ON NORMALIZED MULTIPLICATIVE CASCADES UNDER STRONG DISORDER

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Abstract. The purpose of this note is to determine the structure of the weak limit of a sequence of (normalized) multiplicative cascade measures under strong disorder in terms of the extremes of an associated branching random walk, assuming i.i.d positive, non-lattice bond weights and a second moment condition. The solution is expressed as an almost sure coupling of random probability measures in the disorder parameter $\beta > \beta_c$, that may be viewed in relation to the Aizenmann-Wehr metastates in physics and Aldous’ weak convergence paradigm in probability theory.

1. Introduction

The relationship between branching random walks and multiplicative cascades has a long history, going back to the early work of [11, 25, 31], and of [30], respectively. Recent results from the latter are exploited in the present note to obtain the distribution of the normalized multiplicative cascade probability under strong disorder conditions.

Branching random walks, as discretizations of branching Brownian motion, provide a natural probabilistic structure that as a result of [37] is known to occur, for example, in the context of reaction-dispersion equations of the type introduced by Fisher [23], Kolmogorov, Petrovskii and Piskounov [32]; see [33] and references therein.

Originating in statistical turbulence and other areas in which singular intermittent random distributions arise, multiplicative cascades are random measures that define prototypical models of disorder as well; see [30] for a seminal mathematical formulation whose inspiration is attributed to Benoit Mandelbrot. Much of the early work on multiplicative cascades involved the fine-scale (multifractal) structure of a limiting cascade distribution under conditions that have come to be referred to as weak disorder. In such cases the total mass defines a positive martingale sequence with a non-trivial a.s. limit. In particular, the cascade measure can easily be normalized to a (random) probability measure to obtain an a.s. weak limit. On the other hand, while compactness of the tree boundary ensures tightness, such a.s. weak limits are not expected to exist under strong disorder. However, as shown in [29], a weak limit in probability is possible at critical strong disorder, or the so-called boundary case. In particular a (random) probability can be defined in the infinite path limit at critical strong disorder. A similar connection had been open under strict strong disorder [29, 43]. In [8] the existence of a weak limit under strict strong disorder was obtained in distribution that will also follow from the results presented here. However the the present note provides more detailed analysis of the limit probabilities. Moreover, an almost sure construction is provided that couples the range of strict strong disorder limiting probability distributions.

The approach is to first obtain an explicit formula for the asymptotic distribution of the suitably scaled normalizing constant, or partition function, under conditions of non-critical (or strict) strong disorder, in terms of the extremes of the associated branching random walk defining the (random) energy function. The resulting formula extends the results of [1, 27] for the critical (boundary) case. As a consequence, the genealogy of a naturally associated coalescent also follows from recent results.

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of [14]. Moreover, the existence of normalized cascade in a weak limit is obtained as a corollary under strict strong disorder extending [29]. However, for strict strong disorder the weak convergence is itself in distribution. The physical meaning of such random probability distributions might best be interpreted as a form of metastable in the sense of [4, 39]. The more purely probabilistic meaning of such limits follow the perspective of the convergence in distribution paradigm formulated by [5].

1.1. Multiplicative Cascades. It is convenient to begin with some standard notation defining a multiplicative cascade, and its normalization to a probability. While the results may be more generally formulated for cascades on more general classes of trees, including Galton-Watson supercritical branching processes subject to a Kesten-Stigum condition on the offspring distribution, e.g., [17], we restrict the presentation to directed binary trees for simplicity of exposition.

Consider the infinite binary tree defined by the following set of vertices $T = \bigcup_{n=0}^{\infty} \{-1, +1\}^n$ with edges defined by pairs of vertices of the form $v = (v_1, \ldots, v_n)$, and its parent $v|n-1 = (v_1, \ldots, v_{n-1})$, and rooted at $\emptyset$ in correspondence with $\{-1, 1\}^0$. The boundary of $T$ is defined by $\partial T = \{-1, 1\}^N$, with the product topology. An infinite tree path is denoted by $s = (s_1, s_2, \ldots) \in \partial T$. We will also consider finite tree paths $s = (s_1, \ldots, s_n) \in T \setminus \emptyset$ of length $|s| = n$, and for $s = (s_1, s_2, \ldots) \in \partial T$, continue to use the notation $s|n := (s_1, s_2, \ldots, s_n)$, read “$s$ restricted to $n$”, for truncation. Also, for $v \in T, k = |v| \leq n$ we define

$$\Delta(v) := \{s \in \partial T : s|k = v\} \text{ and } \Delta_n(v) := \{s \in \{-1, +1\}^n : s|k = v\}$$

as the $\infty$-paths passing through the vertex $v$ and the vertices at level $n$ below the vertex $v$, respectively.

