Self-assessment of Local Filters by Non-Gaussianity Measures

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Abstract—The paper deals with state estimation of stochastic nonlinear systems by means of local filters. A new technique is designed to provide a self-assessment of the filter with respect to its estimate quality. It uses a non-Gaussianity measure based on conditional third moment of the state to indicate a possible decrease of estimate quality. The technique is proposed for general local filters with detailed specification for three selected filters in the Kalman filtering framework. An application of the technique is illustrated in a numerical example.

Keywords: state estimation, nonlinear filters, Kalman filtering, non-Gaussianity measures, nonlinearity measures

I. INTRODUCTION

Recursive state estimation of discrete-time stochastic nonlinear systems from noisy measured data has been a subject of considerable research interest for the last four decades.

Within the Bayesian framework, a general solution to recursive state estimation problem is given by the Bayesian recursive relations (BRRs) which compute the probability density functions (PDFs) of the state conditioned by the measurements. These PDFs represent a complete description of the state, which itself cannot be fully measured. A closed-form solution to the BRRs is available only for a few special cases, such as the linear Gaussian system [1]. In other cases, an approximate methods must be used. These methods can be divided with respect to the validity of the resulting estimates into global and local methods [2]–[4].

The global methods provide estimates in the form of conditional PDFs valid within almost whole state space. Global methods are capable to estimate the state of a strongly nonlinear or non-Gaussian system but usually at the cost of substantial computational demands.

The local methods provide computationally feasible estimates predominantly in the form of the conditional mean and covariance matrix\(^1\) but with limited validity (within a close neighbourhood of a point estimate only). These methods offer satisfactory estimation performance (with possibly bounded estimation error) mainly for systems with mild nonlinearities, exact initial condition, and small disturbing noises [5]. The satisfactory performance of local filters (LFS) is also somehow conditioned by the Gaussian-like shape of the conditional PDFs, that means unimodal and non-heavy-tailed PDFs. Indeed, many LFs are based on assumptions of Gaussian conditional PDFs, e.g., those based on various deterministic or stochastic integration rules or Fourier-Hermite expansion [3], [6]–[10].

Knowledge and classification (or assessment) of a system nonlinearity (degree of nonlinearity) [11] and non-Gaussianity [12] is, therefore, important for several reasons; model assessment for selection of a suitable filter, on-line monitoring of alignment with the assumptions to ensure satisfactory estimation performance, and decision whether a filter parameter requires an adaptation [13].

Recently, analysis of nonlinearity and non-Gaussianity measures (NGMs) has been a focal point of several studies. In [11]–[14], a set of local and global measures have been proposed. The global measures aim for assessing the overall nonlinearity of the system or the overall impact of the nonlinearity on the PDFs of the state (without regard to estimation method selection). The global measures are intended for off-line analysis and thus, the conclusions stemming from the analysis are strictly tied with a problem set-up (e.g., initial condition, noise properties). Any change in the set-up results in a necessity to recompute the measures.

On the other hand, the local measures are designed for an on-line monitoring of (local) filters. It means that at each time instant the nonlinear transformation in a filtering or prediction step of the LF is assessed with respect to its severity or its impact on the properties the transformed random variable. The local measures are suitable for self-monitoring of the LFs, but the measures have a very limited capability in monitoring of the transformation cumulative effect (the measures in two subsequent time instants are independent; even the measure in the filtering step is independent of the measure in the prediction step at the same time). As was illustrated in [12], a sequence of mild nonlinearities (with very low values of nonlinearity measures or NGMs\(^2\)) on, for example, an initially Gaussian PDF, might result in a heavy-tailed or multimodal distribution. Subsequently, such a distribution might have adverse effect on the estimate quality.

The goal of the paper is to utilize local NGMs for self-assessment of the LFs. The NGMs will be propagated within the filter similarly to the conditional mean and covariance matrix recursion. Besides a general relation for the recursion, detailed relations for three LFs will also be derived.

The rest of the paper is organized as follows. Section II provides system specification, a generic LF algo-

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\(^1\)First two moments usually do not represent a full description of the immeasurable state.

\(^2\)In this paper, the measure form and its value are not distinguished. Both are simply called measure.
Algorithm 1: Generic Local Filter

3In literature instead of the term 'local filters', terms such as 'Gaussian filters' or 'Kalman filters' can be found.

