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The Generalized Poisson distribution and a model of clustering from Poisson initial conditions

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ABSTRACT
This paper presents a new derivation of the Generalized Poisson distribution. The derivation is based on the barrier crossing statistics of random walks associated with the Poisson distribution. A simple interpretation of this model in terms of a single server queue is also included.

In the astrophysical context, the Generalized Poisson distribution is interesting because it provides a good fit to the evolved, Eulerian counts-in-cells distribution measured in numerical simulations of hierarchical clustering from Poisson initial conditions. The new derivation presented here can be used to construct a useful analytic model of the evolution of clustering measured in these simulations. The model is consistent with the assumption that, as the universe expands and the comoving sizes of regions change as a result of gravitational instability, the number of such expanding and contracting regions is conserved. The model neglects the influence of external tides on the evolution of such regions. Indeed, in the context of this model, the Generalized Poisson distribution can be thought of as arising from a simple variant of the well-studied spherical collapse model, in which tidal effects are also neglected. This has the following implication: Insofar as the Generalized Poisson distribution derived from this model is a reasonable fit to the numerical simulation results, the counts-in-cells statistic must be relatively insensitive to such effects. This may be a consequence of the Poisson initial condition.

The model can be understood as a simple generalization of the excursion set model which has recently been used to estimate the number density of collapsed, virialized halos. The generalization developed here allows one to estimate the evolution of the spatial distribution of these halos, as well as their number density. For example, it provides a framework within which the halo–halo correlation functions, at any epoch, can be computed analytically. In the model, when halos first virialize, they are uncorrelated with each other. This is in good agreement with the simulations. Since it allows one to describe the spatial distribution of the halos and the mass simultaneously, the model allows one to estimate the extent to which these halos are biased tracers of the underlying matter distribution.

Key words: galaxies: clustering – cosmology: theory – dark matter.

1 INTRODUCTION

Consider an initially Poisson distribution of particles that clusters gravitationally as the universe expands. In this paper, the initial Poisson distribution will also be called the initial Lagrangian distribution. As time passes, the particle distribution evolves, as, for example, tightly bound virialized clusters (called halos, or dark matter halos, in this paper) form. Thus, the evolved distribution is different from the initial Lagrangian distribution. In what follows, the evolved distribution will be called the Eulerian distribution. The goal of this paper is to use the properties of the initial Lagrangian distribution to derive a reasonable approximation to the form of the evolved Eulerian distribution. In the absence of a model relating the two distributions, the only constraint is that required by mass conservation: the number of particles in the initial and evolved distributions is the same, so the average density, $\bar{n}$, in the two distributions must be the same. In what follows, quantities measured in the Lagrangian space will have a subscript '0', while those in Eulerian space will not. In this notation, mass conservation implies that $\bar{n}_0 = \bar{n}$.

Studies of clustering from Poisson initial conditions

(Itoh, Inagaki & Saslaw 1993 and references therein) show that when the initial, Lagrangian distribution is Poisson, then the evolved Eulerian distribution is Generalized Poisson. This paper presents a model in which this is so. The model is consistent with three general hypotheses about the evolution of clustering. The first is the hypothesis that, in comoving coordinates, initially denser regions contract more rapidly than less dense regions, and that sufficiently underdense regions expand. The second assumption is that, as the universe evolves, the number of such expanding and contracting regions is conserved—only their comoving size changes. The third is that the influence of external tides on the evolution of such comoving regions can be neglected, if one is only interested in computing statistics such as the mass function of collapsed halos, or the distribution of counts in Eulerian cells. There are no compelling physical arguments for any of these assumptions, and initial particle configurations which violate some or all of these assumptions are relatively easy to construct. That the model predicts a counts-in-cells distribution which is a reasonable approximation to that measured in the numerical simulations suggests that, at least for clustering from Poisson initial conditions, these simple assumptions may also be reasonably accurate.

1.1 The Generalized Poisson distribution

Since it plays a central role in this paper, various known properties of the Generalized Poisson distribution are summarized below.

The Generalized Poisson distribution (Consul 1989) has the form

$$p(N|V,b) = \frac{\bar{N}(1-b)}{N!}\left[\bar{N}(1-b)+Nb\right]^{N-1} \mathrm{e}^{-\bar{N}(1-b)-Nb}. \quad (1)$$

Here $p(N|V,b)$ is the probability that a cell of size V placed randomly within a particle distribution contains exactly N particles. If $\bar{n}$ denotes the average density, then $\bar{N}\equiv\bar{n}V$. In this paper $0\le b<1$, and, for reasons discussed below, it will be supposed that b is not a function of V. The case $b=0$ is the Poisson distribution.

Equation (1) is a Compound Poisson distribution (e.g. Saslaw 1989); it arises if point sized clusters, called halos in the following, have a Poisson spatial distribution, and the probability a randomly chosen halo contains exactly n particles is

$$\eta(n,b) = \frac{(nb)^{n-1}\,\mathrm{e}^{-nb}}{n!}. \quad (2)$$

This is the Borel(b) distribution (Borel 1942). In this paper, equation (2) will be called the halo mass function.

The Generalized Poisson distribution was first discovered in the astrophysical context by Saslaw & Hamilton (1984) (also see Sheth 1995a). It provides a good fit to the distribution of particle counts in randomly placed cells, provided the particle distributions have evolved, as a result of gravitational clustering, from an initially Poisson distribution (Itoh, Inagaki & Saslaw 1993 and references therein). In fact, the fits are significantly improved if b is allowed to increase to an asymptotic value as V increases. This scale dependence is simply a consequence of relaxing the assumption that Borel clusters are point sized, but still requiring that they have some finite size. The asymptotic value of b is that which would have characterized the distribution, had the clusters been point sized (Sheth & Saslaw 1994). For this reason, the asymptotic value of b is fundamental, and the point sized idealization useful. This paper is mainly concerned with the point sized idealization, so that, in what follows, b is independent of V.

The point sized idealization is also motivated by the following observation. To a good approximation, the distribution of bound virialized halos in the numerical simulations is Borel(b). Thus, to a good approximation, clustering from Poisson initial conditions evolves in such a way that, at all times, particles are bound up in Borel(b) halos, and, at the time when they first virialize, these halos have a Poisson distribution. The evolution of clustering is parameterized by the time dependence of b; it is zero initially, and it increases as the universe expands (Zhan 1989; Sheth 1995b and references therein). Therefore, in the remainder of this paper, b will be treated as a pseudo-time variable, and the Borel(b) distribution will often be called the halo mass function at the epoch b.