Suppose one is given a collection $\{X_v : v \in T\}$ of i.i.d. positive random variables indexed by $T$ and defined on a probability space $(\Omega, \mathcal{F}, P)$. Denote by $X$ a generic random variable having the common distribution of each $X_v$ and assume that $E(X) = 1$. Let $\lambda(ds) = \left(\frac{1}{2}\delta_{-1}(ds) + \frac{1}{2}\delta_{1}(ds)\right)^N$ on $(\partial T, \mathcal{B})$, and define the sequence of positive (random) measures $\mu_n(ds), n \geq 1$, absolutely continuous with respect to $\lambda(ds)$, via their sequence of Radon-Nikodym derivatives given by

$$\frac{d\mu_n}{d\lambda}(s) = \prod_{j=1}^{n} X_{s_{|j}}.$$  

(1.1)

Note that $\int_{\partial T} f(s)\mu_n(ds), n \geq 1$, is a bounded martingale for any bounded, continuous function $f$ on $\partial T$. The corresponding sequence of normalized (random) probability measures $\text{prob}_n(ds)$ is defined by

$$\frac{d\text{prob}_n}{d\lambda}(s) = M_n^{-1} \prod_{j=1}^{n} X_{s_{|j}},$$

(1.2)

where the partition function $M_n$ normalizes $\prod_{j=1}^{n} X_{s_{|j}}$, to a probability. Note that

$$M_n = 2^{-n} \sum_{|s|=n} \prod_{j=1}^{n} X_{s_{|j}}$$

(1.3)

has mean 1. The sequence of non-normalized measures $\mu_n(ds), n \geq 1$, is referred to as a multiplicative cascade and is the main object of our analysis.

1.2. Weak and strong disorder. The notions of weak disorder and strong disorder, e.g., see Bolthausen [15] for these notions in the present context, provide a well-known dichotomy defined in terms of the asymptotic behavior of the partition function as follows. First note that the sequence of (normalized) partition functions $M_n, n \geq 1$, is a positive martingale, so $M_\infty := \lim_{n \to \infty} M_n$ exists a.s. in $(\Omega, \mathcal{F}, P)$. By positivity of the factors defining the path probabilities, the event $[M_\infty = 0]$ is a tail event. By Kolmogorov’s zero-one law, $P(M_\infty = 0)$ must equal zero or one. Kahane and Peyrière [30] for multiplicative cascades, and (independently) Biggins, Hammersley and Kingman [11, 25, 31]
for branching random walks, had already obtained the following precise conditions for the disorder dichotomy:

\[ P(M_\infty > 0) = 1 \iff E(X \log X) < \log 2. \]  

(1.4)

In the case for which \([M_\infty > 0]\) a.s., the cascade is said to be in a state of weak disorder, whereas if \([M_\infty = 0]\) a.s., the cascade is in a state of strong disorder. Note that the deterministic environment \(X \equiv 1\) a.s. can be regarded informally as the “weakest” of the weak disorder regimes where \(M_n \equiv 1\) and \(\mu_n(ds) \equiv \lambda(ds)\). The special case

\[ E(X \log X) = \log 2, \]  

(1.5)

belongs to the strong disorder regime as critical disorder, or the boundary case. For example, in the case when \(X = \exp(-\beta N - \beta^2/2)\) with \(N\) being standard normal distributed, the boundary case corresponds to \(\beta = \sqrt{2 \log 2}\), with the strong disorder regime obtained for \(\beta \geq \sqrt{2 \log 2}\).

To describe the limit distribution of the (re-scaled) partition function in the critical case \(E(X \log X) = \log 2\) or \(E(X (\log 2 - \log X)) = 0\), let us recall the derivative martingale in the boundary case of the branching random walk; see [12]. We have that

\[ D_n = 2^{-n} \sum_{|s|=n} n \sum_{j=1}^n (\log 2 - \log X_{sj}) \prod_{j=1}^n X_{sj}, \quad n \geq 1, \]

is an \(L^1\)-bounded martingale having an a.s. positive limit \(D_\infty\), and referred to as the derivative martingale; see [33] for additional historic background in the contexts of branching random walk and branching Brownian motion. Under some natural regularity conditions on \(X\), Aidekon and Shi [3] proved that \(\sqrt{n}M_n/D_n\) converges in probability to \(\sqrt{2/\pi \sigma^2}\) where \(\sigma^2 := E(X(\log 2 - \log X)^2)\).

Additional insight into the relevance of the derivative martingale to multiplicative cascade theory can be obtained by considering the basic stochastic cascade recursion

\[ B \overset{d}{=} A_{-1} B_{-1} + A_{+1} B_{+1}, \]  

(1.6)

where \(B_{\pm 1}\) are i.i.d. non-negative r.v.s having the same distribution as \(B\), and \(A_{\pm 1}\) are i.i.d. non-negative r.v.s having mean \(1/2\), and independent of \(B_{\pm 1}\). The recursion (1.6) is a well-studied recursion in a variety of contexts, see [11,13,21,24,30,34,38], and much is known about the existence and analytic nature of solutions, e.g., asymptotic tail behavior, for the non-lattice distribution of \(X\) case.