Step 1: (initialization) Set the time instant $k = 0$ and define a priori initial condition by the predictive mean $\hat{x}_{0\mid -1} = \bar{x}_0$ and the predictive covariance matrix $P_{0\mid -1} = P_0^x$.

Step 2: (filtering) The state predictive estimate is updated with respect to the last measurement $z_k$ according to

$$\hat{x}_{k\mid k} = \hat{x}_{k\mid k-1} + K_k(z_k - \hat{z}_{k\mid k-1}),$$

$$P_{k\mid k} = P_{k\mid k-1} - K_k P_{k\mid k-1} K_k^T,$$

where $K_k = P_{k\mid k-1} (P_{k\mid k-1})^{-1}$ is the filter gain and

$$z_{k\mid k-1} = E[z_k|z_k^{-1}] = E[h_k(x_k)|z_k^{-1}],$$

$$P_{k\mid k-1} = E[(z_k - z_{k\mid k-1})(z_k - z_{k\mid k-1})^T|z_k^{-1}]$$

$$P_{k\mid k-1} = E[(x_k - \hat{x}_{k\mid k-1})(z_k - \hat{z}_{k\mid k-1})^T|z_k^{-1}].$$

Step 3: (prediction) The predictive statistics are given by

$$\hat{x}_{k+1\mid k} = E[x_{k+1}|z_k^1] = E[f_k(x_k)|z_k^1],$$

$$P_{k+1\mid k} = E[(x_{k+1} - \hat{x}_{k+1\mid k})(x_{k+1} - \hat{x}_{k+1\mid k})^T|z_k^1].$$

Let $k = k + 1$. The algorithm continues by Step 2.

The particular LFs just differ in approximation applied for the solution to the measurement and state predictive statistics (5)–(7) and (8)–(9), respectively.

As the technique for the NGM-based self-assessment will be illustrated for the EKF, UKF, and MCKF, the corresponding approximation techniques will be introduced below, i.e., the first order Taylor expansion based approximation (TE1), the UT, and the Monte-Carlo (MC) based numerical integration. The techniques are illustrated in computation of the measurement predictive statistics (5)–(7). Their application in computation of (8)–(9) is analogous [4].

1) The first order Taylor expansion: This approach approximates the nonlinear function by the TE1 [1]. The TE1 of $h_k(x_k)$ in (2) under assumption of the known state predictive statistics $\hat{x}_{k\mid k-1}$ and $P_{k\mid k-1}$ (defining the approximation point) is of the form

$$z_{k} = h_k(x_k) + v_k,$$

$$\approx h_k(\hat{x}_{k\mid k-1}) + H_k(\hat{x}_{k\mid k-1})(x_k - \hat{x}_{k\mid k-1}) + v_k,$$

where $H_k(\hat{x}_{k\mid k-1}) = \frac{\partial h_k(x_k)}{\partial x_k}|_{x_k = \hat{x}_{k\mid k-1}}$ is Jacobian of $h_k(\cdot)$. The desired statistics are then computed as

$$\hat{z}_{k} = h_k(\hat{x}_{k\mid k-1}),$$

$$P_{k\mid k-1} = H_k P_{k\mid k-1} H_k^T + \Sigma_k,$$

$$P_{k\mid k-1} = P_{k\mid k-1} H_k^T,$$

where $H_k = H_k(\hat{x}_{k\mid k-1})$.

2) Unscented transform: The UT$^5$ is a derivative-free technique for approximate computation of moments of a transformed random variable moments. The UT is based on

$^5$The statistics are generally approximate. Exact values are obtained for a few special functions $h_k(\cdot)$ only, e.g., for a linear one.

$^6$Several other advanced UT versions have been proposed, e.g., higher order, simplified, scaled, orthogonally transformed, or the smart sampling, differing in the $\sigma$-point set computation [16], [18], [19].
specification of a set of deterministic σ-points \( \{ X_i,k|k-1 \}_{i=0}^{2n_x} \) with appropriate weights \( \{ W_i \}_{i=0}^{2n_x} \) according to

\[
X_0,k|k-1 = \tilde{x}_k|k-1, W_0 = \frac{\kappa}{n_x + \kappa},
\]

\[
X_j,k|k-1 = \tilde{x}_k|k-1 + cS_{j,k|k-1}, W_j = \frac{1}{2(n_x + \kappa)},
\]

