur in the geodesics joining all pairs of vertices in W. This is essentially the "smallest" subtree of X containing W.

Suppose now that X is a connected complete $S-$graph. Then for $x \in X$, it follows that D^x is a closed inverse subsemigroup of S which is a union of $\mathcal{D}-$classes. To see this first notice that $D^x = D^y = D^X$ for all $x, y \in X$. Now suppose that $s, t \in D^x$. Then $t \in D^x$ implies $t \in D^{sx}$ and so $ts \in D^x$. Hence by Lemma 3.2, D^X is a closed inverse subsemigroup of S and is a union of $\mathcal{D}-$classes of S.

Let X be a left partial $S-$act. We shall say that S *stabilizes an element* $x \in X$ if $s \in S_x$ if and only if $x \in D_s$.

Lemma 4.1. (cf. [2, Proposition 4.7]) *Let S be an inverse semigroup, T an effective, complete $S-$tree and v a vertex of T. Then S stabilizes a vertex of T if and only if there is an integer n such that the distance from v to each element of S^1v is at most n.*

Proof. Suppose that S stabilizes a vertex v_0 and that the geodesic, p, from v to v_0 has length N. Then for each $s \in D^T$, there is a path, p, sp^{-1} of length $2N$ from v to sv.
Conversely, let T' be the subtree generated by $S^1 v$. Then it follows that T' is an $S-$subtree of T and no reduced path has length greater than $2n$. If T' has at most one edge then every element of T' is $S-$stable and so in particular v is $S-$stable. So we can assume that T' has at least two edges and therefore at least one vertex of T' has degree at least two. Now remove from T' all the leaves of the tree and their incident edges to leave an $S-$subtree, T'', (to see that T'' is an $S-$subtree notice that given an edge e of T'', then $\deg(\iota e), \deg(\tau e) \geq 2$ in T' and so for any $s \in D^T$, $\deg(\iota se), \deg(\tau se) \geq 2$ in T' and hence $se \in T''$) in which no reduced path has length greater than $2n - 2$. Hence by induction, $S-$stabilizes a vertex. $\square$

Corollary 4.1. *If there is a finite $S-$orbit in VT then S stabilizes a vertex of T.*

Corollary 4.2. *A finite inverse semigroup acting completely on a tree must stabilize a vertex.*

If X is a partial $S-$biact, the (left) *Schützenberger* graph of X with respect to a subset T of S, will be denoted $\Gamma = \Gamma(X, T)$, and is the (left) $S-$graph with vertex set $V = X$, edge set $E = \{(x, t) \in X \times T : xt \text{ exists and } xt \neq x\}$ and incidence functions $\iota(x, t) = x, \tau(x, t) = xt$ for all $(x, t) \in E$. The partial action is that induced by the left action of S on X.

$$x \xrightarrow{t} xt$$

Notice that $\Gamma(X, T)$ is a complete $S-$graph.
In particular, we are interested in the case $X = {}_S S_S$. In the following theorem, we shall denote the element $S^1 x$ of the quotient $S \setminus \Gamma(X, T)$ by $\overline{x}$.

Theorem 4.1. *Let X be an $S-$biact and let T be a generating set for S. Then*

(1) $x \,\mathcal{R}^X\, y$ in X if and only if there exists a finite path in $\Gamma(X, T)$ connecting x and y (we include here the null path of length 0),
(2) $x \,\mathcal{L}^X\, y$ in X if and only if $\overline{x} = \overline{y}$ in $S \setminus \Gamma(X, T)$,
(3) $x \,\mathcal{D}^X\, y$ in X if and only if there exists a finite path in $S \setminus \Gamma(X, T)$ connecting $\overline{x}$ to $\overline{y}$.