Under weak disorder \(B = M_\infty\) is the nontrivial solution for \(A_{\pm 1} = \frac{1}{2} X_{\pm 1}\), unique up to positive constant multiples. However, at strong disorder one has \(M_\infty = 0\) a.s.; i.e., a trivial solution to (1.6). Nonetheless there is a nontrivial solution in the (non-lattice) boundary case, namely a constant multiple of \(D_\infty\); see [27,33].

It is generally well-known as a result of early work originating in [21] that under strong disorder the solution (fixed point) of the random recursion (1.6) coincides with a multiple of a Lévy stable process stopped at \(D_\infty\); see [33] for a summary and extensions. The results to follow provide a more detailed analysis of the structure of this solution, through its explicit connections to the extremes of the associated branching random walk, that facilitates the almost sure (coupling) construction of the limit probabilities \(\text{prob}_\infty(ds)\).

1.3. Our result. For purposes of orientation, the main result of this note is essentially as given below. A more comprehensive statement is postponed to Theorem 2.1 with an explicit description of the limiting \(\text{prob}_\infty(ds)\) measure. Let \(X\) be a positive random variable with mean one and satisfying the strict strong disorder condition \(E(X \log X) > \log 2\) for the multiplicative cascade defined in (1.1). By calculations of the type given in [26], it is easy to see that there is a unique \(\alpha \in (0,1)\) such that

\[ E \left( \frac{X^\alpha}{E(X^\alpha)} \log \frac{X^\alpha}{E(X^\alpha)} \right) = \log 2. \]  

(1.7)
Specifically, as observed in ([26], proof of Lemma 3), the function $\rho(\alpha) = \mathbb{E}(\frac{X^\alpha}{\alpha X^\alpha} \log_2 X)$ is continuous and strictly increasing on $(0,1)$, $\rho(0) < 0$. So, one may apply an intermediate value theorem to $\rho(\alpha) - (1 + \log_2 \mathbb{E}X^\alpha)/\alpha, 0 < \alpha < 1$. We define

$$W := \log 2 + \log \mathbb{E}(X^\alpha) - \alpha \log X \tag{1.8}$$

so that $W$ satisfies $\mathbb{E}(e^{-W}) = 1/2$ and $\mathbb{E}(W e^{-W}) = 0$. Now we state our main result.

**Theorem 1.1.** Let $X$ be a positive random variable with $\mathbb{E}(X) = 1$ and $\mathbb{E}(X \log X) > \log 2$. Assume that the distribution of $W$, as defined in (1.8), is non-lattice and the following size-biased moment is finite:

$$\mathbb{E}(W^2 + \log_+ (e^{-W} + W e^{-W})) e^{-W} < \infty \tag{1.9}$$

Then, there is a random probability measure $\text{prob}_\infty(ds)$ on $\partial T$ such that

$$\{\text{prob}_n(\Delta(v)) : v \in T\} \Rightarrow \{\text{prob}_\infty(\Delta(v)) : v \in T\},$$

where $\Rightarrow$ denotes finite-dimensional convergence in distribution. Moreover, $\text{prob}_\infty(ds)$ can be described explicitly in terms of the derivative martingale corresponding to $e^{-W}$ and an $\alpha$-stable random field with skewness intensity 1, where $\alpha \in (0,1)$ is determined by (1.7).

This result can also be viewed in terms of metastate as defined in [4]. In fact, we prove finite-dimensional convergence of the joint distribution of the $\text{prob}_n$’s and the disorder, i.e., the $X_v$’s. This defines a metastate in the Aizenman-Wehr sense [4] by taking this joint limiting distribution and conditioning on the disorder. A metastate in the sense of Newman-Stein [39] requires that one condition on the disorder first, and then obtain the limit of an empirical distribution of the $\text{prob}_{n_k}$’s along some sparse subsequences $n_k$.

It may be noted that similar formulae are known for other models of disorder, such as the random energy model (REM), and generalized random energy model (GREM), introduced by [20, 38, 42] and related by mean-field type formulations. It was shown by [10] as a consequence of [38] that the genealogy of the GREM is given by the Bolthausen-Sznitman coalescent. It is interesting to note the manner in which the asymptotic results for the multiplicative cascade model differ from those of GREM, but remain within the framework of $\Lambda$-coalence. Related comments are included at the close of this note.