As $V\to 0$, most cells in the Lagrangian and Eulerian distributions will be empty. Equation (1) shows that, in this limit, the probability that a cell is not empty is $\bar{N}(1-b)$, and the probability that a non-empty cell contains exactly N particles is given by the Borel(b) distribution. In other words, at the epoch b, the halo mass function is the same as the vanishing-cell-size limit of the Eulerian counts in cells distribution (Sheth 1996a). This fact will be useful later.

The Borel(b) distribution can be derived from a number of different constructions, all of which are related to the Poisson distribution (Epstein 1983; Sheth 1995b; Sheth 1996b; Sheth & Pitman 1997). In the context of this paper, all these constructions can be thought of as providing models that allow one to compute the Eulerian space distribution, in the limit of vanishing cell size, given that the Lagrangian space distribution is Poisson. One of these constructions, based on the statistics of random walk barrier crossings associated with the Poisson distribution, is the excursion set model (Epstein 1983; Sheth 1995b). This paper shows how to derive the Generalized Poisson distribution from a simple generalization of this excursion set model. The generalization shows how to derive the Eulerian space Generalized Poisson distribution from the Lagrangian space Poisson distribution, for all cell sizes, and all times.

1.2 Outline of this paper

Section 2.1 summarizes the random walk, excursion set model which leads to the Borel(b) distribution. Sections 2.2 and 2.3 describe a generalization of this model which leads to a new derivation of the Generalized Poisson distribution. Section 2.4 shows how to describe the spatial distribution of virialized halos within the context of this model. It shows that the model is consistent with the Compound Poisson interpretation of the Generalized Poisson distribution – in the model, Borel(b) halos have a Poisson distribution at the time when they first virialize. Moreover, in the model, the $V\to 0$ limit of the counts-in-cells distribution is, indeed, the halo mass function. This shows explicitly that the excursion set approach developed here is able to reproduce the known properties of the Generalized Poisson distribution. The relation between this model and the well-studied spherical collapse model (outlined in Appendix A) is discussed in Section 2.5.

Section 3 contains a brief digression which relates the excursion set model of the previous section to a simple single server queue system. Section 4 discusses a scaling solution associated with the model that is analogous to the scaling solution found in Section 3.2 of Sheth (1995b). Section 5 discusses the associated two barrier problem. The solution of this problem may provide useful diagnostics in assessing the rate of evolution of the Eulerian statistics computed earlier in the paper.

Clustering from more general initial conditions, using the techniques developed here, is treated in a forthcoming paper.

2 POISSON INITIAL CONDITIONS AND THE GENERALIZED POISSON DISTRIBUTION

This section presents a new derivation of the Generalized Poisson distribution. The derivation uses a simple generalization of the excursion set model studied by Epstein (1983) and Sheth (1995b).

2.1 The excursion set with constant barrier

Suppose that the initial Lagrangian distribution is Poisson, with mean density $\bar{n}$. This means that a volume of size V_0 placed at a random position in this distribution will contain exactly N particles with probability

$$p(N|V_0) = \frac{\bar{N}_0^N\,\mathrm{e}^{-\bar{N}_0}}{N!}, \qquad \text{where } \bar{N}_0 = \bar{n}V_0. \tag{3}$$

Now choose a random particle of this distribution, and compute the density within concentric spheres centred on this position. Call the curve $N(V_0)$ traced out by the number of particles contained within a sphere V_0 centred on this particle, as a function of the sphere size V_0, a trajectory. Then each particle in the Poisson distribution has its associated Lagrangian trajectory. Given δ_{c0}, Epstein (1983) derived an expression for the fraction of Lagrangian trajectories for which $N(V_0) = \bar{n}V_0(1+\delta_{c0})$, and for which $N(V_0') < \bar{n}V_0'(1+\delta_{c0})$ for all $V_0' > V_0$ (also see Sheth 1995b; Sheth & Lemson 1998).

Epstein argued that a given value of $\delta_{c0} \geq 0$ defines a series of volumes $V_0(1) < V_0(2) < \ldots$ for which

$$j/V_0(j) = \bar{n}(1+\delta_{c0}) \equiv \bar{n}/b, \qquad \text{where } 0 \leq b < 1. \tag{4}$$

The final equality defines $b = 1/(1+\delta_{c0})$. The evolution of clustering enters through the time dependence of δ_{c0}, which decreases as time increases. It is in this sense that b is a pseudo-time variable; it is 0 initially, and increases as the universe expands (Sheth 1995b).

Epstein showed that the fraction of trajectories $f_c(j, b_1)$ for which $V_0(j)$ is the largest value of V_0 at which $N(V_0) = \bar{N}_0/b_1$ is

$$f_c(j, b_1) = (1-b_1)\,\frac{(jb_1)^{j-1}\,\mathrm{e}^{-jb_1}}{(j-1)!} = j\,(1-b_1)\,\eta(j, b_1), \tag{5}$$

where $\eta(j, b)$ is the Borel(b) distribution defined earlier. The mean of the Borel(b) distribution is $\sum j\,\eta(j, b) = (1-b)^{-1}$, a fact which will be useful later.

Since $f_c(j, b_1)$ is a statement about the last crossing of the barrier (the dashed line that intersects the origin in Fig. 1) by the random walk, excursion set trajectories, it will sometimes be referred to as the barrier crossing distribution. The subscript 'c' here denotes the fact that this is the distribution of last crossings of a constant boundary; that is, δ_{c0} is independent of V_0.

Let $f_c(j, b_1|N, b_2)$ denote the fraction of trajectories for which $V_0(j)$ is the largest value of V_0 at which $N(V_0) \geq \bar{N}_0/b_1$, given that, at some $V_0' \equiv V_0(N) > V_0(j)$ they have value $N(V_0') = \bar{N}_0/b_2$, with $b_2 \geq b_1$, and that all $V_0 > V_0(N)$ are less dense than $V_0(N)$. Then

$$f_c(j, b_1|N, b_2) = N\left(1-\frac{b_1}{b_2}\right)\binom{N}{j}\frac{j^j}{N^N}\left(\frac{b_1}{b_2}\right)^{j-1} \times \left(N - \frac{jb_1}{b_2}\right)^{N-j-1} \tag{6}$$

(Sheth 1995b).

It is usual to associate these expressions about the statistics of trajectories crossing a constant barrier with statements about the number density of collapsed (point–sized) halos. Thus, the average number density of b_1-halos that contain exactly j particles is

$$\bar{n}_0(j, b_1) = \bar{n}_0\,\frac{f_c(j, b_1)}{j} = \bar{n}(1-b_1)\,\eta(j, b_1). \tag{7}$$

This assignment comes from assuming that the fraction of trajectories $f_c(j, b_1)$ can be equated to the fraction of the Lagrangian space associated with regions containing j particles with overdensity b_1. The final equality comes from equation (5) and using the fact that $\bar{n}_0 \equiv \bar{n}$.