Proof. (1) If $x\,\mathcal{R}^X\,y$ then $y = x \cdot s$ for some $s \in S$ and if $s = t_1 \dots t_n$ for $t_i \in T$ then there are edges in $\Gamma(X,T)$

$$x \xrightarrow{t_1} xt_1 \xrightarrow{t_2} xt_1t_2 \qquad \cdots \qquad \xrightarrow{t_n} xt_1 \dots t_n = y$$

The converse is clear since for any edge e in $\Gamma(X,T)$, $(\iota(e), \tau(e)) \in \mathcal{R}^X$. The other results follow easily. □

Corollary 4.3. *Let S be an inverse semigroup with generating set T. Then S is bisimple if and only if $S \setminus \Gamma = S \setminus \Gamma(S,T)$ is a connected graph.*

Lemma 4.2. *If in $\Gamma(X,T)$, the vertex x is* isolated *(i.e. has no edges incident on it) then $x \cdot t$ does not exists for any $t \in T$. Consequently, if T generates S then there are no isolated vertices in $\Gamma(S,T)$.*

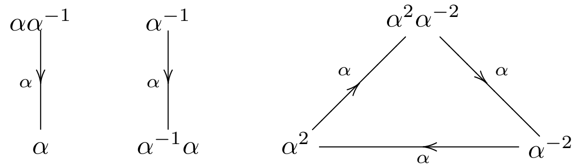
As an illustrative example of some of these constructions, let us consider the inverse subsemigroup S of $\mathcal{J}_{\{1,2,3,4,5\}}$ generated by

$$\alpha = \begin{pmatrix} 1\,2\,3\,4 \\ 2\,3\,1\,5 \end{pmatrix}.$$

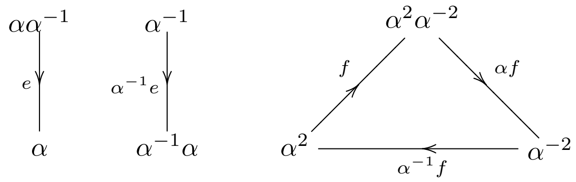
It is easy to calculate that $S = \{\alpha, \alpha^{-1}, \alpha^2, \alpha^{-2}, \alpha\alpha^{-1}, \alpha^{-1}\alpha, \alpha^2\alpha^{-2}\}$ and that $E(S) = \{\alpha\alpha^{-1}, \alpha^{-1}\alpha, \alpha^2\alpha^{-2}\}$.
In fact S has multiplication table given by

	α	α^{-1}	α^2	α^{-2}	$\alpha\alpha^{-1}$	$\alpha^{-1}\alpha$	$\alpha^2\alpha^{-2}$
α	α^2	$\alpha\alpha^{-1}$	$\alpha^2\alpha^{-2}$	α^2	α^{-2}	α	α^{-2}
α^{-1}	$\alpha^{-1}\alpha$	α^{-2}	α^{-2}	$\alpha^2\alpha^{-2}$	α^{-1}	α^2	α^2
α^2	$\alpha^2\alpha^{-2}$	α^{-2}	α^{-2}	$\alpha^2\alpha^{-2}$	α^2	α^2	α^2
α^{-2}	α^2	$\alpha^2\alpha^{-2}$	$\alpha^2\alpha^{-2}$	α^2	α^{-2}	α^{-2}	α^{-2}
$\alpha\alpha^{-1}$	α	α^2	α^2	α^{-2}	$\alpha\alpha^{-1}$	$\alpha^2\alpha^{-2}$	$\alpha^2\alpha^{-2}$
$\alpha^{-1}\alpha$	α^{-2}	α^{-1}	α^2	α^{-2}	$\alpha^2\alpha^{-2}$	$\alpha^{-1}\alpha$	$\alpha^2\alpha^{-2}$
$\alpha^2\alpha^{-2}$	α^{-2}	α^2	α^2	α^{-2}	$\alpha^2\alpha^{-2}$	$\alpha^2\alpha^{-2}$	$\alpha^2\alpha^{-2}$

and so the $\mathcal{R}-$classes are $\{\alpha, \alpha\alpha^{-1}\}, \{\alpha^{-1}, \alpha^{-1}\alpha\}, \{\alpha^2, \alpha^{-2}, \alpha^2\alpha^{-2}\}$.
Let $T = \{\alpha\}$. Then the Schützenberger graph, $\Gamma = \Gamma(S,T)$, of S with respect to T is



The action of S (induced by the Preston-Wagner representation) on the graph is as follows



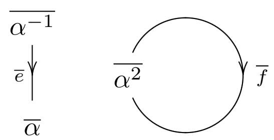
Also, we can calculate the orbits of Γ; the edge orbits are

$$S^1 e = \{e, \alpha^{-1}e\}, S^1 f = \{f, \alpha f, \alpha^{-1}f\}$$

while the vertex orbits are

$S^1\cdot\alpha = \{\alpha, \alpha^{-1}\alpha\}$, $S^1\cdot\alpha^{-1} = \{\alpha^{-1}, \alpha\alpha^{-1}\}$ and $S^1\cdot\alpha^2 = \{\alpha^2, \alpha^{-2}, \alpha^2\alpha^{-2}\}$.