\[
X_{n_x+j}|k|k-1 = \tilde{x}_k|k-1 - cS_{j,k|k-1}, W_{n_x+j} = W_j,
\]

where \( j = 1, \ldots, n_x \), \( S_{j,k|k-1} \) is the \( j \)-th column of the matrix \( S^{XX|k|k-1} \) which is a factor of the covariance matrix \( P^{XX|k|k-1} \) so that \( S^{XX|k|k-1}(S^{XX|k|k-1})^T = P^{XX|k|k-1}, c = \sqrt{(n_x + \kappa)} \), and \( \kappa \) is a scaling parameter. To get approximate characteristics of \( z_k \), each point is propagated through the nonlinear function

\[
Z_i,k|k-1 = h_k(X_i,k|k-1), \forall i.
\]

The resulting UT-based statistics of \( z_k \) are given by

\[
Z^{UT|k|k-1} = \sum_{i=0}^{2n_x} W_i Z_i,k|k-1,
\]

\[
P_{zz,UT|k|k-1} = \sum_{i=0}^{2n_x} W_i (Z_i,k|k-1 - Z^{UT|k|k-1})(\cdot) + \Sigma^y,
\]

\[
P_{xz,UT|k|k-1} = \sum_{i=0}^{2n_x} W_i (X_i,k|k-1 - \tilde{x}_k|k-1)(Z_i,k|k-1 - Z^{UT|k|k-1})^T.
\]

3) Monte-Carlo integration: MC integration is a representative of the numerical integration rules in the LF design [20]. The basic version approaches the predictive state (assumed to be Gaussian \( p(\mathbf{x}_k|z^k) = \mathcal{N}(\mathbf{x}_k|\mathbf{x}_k|k-1, \mathbf{P}_{kk|k-1}) \)) by a set of \( N_s \) equally weighted samples \( \{ \chi_{i,k|k-1} \}_{i=0}^{N_s} \) which are transformed via a nonlinear function as

\[
\xi_i,k|k-1 = h_k(\chi_i,k|k-1), \forall i.
\]

The predictive characteristics of \( z_k \) approximated by the MC-integration are given by

\[
\tilde{z}^{MC|k|k-1} = \frac{1}{N_s} \sum_{i=0}^{N_s} \xi_i,k|k-1,
\]

\[
P_{zz,MC|k|k-1} = \frac{1}{N_s-1} \sum_{i=0}^{N_s} (\xi_i,k|k-1 - \tilde{z}^{MC|k|k-1})(\cdot) + \Sigma^y_k,
\]

\[
P_{xz,MC|k|k-1} = \frac{1}{N_s-1} \sum_{i=0}^{N_s} (X_i,k|k-1 - \tilde{x}_k|k-1)(\xi_i,k|k-1 - \tilde{z}^{MC|k|k-1})^T.
\]

C. Nonlinearity and non-Gaussian measure

The LFs provide reliable results if the approximation point is close to the working point. This is usually true if the nonlinearities in system description are mild, PDFs of the state and measurement noises are Gaussian and the filter and system initial conditions are Gaussian and close to each other. Then, the true conditional PDF of the state estimate is expected to be close to the Gaussian PDF [5] and computationally efficient LFs can be used instead of much more demanding global filters.

Assessment of the system nonlinearity or the resulting state PDF properties is, therefore, a vital part of any filter design procedure. Therefore, several nonlinearity or NGMs have been proposed so far.

The local NGMs [12] assess the impact of the nonlinear transformation on the distribution of the measurement and state predictive estimates at a single time instant assuming the previous predictive and filtering PDFs are Gaussian. As an example of a NGM, the third and higher moments, skewness, kurtosis, or the negentropy of the transformed variable can be mentioned [12]. The local nonlinearity measures, on the other hand, monitor the severity of the nonlinear functions \( f_k(\cdot) \) and \( h_k(\cdot) \) at a given approximation point specified by the state prediction estimate and filtered estimate, respectively. The local measures provide information valid at a given time only without any regard to previous time instants. Thus, the local measures might fail in detection of a sequence of mild nonlinearities resulting in the end in e.g., heavy-tailed PDF, as illustrated in [12].

In this paper, the local NGMs are adjusted to have ability to monitor time behaviour of the measure. That means, that it is an accumulation of the NGM from previous time instants.

