Note

New upper bound formulas with parameters for Ramsey numbers

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Abstract

In this paper, we obtain some new results $R(5, 12) \leq 848$, $R(5, 14) \leq 1461$, etc., and we obtain new upper bound formulas for Ramsey numbers with parameters.

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For two given graphs $G_1$, $G_2$, the Ramsey number $R(G_1, G_2)$ is the smallest positive integer $p + 1$ such that for any graph $G$ of order $p + 1$ either $G$ contains $G_1$ or $G^c$ contains $G_2$, where $G^c$ is the complement of $G$. A graph $H$ of order $p$ is called a $(G_1, G_2; p)$-Ramsey graph if $H$ does not contain $G_1$ and $H^c$ does not contain $G_2$. Let $R(K_m, K_n) =: (m, n; p)$-Ramsey graph. When an edge $e$ is removed from $G$, we denote the graph by $G - e$. Let $d_i$ be the degree of vertex $i$ in $G$ of order $p$, and let $\bar{d}_i = p - 1 - d_i$, where $1 \leq i \leq p$. And let $f(K_r)$ ($g(K_r)$, resp.) denote the number of $K_r$ in $G$ ($G^c$, resp.).

In the following we always assume that $G_1 = K_m$ or $K_m - e$, $G_2 = K_n$ or $K_n - e$, $G^c_1 = K_{m-i}$ or $K_{m-i} - e$, $G^c_2 = K_{n-i}$ or $K_{n-i} - e$, $m \geq 4$ and $n \geq 4$.

Lemma 1 (Goodman [1]). For any graph $G$ of order $p$, we denote $i$ as its vertex, $d_i$ as the degree of vertex $i$, $\bar{d}_i = p - 1 - d_i$, $1 \leq i \leq p$. Then we have

$$f(K_3) + g(K_3) = \left( \frac{p}{3} \right) - \frac{1}{2} \sum_{i=1}^{p} d_i \bar{d}_i.$$ 

Lemma 2 (Huang and Zhang [3]). For any $(G_1, G_2; p)$-Ramsey graph $G$ of order $p$, the following inequalities
must hold:

\[(s + 1) f(K_{s+1}) \leq f(K_s)[R(G_{m-s}^n, G_2) - 1],\]
\[(t + 1)g(K_{t+1}) \leq g(K_t)[R(G_1, G_{n-t}) - 1].\]

**Theorem 1.** Let \(R(G_{m-2}^n, G_2) \leq \alpha + 1, R(G_1, G_{n-2}^2) \leq \beta + 1, R(G_{m-1}^1, G_2) \leq \gamma + 1, R(G_1, G_{n-1}^2) \leq \delta + 1\) and \(t > 0.\) Let \(A = 2\gamma - 2 - \frac{1}{3}(4\alpha + 2\beta), B = (\alpha + \beta + 2)^2 + \frac{1}{3}(\beta - \alpha)^2\) and \(F(t) = \alpha + \beta + 4 - t + \sqrt{\frac{4}{3}t^2 + 2At + B}.\) Then
\[R(G_1, G_2) \leq F(t).\]

In particular, when \(4B - 3A^2 > 0,\) and \(t_0 = \frac{3}{4}(\sqrt{4B - 3A^2} - A) > 0,\) then
\[R(G_1, G_2) \leq F(t_0).\]

**Proof.** Let \(p = R(G_1, G_2) - 1.\) For any \((G_1, G_2; p)\)-Ramsey graph \(G,\) by Lemma 2, we have \(3f(K_3) \leq \frac{1}{2}z\sum_{i=1}^{p} d_i^2 + 3g(K_3) \leq \frac{1}{2}p\sum_{i=1}^{p} \overline{d}_i.\) Combining these two inequalities and Lemma 1, then
\[p(p - 1)(p - 2 - z) \leq \sum_{i=1}^{p} (p - 1 - d_i)(3d_i + \beta - z) \leq \sum_{i=1}^{p} \{-3\overline{d}_i^2 + (3p - 3 + \beta - x + t)\overline{d}_i - t(p - 1) + t\gamma\} \leq \sum_{i=1}^{p} \left\{ \frac{1}{12}(3p - 3 + \beta - x + t)^2 - t(p - 1) + t\gamma \right\}.\]

Thus \(R(G_1, G_2) \leq F(t).\)

From the definition of \(F(t),\) when \(4B - 3A^2 > 0\) and \(t_0 = \frac{3}{4}(\sqrt{4B - 3A^2} - A) > 0,\) we have
\[F'(t_0) = 0, \quad F''(t_0) > 0.\]

Hence we have \(R(G_1, G_2) \leq F(t_0).\) The proof of theorem is completed. \(\square\)

Noting the symmetry of \(\gamma\) and \(\delta,\) we have the following corollary immediately.