Similarly, the average number of (j, b_1)-halos that are within an (N, b_2)-halo is

$$\mathcal{N}(j, b_1|N, b_2) = \binom{N}{j}\,f_c(j, b_1|N, b_2), \tag{8}$$

where $N \geq j$ and $b_2 \geq b_1$.

If the trajectories are not centred on particles, then the barrier crossing distribution is

$$F_c(j, b_1) = b_1\;f_c(j, b_1), \tag{9}$$

and

$$F_c(j, b_1|N, b_2) = (b_1/b_2)\;f_c(j, b_1|N, b_2). \tag{10}$$

However, the number density of associated regions is the same as before (Sheth & Lemson 1998).

Consider an (N, b_2)-halo that is known to contain m b_1 subhalos, of which n_1 are singles, n_2 are doubles and so on. Thus, $\sum_{j=1}^N n_j = m$, and mass conservation requires that $\sum_{j=1}^N j\,n_j = N$. Of course, $b_1 \leq b_2$. Let $\pi[\boldsymbol{n}|N]$ denote one particular partition of N, where $\boldsymbol{n}$ denotes the vector $(n_1, \cdots, n_N)$, and let $p(\boldsymbol{n}; b_1|N; b_2)$ denote the probability that the partition $\pi[\boldsymbol{n}|N]$ occured. Sheth (1996b) shows that

$$p(\boldsymbol{n}; b_1|N; b_2) = \frac{(Nb_{21})^{m-1}\mathrm{e}^{-Nb_{21}}}{\eta(N, b_2)}\prod_{j=1}^{N}\frac{\eta(j, b_1)^{n_j}}{n_j!}, \tag{11}$$

where $b_{21} = (b_2 - b_1)$, and $Nb_2 = \bar{n}V_0(N)$, is consistent with the excursion set model described above (also see Sheth & Pitman 1997, Sheth & Lemson 1998).

For example, the average number of (j, b_1)-halos that are within (N, b_2)-halos is

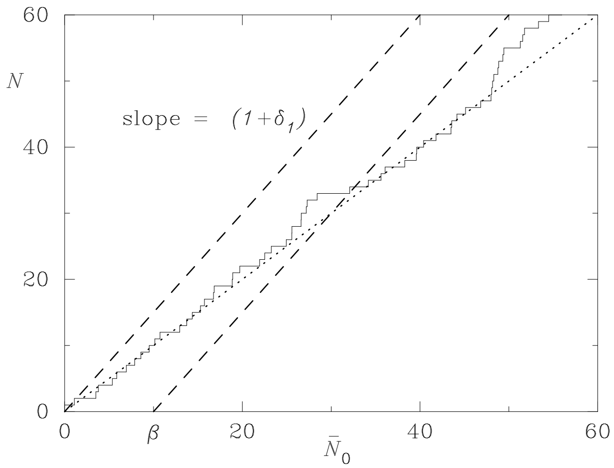

Figure 1. Example of the trajectory (solid line) traced by the number, N, of Poisson–distributed objects within a sphere which contains $\bar{N}_0$ objects on average. The dotted line, which has unit slope, shows the curve traced out by the average density. The dashed line which starts at the origin shows the barrier considered by Epstein (1983) and Sheth (1995b). The dashed line which starts at $\bar{N}_0 = \beta$ is the shifted barrier studied here.

$$\big\langle n_j; b_1|N; b_2\big\rangle = \sum_{\pi[\boldsymbol{n}|N]} n_j\, p(\boldsymbol{n}; b_1|N; b_2) = \mathcal{N}(j, b_1|N, b_2), \quad (12)$$

and the sum is over the set of all distinct ordered partitions of N. The final expression is the same as equation (8) as required; the algebra leading to it is given in Appendix B of Sheth (1996b).

The correlation between (i, b_1)- and (j, b_1)-halos that are within the same (N, b_2)-halo is computed by a similar average over all partitions of N. Thus,

$$\big\langle n_i\, n_j; b_1|N; b_2\big\rangle = \mathcal{N}(j, b_1|N, b_2)\; \mathcal{N}(i, b_1|N - j, b'), \quad (13)$$

where

$$(N - j)b' = Nb_2 - jb_1$$

(Sheth 1996b). This expression reflects the fact that, in the Lagrangian Poisson distribution, non-overlapping volumes are mutually independent (Sheth & Lemson 1998). These expressions will be useful later.

This subsection shows how the statistics of the initial Lagrangian distribution can be used to derive some useful information about the statistics of collapsed halos, and of the distribution of halos within halos. While the language of halos is useful, it is important to remember that an (N, b_2)-halo can also be thought of as a Lagrangian region of size $V_0(N) = Nb_2/\bar{n}$ (Mo & White 1996). Thus, expressions like (13) are related to the average number of (i, b_1)- and (j, b_1)-halos that are both within the same Lagrangian region of size $V_0(N)$. It is in this sense that many of the expressions above will be interpretted later in this paper.

2.2 The excursion set with shifted barrier

The previous subsection considered the distribution of last crossings, by random walk trajectories associated with the Poisson distribution, of a barrier which had shape $\bar{n}V_0(j) = jb$. Instead, suppose that

$$\bar{n}V_0(j) \equiv \beta + jb, \qquad \text{with } 0 \le b < 1, \quad (14)$$

and we seek an expression for the fraction of trajectories $f(j, b, \beta)$ for which $V_0(j)$ is the largest value of V_0 at which $N(V_0) = (\bar{N}_0 - \beta)/b$. This is equivalent to considering the same problem as before, but with the barrier shifted to the right by a constant β (see Fig. 1).