The quotient graph $S \setminus \Gamma$, then looks like



Notice that $S_e = \{\alpha\alpha^{-1}\} = \{\alpha\alpha^{-1}\}\omega, S_f = \{\alpha\alpha^{-1}, \alpha^{-1}\alpha, \alpha^2\alpha^{-2}\} = \{\alpha^2\alpha^{-2}\}\omega$ and so S acts freely on Γ.

In fact we have

Theorem 4.2. *Let S be an inverse semigroup and T a subset of S. Then S acts freely on $\Gamma(S,T)$.*

Proof. Suppose that $p \in S_s$ for some $p, s \in S$. Then $ps = s$ and so $pss^{-1} = ss^{-1}$. Hence $p \geq ss^{-1}$ and $p \in (ss^{-1})\omega$.
Conversely, if $p \in (ss^{-1})\omega$ then $pss^{-1} = ss^{-1}$ and so $ps = s$ and $p \in S_s$. $\square$

4.2. *Graphs of Inverse Semigroups*

An $S-$*transversal* in a partial $S-$act X is a subset Y of X which meets each $S-$orbit exactly once. Hence the composite $Y \subseteq X \to S \setminus X$ is a bijection. Notice that $S \setminus X$ is isomorphic to

$$\dot{\bigcup}_{y\in Y} S/S_y.$$

Lemma 4.3. (cf. [2, Proposition 2.6]) *If X is an $S-$graph and $S \setminus X$ is connected then there exist subsets $Y_0 \subseteq Y \subseteq X$ such that Y is an $S-$transversal in X, Y_0 is a subtree of X, $VY = VY_0$ and for each $e \in EY, \iota e \in VY = VY_0$.*

Proof. Let $\overline{X} = S \setminus X$ and $\overline{x} = S^1 x$ for all $x \in X$. Choose a vertex v_0 of X. By Zorn's Lemma we can choose a maximal subtree Y_0 of X containing v_0 such that the composite $Y_0 \subseteq X \to \overline{X}$ is injective. Let $\overline{Y}_0$ denote the image of Y_0. Then $V\overline{Y}_0 = V\overline{X}$. To see this, suppose that it is not so. Then since $\overline{X}$ is connected, any vertex in $\overline{Y}_0$ is connected to any vertex in $\overline{X} - \overline{Y}_0$ by a path in $\overline{X}$, so some edge $\overline{e}$ of $\overline{X}$ has one vertex $\overline{v}$ in $\overline{Y}_0$ and one vertex in $\overline{X} - \overline{Y}_0$. Here $\overline{v}$ comes from an element v of VY_0 and $\overline{e}$ from an edge e of X; since v lies in the same orbit as a vertex of e, it is a vertex of se for some $s \in S$, and by replacing e with se we may further assume that v is a vertex of e. Let w be the other vertex of e. Notice e, w do not lie in Y_0, since their images do not lie in $\overline{Y}_0$. But $Y_0 \cup \{e, w\}$ contradicts the maximality of Y_0. This proves the claim that $V\overline{Y}_0 = V\overline{X}$.
For each edge $\overline{e}$ in $E\overline{X} - E\overline{Y}_0$, $\overline{\iota}\,\overline{e}$ comes from a unique vertex of Y_0 and as before we can assume $\iota e \in Y_0$. Adjoining the resulting edges to Y_0 gives a subset Y of X such that the composite $Y \subseteq X \to \overline{X}$ is bijective and if $e \in EY$ then $\iota e \in Y$. □

In Lemma 4.3, we call Y a *fundamental $S-$transversal with subtree Y_0*.