**Corollary 1.** Under the assumption of Theorem 1, let \(t > 0, C = 2\delta - 2 - \frac{1}{3}(2\alpha + 4\beta), D = (\alpha + \beta + 2)^2 + \frac{1}{3}(\beta - \alpha)^2\) and \(G(t) = \alpha + \beta + 4 - t + \sqrt{\frac{4}{3}t^2 + 2Ct + D}.\) If \(4D - 3C^2 > 0\) and \(t^* = \frac{3}{4}(\sqrt{4D - 3C^2} - C) > 0,\) then we have:
\[R(G_1, G_2) \leq G(t^*).\]

Now we obtain another new upper bound formula with parameters \(x, y.\)

**Theorem 2.** Let \(m \geq 4, n \geq 4, R(m - 2, n) \leq \alpha + 1, R(m, n - 2) \leq \beta + 1\) and parameter \(x \in (0, 3).\) And let
\[f(x, y) = A + \sqrt{A^2 - B}, \quad g(x, y) = A - \sqrt{A^2 - B},\]
\[A = \frac{3(y + \alpha - \beta) - 2(1 + \alpha)\alpha}{9 - 4\alpha}, \quad B = \frac{(3 - x)(y + \alpha - \beta)^2 + xy^2}{(3 - x)(9 - 4x)}.\]

Then

(a) \(R(m, n) \geq 2 + f(x, y)\) or \(R(m, n) \leq 2 + g(x, y)\) if \(0 < x < \frac{9}{4};\)
(b) \(R(m, n) \leq 2 + f(x, y)\) if \(x \in \left(\frac{9}{4}, 3\right);\)
(c) \(R(m, n) \leq \alpha + \beta + 4 + \frac{2}{3}\sqrt{(\alpha + 2\beta + 3)(2\alpha + \beta + 3) + (\beta - \alpha)^2}\) if \(x = \frac{9}{4}.\)
Especially, when \( m = n \), we obtain K. Walker’s formula once again

\[ R(n, n) \leq 4R(n - 2, n) + 2. \]

**Proof.** Let \( p = R(m, n) - 1 \). Then by using the analogous arguments of Theorem 1, the following inequalities must hold in \((m, n; p)\)-Ramsey graph \( G \):

\[
p(p - 1)(p - 2 - x) \leq \sum_{i=1}^{p} \{-3d_i^2 + (3p - 3 + \beta - x)d_i\}
\]

\[
= \sum_{i=1}^{p} \{-x^2d_i^2 + (3p - 3 + \beta - x - y)d_i - (3 - x)d_i^2 + yd_i\}
\]

\[
\leq \frac{1}{4x} (3p - 3 + \beta - x - y)^2 p + \frac{y^2 p}{4(3 - x)}.
\]

Thus, we have \((9 - 4x)(3 - x)(p - 1)^2 - 2(3 - x)[3(y + x - \beta) - 2(1 + x)x](p - 1) + xy^2 + (3 - x)(y + x - \beta)^2 \geq 0\).

(1) When \( 0 < x < \frac{9}{4} \), (a) follows immediately.

(2) When \( \frac{9}{4} < x < 3 \), since \( B < 0 \), \( g(x, y) < 0 \). Note that in this case \((9 - 4x)(3 - x) < 0 \). Hence \( R(m, n) \leq 2 + f(x, y) \).