To compute $f(j, b, \beta)$, start with an arbitrarily small sphere centred on a particle. Since the distribution is Poisson, counts in different volumes are independent, so

$$f(j, b, \beta) = p\Big(j - 1|V_0(j)\Big)\, f^{\rm E}(j, b, \beta), \quad (15)$$

where $p(j-1|V_0)$ is the Poisson distribution of equation (3), with $V_0(j)$ given by equation (14), and $f^{\rm E}(j, b, \beta)$ denotes the probability that no sphere larger than and concentric to $V_0(j)$ is denser than the threshold value. Now, $f^{\rm E}(j, b, \beta)$ is the same as one minus the probability that $V_0(N)$ is the largest volume denser than the threshold value, summed over all $N \ge j$. As a consequence of the Poisson assumption, $f^{\rm E}(N, b, \beta)$ is independent of N, so it can be written as $f^{\rm E}(b, \beta)$ (e.g. Epstein 1983). This means that

$$1 - f^{\rm E}(b, \beta) = f^{\rm E}(b, \beta) \sum_{N=j}^{\infty} p\Big(N - j|V_0(N) - V_0(j)\Big). \quad (16)$$

Since $\bar{n}[V_0(N) - V_0(j)] = (N - j)b$, setting $m = (N - j)$ means that the sum above is simply

$$\sum_{m=0}^{\infty} \frac{(mb)^m\, {\rm e}^{-mb}}{m!} = \frac{b}{1-b}. \quad (17)$$

The final expression follows from recognizing that the term in the sum is (mb) times the Borel(b) distribution. This implies that

$$f^{\rm E}(j, b, \beta) = f^{\rm E}(b, \beta) = (1 - b), \quad (18)$$

so that

$$f(N, b, \beta) = (1 - b)\, \frac{(\beta + Nb)^{N-1}}{(N-1)!}\, {\rm e}^{-\beta - Nb}. \quad (19)$$

This is the barrier crossing distribution associated with the shifted barrier. Following equation (7), this barrier crossing distribution can be associated with a number density of Lagrangian regions which contain N particles:

$$\bar{n}(N, b, \beta) = \bar{n}\, \frac{f(N, b, \beta)}{N}. \quad (20)$$

Let $F(N, b, \beta)$ denote the barrier crossing distribution if the trajectory is centred on a random position, not necessarily on a particle. Then

$$F(N, b, \beta) = (1 - b)\, p\Big(N|V_0(N)\Big) = \frac{\bar{n}V_0(N)}{N}\, f(N, b, \beta). \quad (21)$$

The number density of associated regions, $\bar{n}(N, b, \beta)$, is related to this fraction analogously to how it is related to $f(N, b, \beta)$. Namely, the barrier crossing distribution should be weighted by the number of trajectories associated with it, so $\bar{n}(N, b, \beta)$ is $F(N, b, \beta)$ times the ratio of the average density $\bar{n}$ to $\bar{n}V_0(N)$, so it is given by equation (20).

2.3 Statistics in the Eulerian space

Imagine partitioning the Eulerian space $V_{\rm tot}$ containing $N_{\rm tot}$ particles into a large number of cells, each of size V. Then the total number of such cells is $V_{\rm tot}/V$. We will be interested in the limit in which $N_{\rm tot}/V_{\rm tot}\to\bar n$ as both $N_{\rm tot}\to\infty$ and $V_{\rm tot}\to\infty$. Let $p(N|V)$ denote the fraction of these cells that contain exactly N particles. If $n(N|V)$ denotes the number of cells containing exactly N particles, then

$$p(N|V)\equiv\frac{n(N|V)}{N_{\rm tot}}=\frac{V\,n(N|V)}{V_{\rm tot}} \qquad (22)$$

is said to be the Eulerian counts-in-cells distribution.

Suppose that $\bar n$ and $V_{\rm tot}$ in the Lagrangian and Eulerian spaces are the same. Then $\bar n_0=\bar n$. Further, suppose that the number of regions which contain a specified number of particles is the same in both the Lagrangian and the Eulerian spaces. Finally, suppose that the parameter β, which controlled the shape of the barrier of the previous subsection, can be related to the Eulerian cell size V. Then equation (20) requires that

$$\frac{V\,n(N|V)}{V_{\rm tot}}\equiv V\,\bar n(N|V)=\frac{\bar nV\,f(N|V)}{N}, \qquad (23)$$

so that

$$p(N|V)=\frac{f(N|V)}{N/\bar N},\qquad \text{where } \bar N\equiv\bar nV. \qquad (24)$$

Equation (24) provides a relation between the barrier crossing distribution $f(N|V)$, which itself depends on the shape of the boundary and the initial Lagrangian field, and the Eulerian counts-in-cells distribution. By hypothesis, the shape of the boundary depends on the Eulerian scale V, so a given relation between β and V implies a specific form for the evolution of the comoving sizes of regions. This is discussed in more detail in Section 2.5. Physically, equation (24) is consistent with the assumption that the difference between the particle distribution in the initial (Lagrangian) and final (Eulerian) spaces arises solely as a consequence of the fact that, although the comoving sizes of regions may change, the number of expanding and contracting regions in the two spaces is conserved.

Let $\Delta\equiv(1+\delta)\equiv N/\bar N$. Then, $p(\Delta|V)$ is the probability distribution function of the density in Eulerian space, and

$$\int_0^\infty p(\Delta|V)\,{\rm d}\Delta=\int_0^\infty\Delta\,p(\Delta|V)\,{\rm d}\Delta=1. \qquad (25)$$

Following equation (24), equation (19) has an associated Eulerian counts in cells distribution:

$$\begin{aligned}p(N|V,b,\beta)\;&\equiv\;\frac{f(N,b,\beta)}{N/\bar N}\\ &=\;\frac{\bar N(1-b)}{N!}\,(\beta+Nb)^{N-1}\,{\rm e}^{-\beta-Nb},\end{aligned} \qquad (26)$$

where $\bar N\equiv\bar nV$. This is the Generalized Poisson distribution (equation 1). Normalization to unity requires that

$$\beta=\bar N(1-b),\qquad \text{where } \bar N\equiv\bar nV, \qquad (27)$$

so the variance is $\bar N/(1-b)^2$, and this distribution is the same as that in equation (1).

Equation (27) shows how the parameter β is related to the Eulerian cell size V. In particular, notice that as $V\to 0$, then the barrier is shifted by $\beta\to0$, so, in this limit, the barrier shape is the same as that in Section 2.1. This shows explicitly that, as $V\to0$, the Eulerian counts in cells distribution is the same as the halo mass function.

2.4 The halo distribution

Recall that the Generalized Poisson distribution with parameter b can be interpretted as arising from a Poisson distribution of Borel(b) halos. This subsection shows that the derivation of the Generalized Poisson distribution presented earlier is consistent with this interpretation.

Fig. 1 shows that the fraction of trajectories which last cross the constant barrier, parameterized by b_1, at j is equal to the fraction of those trajectories which last crossed the shifted barrier (associated with the Eulerian scale V and parameter $b\ge b_1$) at $N\ge j$, that had their last crossing of the constant barrier at j, summed over all $N\ge j$. A little algebra shows that

$$f_{\rm c}(j,b_1)=\sum_{N=j}^{\infty}f_{\rm c}(j,b_1|N,b_2)\,f(N,b,\beta),\quad \text{with } b\ge b_1, \qquad (28)$$

where $f_{\rm c}(j,b_1)$ and $f_{\rm c}(j|N)$ are given by equations (5) and (6), and $b_2=\bar N_0/N=(\beta+Nb)/N$, as required by equation (14). This relation implies that

$$\begin{aligned}\bar n(j,b_1)V\;&=\;\sum_{N\ge j}^{\infty}\mathcal{N}(j,b_1|N,b_2)\;p(N|V,b,\beta)\\ &=\;\bar n_0(j,b_1)V,\end{aligned} \qquad (29)$$

where the final expression is V times equation (7), and follows from setting $b_2=(\beta+Nb)/N$.