Let S be an inverse semigroup and X an $S-$graph such that $S \setminus X$ is connected. Let Y be a fundamental $S-$transversal for X with subtree Y_0. Let $e \in EY$ and consider the $S-$orbits of $\iota e, \tau e$. There is a unique element of $VY, \overline{\iota} e$, which lies in the same $S-$orbit as ιe. Because of the way we construct Y, we can assume that $\iota e = \overline{\iota} e$. In a similar way, there is a unique element of $VY, \overline{\tau} e$, which lies in the same $S-$orbit as τe. This defines functions $\overline{\iota}, \overline{\tau} : EY \to VY$ which makes Y into a graph and it is easy to see that $Y \simeq S \setminus X$.
For each $e \in EY$ we see that τe and $\overline{\tau} e$ lie in the same (transitive) $S-$orbit in EX, so there exists an element $t_e \in S^1$ such that $t_e \overline{\tau} e = \tau e$. If $e \in EY_0$ then $\overline{\tau} e = \tau e$ (by uniqueness) and so we can take $t_e = 1$. Now $S_e \subseteq S_{\iota e}$ and $S_e \subseteq S_{\tau e}$. From Theorem 3.10 we see that $S_{\overline{\tau} e}$ is conjugate to $S_{\tau e}$ and from Theorem 3.8 that if $t_e t_e^{-1}$ is the identity of $S_{\tau e}$ then there is an embedding $S_e \subseteq S_{\tau e} \to t_e^{-1} S_{\tau e} t_e \subseteq S_{\overline{\tau} e}$ given by $x \mapsto t_e^{-1} x t_e$.

We shall refer to this as the *graph of inverse semigroups associated to X* with respect to the fundamental $S-$transversal Y, the maximal subtree Y_0

and the *family of connecting elements* t_e. Notice that it is *sufficient* that $t_e t_e^{-1}$ is the identity of $S_{\tau e}$ in order to guarantee the existence of an embedding $S_e \to S_{\overline{\tau}e}$. At this stage we do not know if it is *necessary*. More generally, by a *graph of inverse semigroups* $(S(-), Y)$, we shall mean a connected graph $(Y, V, E, \bar{\iota}, \overline{\tau})$ together with a function $S(-)$ which assigns to each $v \in V$ an inverse semigroup $S(v)$, and to each edge $e \in E$ a subgroup $S(e)$ of $S(\bar{\iota}e)$ and an inverse semigroup monomorphism $t_e : S(e) \to S(\overline{\tau}e), s \mapsto s^{t_e}$. We shall refer to the semigroups $S(v)$ as the *vertex (inverse) semigroups*, the semigroups $S(e)$ as the *edge (inverse) semigroups* and the maps t_e the *edge functions*.

We present two example of inverse semigroups acting on graphs (in fact trees) and construct the associated graph of inverse semigroups. The first involves the free monogenic inverse semigroup and the second the bicyclic semigroup.

Example 4.1. Let S be the free inverse semigroup on one generator $\{x\}$. From [4,IX.1.1], we see that S is isomorphic to $\{(m, k, n) : m \leq 0 \leq n, m \leq k \leq n, m < n\}$ with multiplication given by $(m, k, n)(m', k', n') = (\min\{m, k + m'\}, k + k', \max\{n, k + n'\})$. Under this isomorphism, x is mapped to $(0, 1, 1)$ and $(m, k, n)^{-1} = (m - k, -k, n - k)$.

Lemma 4.4. *Let S be as above.*

(1) $(m, k, n)\,\mathcal{R}\,(p, l, q)$ *if and only if* $m = p$ *and* $n = q$.
(2) $(m, k, n)\,\mathcal{L}\,(p, l, q)$ *if and only if* $m - k = p - l$ *and* $n - k = q - l$.
(3) $(m, k, n) \in (p, l, q)^{-1}(p, l, q)S$ *if and only if* $m \leq p - l < q - l \leq n$.
(4) *If* $(m, k, n) \in (p, l, q)^{-1}(p, l, q)S$ *then* $(p, l, q)(m, k, n) = (m+l, k+l, n+l)$.
(5) $(m, k, n) \in S(p, l, q)(p, l, q)^{-1}$ *if and only if* $m - k \leq p < q \leq n - k$.
(6) *If* $(m, k, n) \in S(p, l, q)(p, l, q)^{-1}$ *then* $(m, k, n)(p, l, q) = (m, k + l, n)$.
(7) $(m, k, n) \leq (p, l, q)$ *if and only if* $l = k, p \geq m$ *and* $q \leq n$.

Proof.