### 2. Background, Preliminaries and Notation

In recent years there has been a rapidly growing literature on the asymptotics of the extremes of branching random walks. Relatively long, technical papers have provided a refined understanding of the behavior of the right (or left) most particles in branching random walks; e.g., [2, 7, 9, 13, 16, 27, 28, 35]. This theory will be exploited to provide a rather simple derivation of an illuminating formula for the asymptotic distribution of the partition function, suitably scaled, for a general class of multiplicative cascades under strong disorder and non-lattice energy distributions. In particular, two essential structures underlying the results here are (a) Brunet-Derrida’s notion of superposability, and (b) Biggins-Kyprianou’s version of the derivative martingale; see [16] and [12], respectively, where these ideas arise in connection with the extremes of branching random walks. For ease of reference and in an effort to make the presentation self-contained, key results in this regard will be explicitly stated along the way. This is the essential element of the a.s. coupling construction of the weak limits in distribution of the normalized cascade probabilities.

Let us recall the notion of multiplicative cascade from Section 1.1. When $X = \exp(-\beta W)/\mathbb{E}(e^{-\beta W})$ for some $\beta > 0$, the sequence of random probability measures $\{\text{prob}_n(ds) : n \geq 1\}$ is also referred to as a tree polymer on $(\partial T, \mathcal{B})$ at inverse temperature $\beta$. In the tree polymer case, we will always assume the boundary case for $W$ (as explained in Section 2.1), i.e.,

$$\mathbb{E}(e^{-W}) = 1/2 \text{ and } \mathbb{E}(W e^{-W}) = 0$$

and the recipe for obtaining $W$ from $X$ is given in (1.8).
2.1. Tree Polymer. Assume that $W$ is a random variable with $\varphi(\beta) := E(e^{-\beta W}) < \infty$ for all $\beta \geq 0$. Then the dichotomy (1.4) for the r.v. $X = \varphi(\beta)^{-1}e^{-\beta W}$ gives the critical disorder as $\beta = \beta_c$ where $(\beta^{-1}\log(2\varphi(\beta)))'_{|\beta=\beta_c} = 0$ and the weak disorder as $\beta < \beta_c$. By centering and scaling appropriately, i.e., working with $\beta_c W + \log(2\varphi(\beta_c)) = \log(2) + \log(2\varphi(\beta_c))$ instead of $W$, without loss of generality we can assume that

$$E(e^{-W}) = \frac{1}{2} \text{ and } E(We^{-W}) = 0. \quad (2.1)$$

Thus with $X_v = \varphi(\beta)^{-1}e^{-\beta W_v}, v \in T$ the strong disorder corresponds to $\beta \geq 1$. The exponent defined by (1.7) is given by $\alpha = 1/\beta$ in this framework.

We define the energy of a finite path $s \in T$ as

$$H(s) = \sum_{i=1}^{|s|} W_{s|v} \text{ for } s \in T$$

and will sometimes use $H_n(s)$ instead of $H(s)$ when $|s| = n$ to emphasize the dependence on $n$. Also, in this context the partition function is defined as $Z_n(\beta) := \sum_{|s|=n} e^{-\beta H(s)}$. We also define, for $v \in T, |v| \leq n$

$$Z_n(\beta; v) = e^{-\beta H(v)} \sum_{s \in \Delta_n(v)} e^{-\beta[H(s) - H(v)]}. \quad (2.2)$$

Then (2.2) can be understood as the partition function at the vertex $v$. Clearly $Z_n(\beta) = Z_n(\beta; \emptyset)$.

When $X = \varphi(\beta)^{-1}e^{-\beta W}$, we will use $\mu_{n,\beta}$ and $\text{prob}_{n,\beta}$ for (1.1) and (1.2), respectively. Note that the normalization constant $M_n$ in (1.3), is the same as $(2\varphi(\beta))^{-n} Z_n(\beta)$. Also, we have for $v \in T, |v| < n$

$$\mu_{n,\beta}(\Delta(v)) = (2\varphi(\beta))^{-n} e^{-\beta H(v)} Z_n(\beta; v)$$

and $\text{prob}_{n,\beta}(\Delta(v)) = e^{-\beta H(v)} Z_n(\beta; v) / Z_n(\beta)$.

2.2. Coupled limits of derivative martingales. We first define a collection of random variables indexed by the vertices of the infinite binary tree that will appear in the joint convergence of the partition functions at different vertices at the critical disorder, i.e., $\beta = 1$. Recall that, the derivative martingale is defined as

$$D_n := \sum_{|s|=n} H(s)e^{-H(s)}$$

and has an a.s. positive limit $D_\infty$ satisfying the distributional recursion (1.6), i.e., $e^{-W-1}D_\infty(-1) + e^{-W+1}D_\infty(+1) \overset{d}{=} D_\infty$ where $D_\infty(\pm 1)$ are i.i.d. $\sim D_\infty$.