There are two reasons for writing this calculation out in detail. The first is simply to show explicitly that the mean number density of (j,b_1)-halos in the Eulerian space is the same as in the Lagrangian space, as required. The second is that it shows how statistics in the Lagrangian space can be used to compute statistics in the Eulerian space. Recall that the number of regions containing N particles is the same in both spaces, though the sizes V_0 and V of the regions may be different. In particular, the Lagrangian scale associated with an Eulerian region of size V depends on the number of particles N within it: $V_0(N)=(\beta+Nb)/\bar n$ (equation 14). So, to compute averages over Eulerian cells V, one simply needs to sum over the relevant Lagrangian regions $V_0(N)$ that now have Eulerian scale V, and weight by the Eulerian probability $p(N|V)$ that the Eulerian region V contains N particles.

Thus, the cross-correlation between halos and mass, averaged over Eulerian cells V, can be computed as follows. Define

$$\delta_{\rm h}(j,b_1|N,b_2)=\frac{\mathcal{N}(j,b_1|N,b_2)}{\bar n_0(j,b_2)V}-1. \qquad (30)$$

This is the number of (j,b_1)-halos that are within Lagrangian regions of size $V_0(N)=Nb_2/\bar n$ and which contain exactly N particles, divided by the average number of (j,b_1)-halos that are within Eulerian volumes of size V, minus one. By hypothesis, the number of such Lagrangian regions is constant, only their size has changed. However, if we now require that $b_2=(\beta+Nb)/N$, then this means that the Eulerian size of such a Lagrangian region is V. So, if we require that $b_2=(\beta+Nb)/N$, then equation (30) is the number of (j,b_1)-halos in Eulerian cells V that contain exactly N particles, relative to the average number of (j,b_1)-halos in Eulerian cells V, minus one.

The cross correlation function between (j,b_1)-halos and mass, averaged over all Eulerian cells V, is $\delta_{\rm h}(j,b_1|N,b_2)$, with $Nb_2=\beta+Nb$, times $\delta\equiv(N-\bar N)/\bar N$ times the probability that an Eulerian region V contains exactly N particles, summed over all N. Thus,

$$\begin{aligned}\bar\xi_{\rm hm}(j,b_1,\beta) &\equiv \Big\langle \delta_{\rm h}(j,b_1|N,b_2)\,\delta\Big\rangle \\ &= \sum_{N=j}^{\infty}\left(\frac{N}{\bar N}-1\right)\frac{\mathcal{N}(j,b_1|N,b_2)}{\bar n(j,b_1)V}\,p(N|V,b,\beta) \\ &= \sum_{N=j}^{\infty}\left(\frac{N}{\bar N}-1\right)\frac{f_{\rm c}(j,b_1|N,b_2)}{f_{\rm c}(j,b_1)}\,f(N,b,\beta) \\ &= \frac{j}{\bar N}\left(\frac{1-b_1}{1-b}\right)+\frac{(b-b_1)}{\bar N(1-b)^2(1-b_1)}, \qquad (31)\end{aligned}$$

where $\mathcal{N}(j|N)$ is given by equation (8), $\bar n(j,b_1)$ by equation (29), and $p(N|V,b,\beta)$ by equation (26). The second line follows from equation (30), and the fact that $\langle\Delta\rangle=\langle1+\delta\rangle=1$ (equation 25), so $\langle\delta\rangle=0$. The third line follows from equations (7), (8) and (24), and the last line from doing the sum, after using the fact that $Nb_2=(\beta+Nb)$. Notice that when $b=b_1$ then $\bar\xi_{\rm hm}=j/\bar N$.

The correlation between b_1-halos of mass i and j, averaged over Eulerian cells V, arises as a result of two averages. The first is over all possible ways the N particles in an Eulerian cell V could have been partitioned into b_1-halos, and the second is over all possible values of N. The assumption that an Eulerian cell V is simply a Lagrangian region that has changed size allows us to assume that the first average (over all partitions of N) is the same in the two spaces. So, the result of this average is $\langle n_i\,n_j;b_1|N;b_2\rangle$ of equation (13), provided we set $b_2=(\beta+Nb)/N$. All that remains is to average this quantity over N, and then divide out the factors expected on average. Thus,

$$\bar\xi_{\rm hh}(i,j,b_1|V)=\sum_{N=i+j}^{\infty}\frac{\big\langle n_i\,n_j;b_1|N;b_2\big\rangle}{\bar n(i,b_1)V\,\bar n(j,b_1)V}\,p(N|V,b,\beta)-1, \qquad (32)$$

where

$$Nb_2=\beta+Nb, \qquad \text{and} \quad (N-j)b'=Nb_2-jb_1.$$

This sum can be solved analytically:

$$\bar\xi_{\rm hh}(i,j,b_1|V)=\frac{(i+j)(b-b_1)}{\bar N(1-b)}+\frac{(b-b_1)^2}{\bar N(1-b)^2(1-b_1)^2}. \qquad (33)$$

When $b=b_1$, $\bar\xi_{\rm hh}=0$, for all i,j, and V, which implies that the halos have a Poisson distribution. This is consistent with the fact that equation (26) is a Compound Poisson distribution which arises if Borel(b) clusters have a Poisson spatial distribution (Saslaw 1989; Sheth & Saslaw 1994). When $b_1=0$, then all halos are single particles, so $i=j=1$, and this expression gives the second factorial moment of the single particle distribution. Simple algebra shows that, in this limit, it is equal to the second factorial moment of equation (26). Further, notice that when $b_1\le b$, then correlations between halos only depend on the sum of the halo masses, not on the masses of the individual halos themselves. This suggests that, in this model, halo-halo correlations arise because of volume exclusion effects only. That is, halo-halo correlations arise only because, initially, a (j,b_1)-halo occupies a region $V_0(j)=jb_1/\bar n$, so no other halos can occupy this region. As time passes, such an object collapses to a region of zero Eulerian size, so the volume excluded by it becomes negligibly small.