(1) $(m, k, n)\,\mathcal{R}\,(p, l, q)$ if and only if $(m, k, n)(m, k, n)^{-1} = (p, l, q)(p, l, q)^{-1}$ if and only if $(m, k, n)(m-k, -k, n-k) = (p, l, q)(p-l, -l, q-l)$ if and only if $(m, 0, n) = (p, 0, q)$.
(2) $(m, k, n)\,\mathcal{L}\,(p, l, q)$ if and only if $(m, k, n)^{-1}(m, k, n) = (p, l, q)^{-1}(p, l, q)$ if and only if $(m-k, -k, n-k)(m, k, n) = (p-l, -l, q-l)(p, l, q)$ if and only if $(m - k, 0, n - k) = (p - l, 0, q - l)$.

(3) $(p,l,q)^{-1}(p,l,q) = (p-l,-l,q-l)(p,l,q) = (p-l,0,q-l)$. So $(m,k,n) \in (p,l,q)^{-1}(p,l,q)S$ if and only if there exists $(a,b,c) \in S$ with $(m,k,n) = (p-l,0,q-l)(a,b,c) = (\min\{p-l,a\}, b, \max\{q-l,c\})$. But this is true if and only if $m \leq p-l$ and $n \geq q-l$ (take $a = m, b = k, c = n$).
(4) By (3), it follows that $m \leq p-l$ and $n \geq q-l$ and so $(p,l,q)(m,k,n) = (m+l, k+l, n+l)$.
(5) $(p,l,q)(p,l,q)^{-1} = (p,l,q)(p-l,-l,q-l) = (p,0,q)$. So $(m,l,n) \in S(p,l,q)(p,l,q)^{-1}$ if and only if there exists $(a,b,c) \in S$ with $(m,k,n) = (a,b,c)(p,0,q) = (\min\{a,b+p\}, b, \max\{c,b+q\})$. But this is true if and only if $m \leq p+k$ and $n \geq q+k$ (take $a = m, b = k, c = n$).
(6) By (5), it follows that $m \leq p+k, n \geq q+k$ and so $(m,k,n)(p,l,q) = (m, k+l, n)$.
(7) $(m,k,n) \leq (p,l,q)$ if and only if $(m,k,n) = (m,k,n)(m,k,n)^{-1}(p,l,q)$ if and only if $(m,k,n) = (m,0,n)(p,l,q) = (\min(p,m), l, \max(q,n))$ if and only if $l = k, p \geq m$ and $q \leq n$. □

Notice then that the (right) orbit $(m,k,n)S^1$ of an element $(m,k,n) \in S$ is given by

$$(m,k,n)S^1 = \{(m,l,n) : l \geq m\}.$$

Notice then that $(m,k,n)S^1 = (m,m,n)S^1$.
The left orbit $S^1(m,k,n)$ is given by

$$S^1(m,k,n) = \{(m+l, k+l, n+l) : l \leq -m\}$$

and so $S^1(0, k-m, n-m) = S^1(m,k,n)$
Now $(m,k,n)\,\mathcal{R}\,(m,k,n)(0,1,1)$ if and only if $k \leq n-1$ and so the Schützenberger graph, Γ, of S with respect to $T = \{x\}$ consists of all finite chains of the form

$$(m,m,n) \xrightarrow{x} (m,m+1,n) \xrightarrow{x} \quad \cdots \quad \xrightarrow{x} (m,n,n)$$

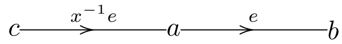
for all $m \leq 0 \leq n, m < n$.
Moreover, since $(m,k,n)\,\mathcal{L}\,(0, k-m, n-m)$ then on writing the $S-$orbit, $(m,k,n)S^1$ as $\overline{(m,k,n)}$, we see that the quotient graph, $S \setminus \Gamma$ consists of the finite chains

$$\overline{(0,0,n)} \longrightarrow \overline{(0,1,n)} \longrightarrow \quad \cdots \quad \longrightarrow \overline{(0,n,n)}$$

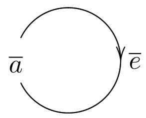
for $n \geq 1$.

Consider now the $3-$element set $V = \{a,b,c\}$ and define a partial action on V from the representation $S \to \mathcal{I}_V$ generated by $x \to \rho_x$ where $\rho_x =$

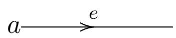
$\begin{pmatrix} a & c \\ b & a \end{pmatrix}$. Define an $S-$graph G, as follows



and note that the quotient graph, $S \setminus G$ is

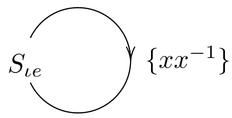


with a fundamental transversal Y



It is worth noting that the $S-$graph G is not a complete $S-$graph.