Assume that $\{W_v : v \in T\}$ is a collection of i.i.d. random variables each distributed as $W$ and indexed by $T$. Fix a positive integer $k$. For $v \in T, |v| = k$, let $D(v)$ be i.i.d. copies of $D_\infty$. Now inductively for $i = k-1, k-2, \ldots, 0$, define

$$D(v) := D(v, -1)e^{-W_v, -1} + D(v, +1)e^{-W_v, +1} \text{ for } v \in T, |v| = i. \quad (2.3)$$

It is easy to see that for any fixed $i \leq k$, $\{D(v), |v| = i\}$ are i.i.d. copies of $D_\infty$ and thus $\{D(v) : |v| \leq k\}$ is a consistent family of distributions. By Kolmogorov’s consistency theorem there exists a (denumerable) tree-indexed collection of random vectors

$$D_\infty := \{(W_v, D_\infty(v)) : v \in T\}$$

such that the finite-dimensional distribution restricted to $\{v : |v| \leq k\}$ is given by the above construction (2.3).

Now define the interval $I(\emptyset) = [0, D_\infty(\emptyset))$. One can think of $D_\infty$ as a providing a way in which to partition the interval $I(\emptyset)$ into successively smaller intervals. Define

$$I(-1) = [0, e^{-W-1}D_\infty(-1))$$

and $I(+1) = [e^{-W-1}D_\infty(-1), e^{-W+1}D_\infty(+1) + e^{-W-1}D_\infty(-1))$. 
Note that $D_{\infty}(\emptyset) = e^{-W-1}D_{\infty}(-1) + e^{-W+1}D_{\infty}(+1)$ a.s. by construction and thus $I(+1), I(-1)$ is a partition of $I(\emptyset)$. Now to define $I(v)$ for $v \in T, |v| = k$, consider the lexicographic ordering on \{-1, +1\}^k, i.e., for $u, v \in \{-1, +1\}^k$, $u < v$ iff there exists $i \in \{0, 1, \ldots, k\}$ such that $u[i] = v[i]$ and $u[i+1] < v[i+1]$. Now, for $v \in T, |v| = k$ define

$$I(v) := \left[ \sum_{u < v} e^{-H(u)}D_{\infty}(u), e^{-H(v)}D_{\infty}(v) + \sum_{u < v} e^{-H(u)}D_{\infty}(u) \right].$$

One can easily check that the collection of intervals $\{I(v) : v \in T\}$ respects the tree structure in terms of set-inclusion, i.e., if $u < v$ then $I(u) \subseteq I(v)$.

Here we mention that, any infinite path $s \in \partial T$ can be represented by a point $t(s) \in \partial(T)$ and conversely any point $t_0 \in \partial(T)$ corresponds to a unique path $s = s(t_0) \in \partial T$ in the sense that $\{t\} = \bigcap_{k=1}^{\infty} I(s|k)$.

### 2.3. Coupling at the strong disorder.

Let $\theta > 0$ be a fixed real number. Consider a decorated Poisson point process $N \times [0, \infty)$ with intensity measure $e^{-t}dt$ and the decoration at the point $(x, t)$ given by $V_{x, t}$ which are i.i.d. copies of a point process $V$. Let $\{(W_v, D_v) : v \in T\}$ be a collection of random variables indexed by the vertices of $T$ as constructed in Section 2.2, and independent of $N$. Fix a real number $\theta > 0$.

Now for any $\alpha \in (0, 1)$ and $v \in T$, define

$$I_\alpha(v) = \sum_{(x,t) \in N} e^{-x/\alpha}I_{t/\theta \in I(v)} \sum_{y \in V_{x, t}} e^{-y/\alpha}. \quad (2.4)$$

The main result of this note may now be stated as follows.

**Theorem 2.1.** Assume that the distribution of $W$ is non-lattice, satisfies the boundary condition (2.1) and the following size-biased moment is finite:

$$E(W^2 + \log+ (e^{-W} + We^{-W}))e^{-W} < \infty \quad (2.5)$$

Then, we have for any $\beta_1, \beta_2, \ldots, \beta_k \in (1, \infty)$ and $v_1, v_2, \ldots, v_k \in T$

$$\{n^{\beta_i/2}e^{-\beta H(v_i)}Z_0(\beta_i; v_i) : i = 1, 2, \ldots, k\} \Rightarrow \{I_{\theta, \beta_i}(v_i) : i = 1, 2, \ldots, k\}$$

in the sense of convergence in distribution for some $\theta > 0$ and some point process $V$. In particular, there is a random probability measure $\text{prob}_{\infty, \beta}(ds)$ on $\partial T$ such that

$$\{\text{prob}_{n, \beta}(\Delta(v)) : v \in T\} \Rightarrow \{\text{prob}_{\infty, \beta}(\Delta(v)) : v \in T\},$$

where $\Rightarrow$ denotes finite-dimensional convergence in distribution and

$$\text{prob}_{\infty, \beta}(\Delta(v)) := I_{1/\beta}(v)/I_{1/\beta}(\emptyset), v \in T.$$