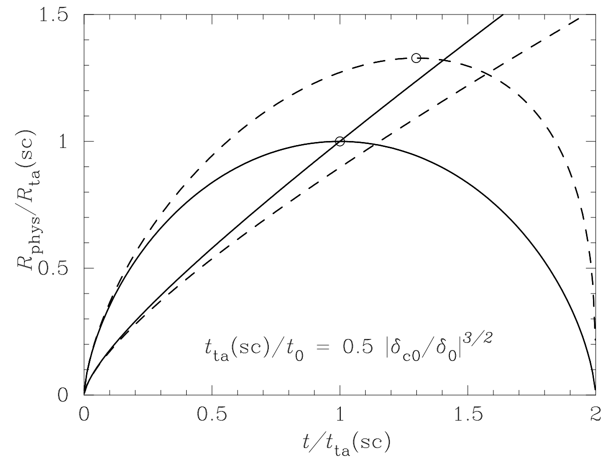

Figure 2. The physical radius of a perturbation in units of the spherical model turnaround radius, as a function of time in units of the spherical model turnaround time. The solid curves show the spherical model, and dashed curves show the model developed here. The two curves for each line type are for denser perturbations (which recollapse) and under-dense perturbations (which do not).

2.5 Relation to the spherical collapse model

In the spherical Poisson model, b is related to the critical overdensity required for collapse: $b=1/(1+\delta_{\rm c0})$ (equation 4). This relation for b, with equation (27) for β and equation (14), imply that, as N changes, the height relative to the average density of the shifted barrier considered here is

$$\delta_0(N|V)\equiv\frac{N}{\bar N_0}-1=\frac{(\bar N_0-\beta)}{\bar N_0 b}-1=\delta_{\rm c0}-\frac{\bar N}{\bar N_0}\frac{(1-b)}{b}. \qquad (34)$$

When $b\to1$, and $N\gg1$, then $\delta_{\rm c0}\ll1$ and $\bar N/\bar N_0\to\bar N/N$, so

$$\delta_0\to\delta_{\rm c0}-\frac{\delta_{\rm c0}}{1+\delta}. \qquad (35)$$

In this limit, the relation between δ_0 and δ is independent of V. This relation should be compared with that for the spherical collapse model given in Appendix A.

In the limits $\delta_{\rm c0}\ll1$ and $N\gg1$, equation (35) suggests the following model for the collapse of objects. Let $R(z)$ denote the comoving size of an object at the epoch z. Then $R(z)=R_0$ initially. If the object is in an underdense region, then its comoving size will increase, else it will decrease. Trajectories with extrapolated linear overdensity δ_0 greater than $\delta_{\rm c0}$ are associated with collapsed objects. Collapsed objects have $R(z)=0$. Until collapse

$$\frac{V(z)}{V_0}=\frac{R^3(z)}{R_0^3}=1-\frac{\delta_0/(1+z)}{\delta_{\rm c0}}. \qquad (36)$$

The radius of an object in proper, physical coordinates is $R_{\rm p}(z)=R(z)/(1+z)$. Objects which collapse have a turnaround radius–the maximum value that $R_{\rm p}(z)$ attains. This occurs at

$$(1+z_{\rm ta}) = \frac{4}{3}\frac{\delta_0}{\delta_{\rm c0}}, \tag{37}$$

at which time

$$\frac{R(z_{\rm ta})}{R_0} = \frac{1}{4^{1/3}}. \tag{38}$$

Figure 2 shows that, for overdense perturbations, turn-around in this model (dashed lines) occurs later, and at a larger radius, than in the spherical model (solid lines). In contrast, underdense regions expand less rapidly in this model than in the spherical model.

3 THE ASSOCIATED QUEUE

The excursion set problem studied above can be understood in terms of a simple queue system. A single server queue with deterministic service time $0 \le b \le 1$ and Poisson arrivals with unit mean, which starts with one customer at the initial time, can be expressed in terms of the random walk problem studied by Epstein (1983) and Sheth (1995b). The time parameter in the queue system is like the cell size parameter in the excursion set model.

Consider the probability $B(j,b)$ that, after exactly j customers have been served, the queue is empty for the first time. Then $f(j,b)$ of equation (5) is j times this probability times $(1-b)$. The $(1-b)$ factor simply comes from the additional constraint in the excursion problem on the number of particles within volumes larger than the critical volume $V_0(j)$. This constraint is not present in the queue model, since we have made no assumption about what happens in the queue after the end of the first busy period.

Figure 1 shows clearly that the same queue system, started with m customers at the initial time, is related to the excursion set problem studied in the previous section. Let $B(N,b|m)$ denote the probability that exactly N customers were served in the first busy period, given that there were exactly m customers in the queue initially. Tanner (1953, 1961) shows that

$$B(N,b|m) = \frac{m}{N}\,\frac{(Nb)^{N-m}\,{\rm e}^{-Nb}}{(N-m)!}, \tag{39}$$

and he also discusses the origin of the (m/N) term. Notice that $B(N,b|1)$ is the Borel(b) distribution.

The excursion set probability $f(N,b,\beta)$ of equation (19) is the same as the probability that there were exactly m customers at the start, times $(N/m)\,B(N,b|m)$, times $(1-b)$, summed over all possible values of m. In the excursion set problem, the probability that there are exactly m customers at the start is simply

$$p(m,\beta) = \frac{\beta^{m-1}\,{\rm e}^{-\beta}}{(m-1)!} \tag{40}$$

(Figure 1), so that

$$f(N,b,\beta) = \sum_{m=0}^{N} (1-b)\,(N/m)\,B(N,b|m)\,p(m,\beta). \tag{41}$$

Appendix D of Sheth (1996b) shows that the right hand side of this expression is the same as that on the right hand side equation (19). The corresponding expression for trajectories not necessarily centred on a particle, $F(N,b,\beta)$, can also be derived in this context. Simply set $p(m,\beta)$ to (β/m) times the expression above, and do the sum. Sheth (1996b) discusses a branching process derivation of this formula. Thus, this section shows how that branching process, this queue model, and the excursion set model of the previous section are all interrelated.

4 A SCALING SOLUTION

This subsection extends the results of Sheth (1995b), Section 3, for a constant barrier to the shifted barrier considered in this paper. In particular, it shows that the shifted barrier problem has a scaling solution that is analogous to the one associated with the constant barrier.

Suppose that the underlying distribution is not Poisson, but is Compound Poisson. This means that equation (3) must be replaced with $P_{\rm CP}[n|V_0(n)]$.