To construct the associated graph of inverse semigroups, notice that $\iota e = a, \tau e = b, \overline{\tau}(e) = a$, $S_e = \{xx^{-1}\}, S_{\iota e} = \{x^{-1}x, xx^{-1}, xx^{-1}x^{-1}x\} = S_{\overline{\tau}(e)}$, $S_{\tau e} = \{x^2x^{-2}, xx^{-1}\}$ and that xx^{-1} is the identity of $S_{\tau e}$. Hence the graph of inverse semigroups is given by

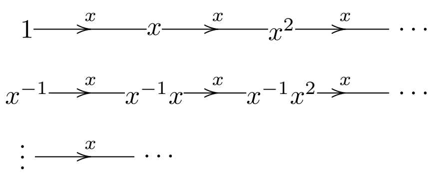


and there is an embedding $\{xx^{-1}\} \to S_{\iota e}$ given by $xx^{-1} \mapsto x^{-1}(xx^{-1})x = x^{-1}x$.
Notice that $x^{-1}x$ is *not* the identity in $S_{\overline{\tau}e}$.

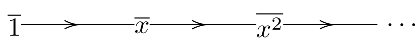
Example 4.2. Let S be the bicyclic semigroup with presentation given by

$$S = \mathrm{Inv}\langle x | xx^{-1}x^{-1}x = x^{-1}x \rangle.$$

The Schützenberger graph of S with respect to $T = \{x\}$ is



and the quotient graph $S \setminus \Gamma$ is



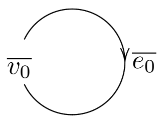
with corresponding graph of inverse semigroups

$$\{1\}\xrightarrow{\{1\}}\{1\}\xrightarrow{\{1\}}\{1\}\xrightarrow{\{1\}}\cdots$$

Consider now the sets $V = \{v_1, v_0, v_{-1}, v_{-2}, \ldots\}, E = \{e_0, e_{-1}, e_{-2}, \ldots\}$ and define partial actions on V and E from the representations $\rho : S \to \mathcal{I}_V, \varphi : S \to \mathcal{I}_E$ generated by $x \mapsto \rho_x, x \mapsto \varphi_x$ respectively, where $\rho_x = \begin{pmatrix} v_0 & v_{-1} & v_{-2} & \cdots \\ v_1 & v_0 & v_{-1} & \cdots \end{pmatrix}$ and $\varphi_x = \begin{pmatrix} e_{-1} & e_{-2} & \cdots \\ e_0 & e_{-1} & \cdots \end{pmatrix}$. This defines for us an $S-$graph $G = (G, V, E, \iota, \tau)$, with $\iota e_n = v_n, \tau e_n = v_{n+1}$.

$$\cdots\xrightarrow{x^{-2}e_0}v_{-1}\xrightarrow{x^{-1}e_0}v_0\xrightarrow{e_0}v_1$$

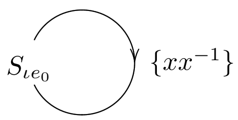
and note that the quotient graph, $S \setminus G$ is



with a fundamental transversal Y

$$v_0\xrightarrow{e_0}$$

To construct the associated graph of inverse semigroups, notice that $xx^{-1} = 1, \iota e_0 = v_0, \tau e_0 = v_1, \overline{\tau} e_0 = v_0$, $S_{e_0} = \{1\}, S_{\iota e_0} = \{1, x^{-1}x\} = S_{\overline{\tau} e_0}$, $S_{\tau e_0} = \{1\}$ and that xx^{-1} is the identity of $S_{\tau e_0}$. Hence the graph of inverse semigroups is given by



and there is an embedding $\{1\} \to S_{\iota e_0}$ given by $1 \mapsto x^{-1}(1)x = x^{-1}x$.

5. Summary

The contents of this paper are a start at developing a theory of (partial) actions of inverse semigroups on sets along the lines of the group theory approach. It is clear that much more work needs to be done on this and we hope to expand on this work in a future paper [5]. In particular we hope to address the question of a 'structure theorem' for partial actions on trees and consider examples of amalgamated free products of inverse semigroups.

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