(3) When \( x = \frac{9}{4} \), we have

\[ R(m, n) \leq 2 + \frac{4y^2 - 2(\beta - x)y + (\beta - x)^2}{6y - 3x - 6\beta - 9} =: 2 + f(y). \]

It is easy to check that when \( y_0 = \frac{1}{2} \left( x + 2\beta + 3 + \sqrt{(x + 2\beta + 3)(2x + \beta + 3)} \right) \),

\[
\min f(y) = f(y_0) = x + \beta + 4 + \frac{2}{3} \sqrt{(x + 2\beta + 3)(2x + \beta + 3) + (\beta - x)^2}.
\]

Hence (c) follows. \( \square \)

It is not difficult to generalize the results to \( R(G_1, G_2) \) for \( G_1 = K_m \) or \( K_m - e \), and \( G_2 = K_n \) or \( K_n - e \). Hence using (c) of the generalized Theorem 2, taking \( x = 20, \beta = 35 \), we have \( R(K_6 - e, K_6) \leq 116 \) once more, which appears in [2].

Noting the symmetry of \( x \) and \( \beta \), we have the following corollary immediately.

**Corollary 2.** Under the hypotheses of Theorem 2, let \( F(x, y) = C + \sqrt{C^2 - D}, G(x, y) = C - \sqrt{C^2 - D}, C = (3(y + \beta - x) - 2(1 + \beta)x)/(9 - 4x) \) and \( D = (y + \beta - x)^2/(9 - 4x) + xy^2/((3 - x)(9 - 4x)) \). Thus we have:

(1) If \( 0 < x < \frac{9}{4} \), then \( R(m, n) \geq 2 + F(x, y) \) or \( R(m, n) \leq 2 + G(x, y) \).

(2) If \( \frac{9}{4} < x < 3 \), then \( R(m, n) \leq 2 + F(x, y) \).

Note that there is the well-known formula:

\[ R(m, n) \leq R(m - 1, n) + R(m, n - 1), \]  \hspace{1cm} (3)

and its generalized formula in [3]:

\[ R(G_1, G_2) \leq R(G_1^{m-1}, G_2) + R(G_1, G_2^{n-1}), \]  \hspace{1cm} (4)

where \( G_1 = K_m \) or \( K_m - e \), \( G_2 = K_n \) or \( K_n - e \).

Up to date upper and lower bounds on Ramsey numbers are listed in [5]. Using these tables and (3), (4), 18 new upper bounds of \( R(m, n) \) obtained by (1) are shown in Table 1, where \(-, -, -; -) = (x, \beta, \gamma; t_0)\), and the number with \( * \) is obtained by (3).
### Table 1

<table>
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<th>$n$</th>
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<th>$8$</th>
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<td>1804*</td>
<td>4553</td>
<td>(632, 1712, 10630)</td>
<td>(1803, 3582, 22325)</td>
<td>(1803, 1756.4, 10629, 1162.1)</td>
</tr>
<tr>
<td>12</td>
<td>848 (58, 441, 237; 735.3)</td>
<td>2566 (237, 1170, 847; 1679.1)</td>
<td>6954 (847, 2825, 2565; 3488.4)</td>
<td>16944 (2565, 6089, 10629, 16943, 6953, 5585.5)</td>
<td>39025 (6953, 12676, 16943, 9110.7)</td>
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<tr>
<td>13</td>
<td>1139*</td>
<td>3705*</td>
<td>10581 (1138, 4552, 3704; 4215.9)</td>
<td>27490 (3704, 10629, 10580, 27489, 4215.9)</td>
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<tr>
<td>14</td>
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<td>5033 (348, 2565, 1460, 1404, 6911; 4118.6)</td>
<td>15263 (1460, 6953, 5032, 10099.1)</td>
<td>41525 (5032, 16943, 15262, 21087.03)</td>
<td>89203 (15262, 39024, 41524, 41657.1)</td>
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<tr>
<td>15</td>
<td>1878*</td>
<td>6911*</td>
<td>22116 (1877, 10580, 6910; 16179.1)</td>
<td>63620 (6910, 27489, 22115, 45750.3)</td>
<td>12059.2</td>
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</table>

Remarks. (1) Theorems 1 and 2 can be generalized by using the ideas in [3,4]. (2) Taking $(\alpha, \beta, \gamma, t_0) = (33, 66, 87, 45.9)$, we have $R(K_6 - e, K_7) \leq 202$ once more, which appears in [2].

References