Then

$$f_{\rm CP}(n,\delta) = f^{\rm I}_{\rm CP}(n,\delta)\; f^{\rm E}_{\rm CP}(n,\delta), \tag{42}$$

where the first term is the probability that there are exactly n particles within $V_0(n)$, and the second is the probability that all volumes larger than $V_0(n)$ are less dense than it. The second term is obtained by noting that the argument leading to equation (18) holds here also. Therefore, when $P_{\rm CP}(n|V_0,b_0)$ is the Generalized Poisson distribution with parameter b_0, and $V_0(n)$ is given by equation (14), i.e.,

$$\bar nV_0(n) \equiv \beta + nb = \bar N(1-b) + nb \equiv \bar N_n,$$

then $f^{\rm E}_{\rm GPD}$ is given by an expression like (16) but with p replaced by $P_{\rm CP}$. Since $\bar n[V_0(n)-V_0(j)] = (n-j)b$, setting $m\equiv n-j$ and

$$B = b_0 + b\,(1-b_0) \tag{43}$$

means that

$$\frac{1}{f^{\rm E}_{\rm GPD}} = 1 + \sum_{m=0}^{\infty} mb(1-b_0)\,\frac{(mB)^{m-1}{\rm e}^{-mB}}{m!}. \tag{44}$$

This sum is similar to that in equation (17). Thus,

$$f^{\rm E}_{\rm GPD} = \left(1+\frac{b(1-b_0)}{1-B}\right)^{-1} = 1-b. \tag{45}$$

The other term is slightly more complicated. Recall that the Generalized Poisson distribution with parameter b_0 can be understood as describing a Poisson distribution of point–sized clusters, where $\eta(m,b_0)$, the probability that a randomly chosen cluster contains exactly m particles is the Borel(b_0) distribution. Since the mean of the Borel(b_0) distribution is $(1-b_0)^{-1}$, the ratio of the density of cluster centres to that of particles is $\bar n_{\rm clus}/\bar n = (1-b_0)$, and

$$\begin{aligned} f^{\rm I}_{\rm GPD}(n) \;&=\; \sum_{m=1}^{n} \frac{\bar n_{\rm clus}}{\bar n}\, m\,\eta(m,b_0) \\ &\qquad \times\; P_{\rm GPD}\Big(n-m\Big|V_0(n),b_0\Big) \end{aligned} \tag{46}$$

(Sheth 1995b equation 26). Abel's generalization of the Binomial formula (e.g. Sheth 1995b equation 30) reduces this to

$$f_{\rm GPD}(n) = \frac{(1-B)}{(n-1)!}\,\Big[\theta+nB\Big]^{n-1}\,{\rm e}^{-\theta-nB}, \tag{47}$$

where

$$1 - B \equiv (1 - b)(1 - b_0) \qquad \text{and} \qquad \theta \equiv \bar{N}(1 - B).$$

This has the same form as equation (19). Notice that when $b_0 = 0$, then $B = b$, and this expression is exactly the same as equation (19). This is sensible, since a Generalized Poisson distribution with parameter $b_0 = 0$ is just a Poisson distribution.

Since b is a pseudo-time variable, the new definition of B simply means that the time parameter in this model is slightly different from that in the case of Poisson initial conditions. In other words, if the initial Lagrangian distribution is Generalized Poisson, rather than Poisson, then, except for the appropriate rescalings of the time-dependent parameters, none of the results of Section 2 are changed.

5 THE TWO BARRIER PROBLEM

Suppose $b_1 \le b_2$. Let $f(j, b_1|k, b_2)$ denote the fraction of trajectories, centred on a particle, which have j particles when they last crossed the barrier with index b_1, when it is known that they have exactly k particles when they last cross the barrier with index b_2. When $k \ge \bar{N}$, then the results of Sheth (1995b) imply that

$$f(j, b_1|k, b_2) = \binom{k-1}{j-1} \frac{{}_2V_k - {}_1V_k}{{}_2V_k^{k-1}} \times ({}_1V_j)^{j-1} ({}_2V_k - {}_1V_j)^{k-j-1}, \quad (48)$$

where $j \le k$,

$${}_1V_j = \bar{N}(1 - b_1) + jb_1, \qquad {}_1V_k = \bar{N}(1 - b_1) + kb_1,$$

and ${}_2V_k = \bar{N}(1 - b_2) + kb_2$.

(Notice that equation 6 is this expression with ${}_1V_j = jb_1$, ${}_1V_k = kb_1$, and ${}_2V_k = kb_2$.) This reflects that fact that a comoving volume which is denser than the average density at time b_2 will have been less dense at an earlier time. When $k < \bar{N}$, then

$$f(k, b_2|j, b_1) = \binom{j-1}{k-1} \frac{{}_1V_j - {}_2V_j}{{}_1V_j^{j-1}} \times ({}_2V_k)^{k-1} ({}_1V_j - {}_2V_k)^{j-k-1}, \quad (49)$$

but now $k \le j$, since a comoving volume that is less dense than the average at some late time b_2 must also have been underdense at the earlier time $b_1 \le b_2$, and its density will have decreased since the earlier time. This expression is the probability that a cell contains exactly k particles at time b_2 given that at some earlier time $b_1 \le b_2$ it contained exactly j particles. These expressions are the analogues of equation (6).

In the limits $k \gg \bar{N}$ and $k \gg j$, the use of Stirling's approximation shows that

$$f(j|k) \to f(j, B, \beta'), \quad (50)$$

where $f(j, B, \beta')$ has the same form as equation (19) with

$$B = \frac{kb_1}{\bar{n}\, {}_2V_k} \qquad \text{and} \qquad \beta' = \frac{b_1\, \bar{N}(1 - b_1)}{\bar{n}\, {}_2V_k}. \quad (51)$$

This is similar to the rescaling associated with the constant barrier: When $N \gg j$ and $b_2 \ge b_1$, then

$$f_c(j, b_1|N, b_2) \to f_c(j, b_1/b_2). \quad (52)$$

This rescaling, and the scaling solution of the previous section, suggest that there may be a merger–fragmentation model of the type described by Sheth & Pitman (1997) associated with the Generalized Poisson distribution.

For trajectories that are not necessarily centred on particles, the expression corresponding to equation (48) is

$$F(j, b_1|k, b_2) = \binom{k}{j} \frac{{}_2V_k - {}_1V_k}{({}_2V_k)^k} ({}_1V_j)^j ({}_2V_k - {}_1V_j)^{k-j-1}, \quad (53)$$

when $k \ge \bar{N}$, and $F(k|j)$ is related to $f(k|j)$ similarly. These expressions follow from arguments given in Sheth & Lemson (1998). Identities associated with Abel's generalization of the Binomial theorem show that all these expressions are normalized to unity.