The scaling of the partition function implies a certain centering of the branching random walkers induced by the path energies $H_n(s), |s| = n$, that may explicitly be expressed as follows:

$$n^{\frac{3}{2}}\beta Z_n(\beta) = \sum_{|s| = n} e^{-\beta(H_n(s) - \frac{3}{2} \log n)},$$

**Remark 2.2.** Here we remark that the limiting measures $(\text{prob}_{\infty, \beta}, \beta > 1)$ are defined on the same probability space and are absolutely continuous w.r.t each other. By the definition of the intervals $\{I(v) : v \in T\}$, any infinite path $s \in \{-1, +1\}^\infty$ in the binary tree will be represented by a point $t(s)$ in the interval $I(\emptyset)$. Now for the Poisson process, which is independent of the intervals, we can define the $\beta$-contribution for a single point $t_0$ as

$$C_\beta(t_0) = \sum_{(x,t) \in N: t = t_0} e^{-\beta x} \sum_{y \in V_{x, t}} e^{-\beta y}.$$
This is nonzero for countably infinitely many t’s and the support set for $t_0$, projection of $\mathcal{N}$ on the second co-ordinate, is independent of $\beta$. It is easy to see that the Radon-Nikodym derivative of $\text{prob}_{\infty, \beta_1}$ w.r.t. $\text{prob}_{\infty, \beta_2}$ at the infinite path $s$ (with corresponding time point $t(s)$) is

$$\frac{C_{\beta_1}(t(s))\mathcal{I}_{1/\beta_2}(0)}{C_{\beta_2}(t(s))\mathcal{I}_{1/\beta_1}(0)}.$$  

2.4. Superposability. A key idea for the results here is that of superposability of extremal point processes introduced by [16], and the corresponding representation as a Poisson cluster process; also referred to as a “decorated Poisson process” in the literature. Specifically,

**Definition 2.1.** A point process $N$ on $\mathbb{R}$ is said to be superposable if, for an independent copy $N'$ and any $a, b \in \mathbb{R}$ such that $e^{-a} + e^{-b} = 1$,

$$T_a N + T_b N' \overset{d}{=} N,$$

where $T_x \left( \sum_y \delta_y \right) = \sum_y \delta_{y + x}, \ x \in \mathbb{R}$.

The basic example of a superposable point process is the Poisson process on $\mathbb{R}$ with intensity $e^x dx$. This is the well-known point process of extremes of a centered and scaled i.i.d. Gaussian sequence; see [41]. More generally, a superposable point process is infinitely divisible and, therefore, it follows that it must be a Poisson cluster point process. Based on analogous results for branching Brownian motion, it had been conjectured in [16] that the only superposable point processes were Poisson cluster processes with Poisson intensity $\theta e^x dx, \theta > 0$. This was recently proven as a consequence of infinitely divisibility, and also as a consequence of LePage representation theory, see [36]. In the context of the present note it may also be interesting to note that Poisson cluster processes must be associated in the sense of positive dependence (or FKG inequalities); [18, 22].

Another conjecture by [16] was recently proven in [35] extending the above quoted result for i.i.d. Gaussian extremes to the extremes of the energies $H_n(s), |s| = n$, centered and scaled. In particular, it is shown that in the boundary case the point process of extremes is superposable. More specifically, in the notation of the present article,

**Theorem 2.3 (Theorem 1.1 in [35]).** Assume that the distribution of $W$ satisfies the condition of Theorem 2.1. Let $N_n = \sum_{|s| = n} \delta_{H(s)} - \frac{\beta}{2} \log n + \log D_\infty$. Then $(N_n, D_n)$ converge jointly in distribution to $(N_\infty, D_\infty)$ where $N_\infty = \sum_{k \geq 1} \sum_{y \in V_k} \delta_{x_k + y}$ is a Poisson cluster point process on $\mathbb{R}$ with Poisson center process $\{x_k : k \geq 1\}$ having intensity $\theta e^x dx, x \in \mathbb{R}$ for some $\theta > 0$, $V_k$’s are i.i.d. copies of some point process $V$ and $D_\infty$ is independent of $N_\infty$.

2.5. Genealogy. As a byproduct of Theorem 2.1 one can easily find the limiting distribution of the genealogical tree of randomly chosen $k$ vertices in $\{-1, +1\}^n$ from the distribution prob$_{n, \beta}$. Recall that for $v = (v_1, v_2, \ldots, v_n) \in T$ and an integer $k \leq n$, we have $v | k = (v_1, v_2, \ldots, v_k)$. Consider the decorated Poisson process as given in Section 2.3. Let $\nu$ be the (random) probability measure supported on $I(\theta)$ so that

$$\nu'_\beta(t_0) = \frac{1}{\mathcal{I}_I(\beta)(\theta)} \sum_{(x, t) \in N : t = t_0} e^{-\beta x} \sum_{y \in V_{x, t}} e^{-\beta y}.$$  

Let $\nu_\beta$ be the probability measure on $\partial T$ such that $t(s) \sim \nu'_\beta$ when $s \sim \nu_\beta$. The following corollary follows easily from Theorem 2.1.