This last expression is related to the following problem. Choose a random Eulerian cell of comoving size V in an N-body simulation, and study the evolution of the mass within it. Let $p(j, b_1|k, b_2)$ denote the probability that at time b_1 there are exactly j particles within it, given that at some time $b_2 \ge b_1$ it is known to contain exactly $k \ge j$ particles. Then $p(j, b_1|k, b_2) = F(j, b_1|k, b_2)$. These expressions show explicitly that, for some Eulerian cells, it may happen that the number of particles within the cell decreases initially and increases later. In other words, in the model, matter can flow in and out of Eulerian cells.

6 DISCUSSION

This paper presents a new derivation of the Generalized Poisson distribution. The derivation allows one to construct a useful model of hierarchical clustering from Poisson initial conditions. The resulting model is useful because the Poisson assumption allows one to solve many problems that, at present, have no solution if more realistic initial conditions are used.

The model is a simple generalization of the excursion set model developed by Bond et al. (1991). Their approach allows one to estimate the mass function of collapsed halos; the generalization presented here allows one to describe the spatial distribution of these halos as well. The model can also be thought of as a simple variant of the spherical collapse model. In the model, initially denser regions contract more rapidly than less dense regions, sufficiently underdense regions expand, the influence of external tides on the evolution of such regions is ignored, and the number of expanding and contracting regions is assumed to be conserved. Strictly speaking, none of these assumptions can be justified physically. However, these simplifications mean that the model can be worked out relatively easily. Moreover, the Generalized Poisson distribution, derived after making these assumptions, is a reasonably accurate fit to the Eulerian counts-in-cells distribution measured in numerical simulations of clustering from Poisson initial conditions. This suggests that, at least for estimating the evolution of the counts-in-cells statistic from such initial conditions, these simplifications are justified.

In the model, a collapsed halo occupies a vanishingly small volume. In the simulations, collapsed halos have non-zero sizes—any given halo virializes at some fraction, typically about one half, of its turnaround radius. This means that on scales smaller than that of a typical halo, the counts-in-cells distribution computed here will cease to be a good approximation to that measured in the simulations. As discussed in the introduction, the fact that halos have non-trivial density profiles means that the b parameter in equation (1) depends on scale. A reasonable approximation to the effects of this scale dependence can be computed from models, such as those proposed by Navarro, Frenk & White (1996), of the density profiles of collapsed halos (see Sheth & Saslaw 1994 for details). As Poisson initial conditions are not realistic anyway, this seems an unnecessary refinement to an already idealized model.

As the basic model has worked out so easily, as it allows one to estimate the extent to which halos are biased tracers of the mass, and, most importantly, as it provides a reasonably accurate description of the evolution of clustering measured in numerical simulations, it seems worth extending it to describe clustering from more general initial conditions. This extension is in progress.

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REFERENCES

Borel E., 1942, Compt. rend., 214, 452
Consul P. C., 1989, Generalized Poisson Distributions: Properties and Applications. M. Dekker, New York
Epstein R. I., 1983, MNRAS, 205, 207
Itoh M., Inagaki S., Saslaw W. C., 1993, ApJ, 403, 459
Mo H. J., White S. D. M., 1996, MNRAS, 282, 347
Navarro J., Frenk C., White S. D. M., 1996, ApJ, 462, 563
Peebles P. J. E., 1980, The Large Scale Structure of the Universe. Princeton Univ. Press, Princeton
Saslaw W. C., 1989, ApJ, 341, 588
Saslaw W. C., Hamilton A. J. S., 1984, ApJ, 276, 13
Sheth R. K., 1995a, MNRAS, 274, 213
Sheth R. K., 1995b, MNRAS, 276, 796
Sheth R. K., 1996a, MNRAS, 281, 1124
Sheth R. K., 1996b, MNRAS, 281, 1277
Sheth R. K., Lemson G., 1998, in preparation
Sheth R. K., Pitman J., 1997, MNRAS, 289, 66
Sheth R. K., Saslaw W. C., 1994, ApJ, 437, 35
Tanner J. C., 1953, Biometrika, 40, 58
Tanner J. C., 1961, Biometrika, 48, 222
Zhan Y., 1989, ApJ, 340, 23

APPENDIX A: THE SPHERICAL COLLAPSE MODEL

Consider a region of initial, comoving Lagrangian size R_0. Let δ_0 denote the extrapolated linear overdensity of this object. In units where the average comoving density is unity, there is a deterministic relation between the mass M_0 within R_0: $M_0 \propto R_0^3$ provided $\delta_0 \ll 1$. As the Universe evolves, the size of this region changes. Let R denote the size of the region at some later time. Then the density within the region is simply $(R_0/R)^3 \equiv (1+\delta)$. In the spherical collapse model there is a deterministic relation between the initial Lagrangian size R_0 and density of an object, and its Eulerian size R at any subsequent time. For an Einstein–de Sitter universe,

$$\begin{aligned} \frac{R_{\rm p}(z)}{R_0} &= \frac{3}{10}\frac{1-\cos\theta}{|\delta_0|} \\ \frac{1}{1+z} &= \frac{3\times 6^{2/3}}{20}\frac{(\theta-\sin\theta)^{2/3}}{|\delta_0|} \end{aligned} \tag{A1}$$

(e.g. Peebles 1980). If $\delta < 0$, $(1-\cos\theta)$ should be replaced with $(\cosh\theta - 1)$ and $(\theta-\sin\theta)$ with $(\sinh\theta-\theta)$. In this model, collapsing objects reach turnaround at

$$(1+z_{\rm ta}) = 4^{1/3}\,\frac{\delta_0}{\delta_{\rm c0}}, \tag{A2}$$

at which time

$$\frac{R(z_{\rm ta})}{R_0} = \frac{(1+z_{\rm ta})\,R_{\rm p}(z_{\rm ta})}{R_0} = \frac{6}{10}\,\frac{4^{1/3}}{\delta_{\rm c0}}. \tag{A3}$$

For simplicity, consider the epoch when $z=0$. Since $(1+\delta) = (R/R_0)^3$, this means that there is a relation between δ_0 and $(1+\delta)$ that is independent of R. Mo & White (1996) give the following approximation to this relation:

$$\begin{aligned} \delta_0 &= 1.68647 - 1.35(1+\delta)^{-2/3} - 1.12431(1+\delta)^{-1/2} \\ &+ 0.78785(1+\delta)^{-0.58661}. \end{aligned} \tag{A4}$$

These relations imply that to every pair (R,z) there is an associated curve in the (δ_0, R_0) plane, so there is a corresponding curve in the (δ_0, V_0) plane. For a given Eulerian scale R, and a specified epoch z, equation (A4) gives what is effectively the boundary $\delta_{\rm sc}(V_0|R)$ associated with the spherical collapse model. This barrier should be compared with that given by equation (35).