**Corollary 2.4.** Let $v_1, v_2, \ldots, v_k$ be $k$ many i.i.d. vertices from the probability measure prob$_{n, \beta}$ on $\{-1, +1\}^n$. Let $u_1, u_2, \ldots, u_k$ be $k$ many i.i.d. vertices from the probability measure $\nu_\beta$. Then for any fixed integer $k$ we have

$$(v_1 | k, v_2 | k, \ldots, v_k | k) \xrightarrow{w} (u_1 | k, u_2 | k, \ldots, u_k | k)$$  

as $n \to \infty$. 

This implies local convergence of the genealogical tree for randomly chosen \( k \) vertices from \( \text{prob}_{n,\beta} \) near the root.

3. Proof of Main Result

An easy consequence of Theorem 2.3 (Theorem 2.4 in [35]) and the fact that

\[
n^{3/2}Z_n(\beta) = D^\beta \sum_{|s|=n} e^{-\beta(H(s)-\frac{3}{2}\log n + \log D_\infty)} = D^\beta \int_{\mathbb{R}} e^{-\beta z} N_n(dz),
\]

is that for any fixed \( \beta_1, \beta_2, \ldots, \beta_k \in (1, \infty) \) we have

\[
\left(n^{3\beta_k/2}Z_n(\beta_i), i = 1, 2, \ldots, k \right) \implies \left(D^\beta_k \sum_{k \geq 1} e^{-\beta_i x_k} \sum_{y \in V_k} e^{-\beta_i y}, i = 1, 2, \ldots, k \right)
\]

where \( (x_k, k \geq 1) \) are points of a Poisson point process with intensity \( \theta e^x dx, x \in \mathbb{R} \), \( V_k \)'s are i.i.d. copies of some process \( V \) and \( D_\infty \) is the limiting derivative martingale independent of everything else.

Now let \( \mathcal{N} \) be a Poisson point process in \( \mathbb{R} \times [0, \infty) \) with intensity measure \( e^x dt dx \), \( (x, t) \in \mathbb{R} \times [0, \infty) \).

It is easy to see that for a finite interval \( I \subset [0, \infty) \), the point process \( \{x : (x, t) \in \mathcal{N}, t \in I\} \) is Poisson point process with intensity \( |I| e^x dx \), which has the same distribution as \( \{x_k - \log |I|, k \geq 1\} \) where \( \{x_k : k \geq 1\} \) is a Poisson point process with intensity \( e^x dx \). Thus for an interval \( I \) of length \( \theta D_\infty \) independent of \( \mathcal{N} \), we have

\[
\left(\sum_{k \geq 1} e^{-\beta_i (x_k - \log D_\infty)} \sum_{y \in V_k} e^{-\beta_i y}, i = 1, 2, \ldots, k \right)
\]

\[
\overset{d}{=} \left(\sum_{(x,t) \in \mathcal{N}, t \in I} e^{-\beta_i x} \sum_{y \in V(x,t)} e^{-\beta_i y}, i = 1, 2, \ldots, k \right)
\]

Now fix an integer \( k \geq 1 \) and consider the set of vertices \( v \in T, |v| = k \) in the tree \( T \) at level \( k \). Consider the collection of random variables \( (n^{3/2}Z_n(\beta; v), |v| = k) \) which clearly are i.i.d. and by the above reasoning has the limit (in distribution)

\[
\left(\sum_{(x,t) \in \mathcal{N}, t \in I(v)} e^{-\beta x} \sum_{y \in V(x,t)} e^{-\beta y} |v| = k \right)
\]

where \( I(v), |v| = k \) are mutually disjoint intervals of length \( \theta D_\infty \) and \( \{D_\infty(v), |v| = k\} \) are i.i.d. copies of \( D_\infty \). From here the proof of Theorem 2.1 follows easily. □

A consequence of Theorem 2.1 is that the limiting distribution of \( \Gamma(1-1/\beta)^{-\beta} n^{3\beta/2} Z_n(\beta) \) is the same as a \( \alpha \)-stable subordinator \( T^{(\alpha)}_T \) stopped at an independent r.v. \( \theta D_\infty \mathbb{E}(||\{e^{-y}, y \in V\}||_{\beta}) \), where \( \alpha = 1/\beta \), and \( ||g(y), y \in V||_{\beta}, \beta > 1 \), denotes the usual \( L^\beta \)-norm, \( \left(\int_{\mathbb{R}} e^{-y^2} V(dy)\right)^{1/2} \) with respect to the decorating points. Note that we have

\[
\lim_{\beta \to \infty} \Gamma(1-1/\beta)^{-1} n^{3/2} Z_n(\beta)^{1/\beta} = n^{3/2} e^{-\min_{|s|=n} H(s)}
\]

and

\[
\lim_{\beta \to \infty} \mathbb{E}(||\{e^{-y}, y \in V\}||_{\beta}) \left(T^{(1/\beta)}_{\theta D_\infty}\right)^{1/\beta} \overset{d}{=} \mathbb{E}(\max_{y \in V} y) \cdot D_\infty \cdot G
\]

where \( -\log G \) has Gumble distribution. For the corollary, note that for \( |v| < n \), one has

\[
\text{prob}_{n,\beta}(\Delta(v)) = Z_n(\beta; \emptyset)^{-1} e^{-\beta H(v)} Z_n(\beta; v).
\]

Thus the following weak limit in distribution follows easily

\[
\text{prob}_{\infty,\beta}(\Delta(v)) = \mathcal{I}_{1/\beta}(v) / \mathcal{I}_{1/\beta}(\emptyset).
\]
**Corollary 3.1.** At any strong disorder \( \beta > 1 \), for each finite set \( F \subseteq \mathbb{N} \)

\[
\hat{\text{prob}}_{n,\beta}(F) \Rightarrow \hat{\text{prob}}_{\infty,\beta}(F) \text{ in distribution},
\]

where \( \hat{\text{prob}}_{n,\beta}, n \geq 1, \hat{\text{prob}}_{\infty,\beta} \) denote their respective Fourier transforms as probabilities on the compact abelian multiplicative group \( \partial T \) for the product topology.

**Proof.** The continuous characters of the group \( \partial T \) are given by \( \chi_F(t) = \prod_{j \in F} t_j \) for finite sets \( F \subseteq \mathbb{N} \).

In particular there are only countably many characters of \( \partial T \). From standard Fourier analysis it follows that we need only show that

\[
\lim_{n \to \infty} E_{\hat{\text{prob}}_{n,\beta}} \chi_F = E_{\hat{\text{prob}}_{\infty,\beta}} \chi_F \text{ in distribution}
\]

for each finite set \( F \subseteq \mathbb{N} \). Let \( m = \max\{k : k \in F\} \). Then for \( n > m \),

\[
E_{\hat{\text{prob}}_{n,\beta}} \chi_F = \int_{\partial T} \chi_F(s) \frac{d\hat{\text{prob}}_{n,\beta}}{d\lambda}(s) \lambda(ds)
= \sum_{|v|=m} \prod_{j \in F} v_j \cdot Z_n(\beta; \emptyset)^{-1} e^{-\beta H(v)} Z_n(\beta; v)
= \sum_{|v|=m} \prod_{j \in F} v_j \cdot e^{-\beta H(v)} \cdot \frac{n^{3\beta/2}}{n^{3\beta/2} Z_n(\beta; \emptyset)} \Rightarrow E_{\hat{\text{prob}}_{\infty,\beta}} \chi_F
\]

where the convergence is in distribution.

As a closing remark one may view the “genealogical structure” of the resulting a.s. defined strong disorder cascade probability limit as follows: If vertices are chosen from the \( n \)-th level according to the cascade measure in strong disorder, most of the branching occurs either within distance \( o(n) \) from the root or within distance \( o(n) \) from the \( n \)-th level. The branching near the \( n \)-th level gives rise to the decoration Point process in the limiting decorated Poisson process, whereas the Poisson process arises out of the time spent without any branching; see [6,19] for comparison with branching Brownian motion. Our result gives the structure near the root within distance \( O(1) \), as discussed in Section 2.5.

See Figure 3.1 for a graphical depiction.

Another genealogical structure can be identified in terms of the Lévy stable subordinator \( \{T_s^{(\alpha)} : s \geq 0\} \) by viewing it as a continuous state branching process (csbp), in a manner as was done in [10] in describing the genealogy of Neveu’s csbp associated with another disordered system; namely, Derrida’s generalized random energy model (GREM). In particular it was shown in ([10], Theorem 4) that the genealogy of Neveu’s csbp defines a Bolthausen-Sznitman coalescent (BSC). This could be accomplished by exploiting an alternative cascade version of GREM, due to Ruelle in [42]. Now observe that the (BSC) is a \( \Lambda \)-coalescent for a uniform distribution \( \Lambda \) on \([0,1]\); see [40]. So, in view of recent results of [14], the genealogy of \( \{T_s^{(\alpha)} : s \geq 0\} \) is that of a \( \Lambda \)-coalescent for which \( \Lambda \) is a Beta distribution with parameters \( \beta_c/\beta \) and \( 1 - \beta_c/\beta \). Since \( \beta_c/\beta < 1 \) under strict strong disorder, the results here establish interesting points of contrast and comparison for these respective models of disorder; also see [19] for other observations in this regard.
References


[34] Q. Liu, Fixed points of a generalized smoothing transformation and applications to the branching random walk, Adv. in Appl. Probab. 30 (1998), no. 1, 85–112. MR1618888