III. DESIGN OF SELF-ASSESSMENT FOR LFS

All the NGMs are based on comparison of higher moments\(^6\) or their functions. The computed higher moments of the transformed random variable are compared with the expected moments computed on the basis of the covariance matrix of the transformed random variable assuming its Gaussian distribution\(^7\).

Therefore, the main idea of the proposed concept for the self-assessment of a LF is to propagate not only the mean and covariance matrix, but also higher moment(s), and then to check their consistency. In this paper, the third moment is selected.

A. Recursive computation of the third moment of the state

To compute the conditional third moment, recursive formulas for filtering and predictive moment have to be proposed. Let the filtering third moment be defined as

\[
M^{xxx|k|k} = E[(x_k - \tilde{x}_k|k)(x_k - \tilde{x}_k|k)^T \otimes (x_k - \tilde{x}_k|k)^T],
\]

\[
M^{xxx|k|k} = \hat{E}[(x_k - \tilde{x}_k|k)(x_k - \tilde{x}_k|k)^T \otimes (x_k - \tilde{x}_k|k)^T],
\]

where the symbol \( \otimes \) stands for the Kronecker product [21]. Then, substituting (3) into (25) and defining the error variables \( \tilde{x}_k|k-1 = x_k - \hat{x}_k|k-1 \) and \( \tilde{z}_k|k-1 = z_k - \hat{z}_k|k-1 \), equation (25) can be further treated as:

\[
M^{xxx\hat{E}} = \hat{E}[(\tilde{x} - K\tilde{z})(\tilde{x} - K\tilde{z})^T \otimes (\tilde{x} - K\tilde{z})^T]
\]

\[
= E[\tilde{x}\tilde{z}^T \otimes \tilde{x}\tilde{z}^T \otimes \tilde{x}\tilde{z}^T \tilde{x}^T \otimes \tilde{x}^T] + K\tilde{z}\tilde{z}^T + K\tilde{z}\tilde{z}^T \tilde{x}^T \tilde{x}^T - K\tilde{z}\tilde{z}^T \tilde{x}^T \tilde{x}^T - K\tilde{z}\tilde{z}^T \tilde{x}^T \tilde{x}^T + K\tilde{z}\tilde{z}^T \tilde{x}^T \tilde{x}^T \tilde{z}^T \tilde{x}^T \tilde{z}^T \tilde{x}^T \tilde{z}^T \tilde{x}^T \tilde{z}^T
\]

\[
= K\tilde{z}\tilde{z}^T \tilde{x}^T \tilde{x}^T - K\tilde{z}\tilde{z}^T \tilde{x}^T \tilde{x}^T + K\tilde{z}\tilde{z}^T \tilde{x}^T \tilde{x}^T \tilde{z}^T \tilde{x}^T \tilde{z}^T \tilde{x}^T \tilde{z}^T
\]

\[
\hat{E}[(\tilde{x} - K\tilde{z})(\tilde{x} - K\tilde{z})^T \otimes (\tilde{x} - K\tilde{z})^T]
\]

6The term higher moments means the moments of order three or higher.

7All odd higher moments of a Gaussian random variable are zero and all even moments are function of its covariance matrix.
Note that the time indices were omitted for the sake of brevity. The terms of (26) can be rewritten (for the ease of further derivations) using the relation $\text{AD} \otimes \text{BG} = (\text{A} \otimes \text{B})(\text{D} \otimes \text{G})$ [21]. For example the third term of (26) can be rewritten as

$$\tilde{z}^T K^T \otimes \tilde{x}^T = (\tilde{z} \otimes \tilde{x}^T)(K^T \otimes I_{n_x})$$

where $I_{n_x}$ denotes an $n_x \times n_x$ identity matrix. The final form of the filtering third moment (26) is

$$M^{z\tilde{z}}_{k|k} = M^{z\tilde{z}}_{k|k-1} - K_k M^{z\tilde{z}}_{k|k-1} (K_k^T \otimes I_{n_x}) + K_k M^{z\tilde{z}}_{k|k-1} (K_k^T \otimes I_{n_x}) - M^{z\tilde{z}}_{k|k-1} (I_{n_x} \otimes K_k^T) + K_k M^{z\tilde{z}}_{k|k-1} (I_{n_x} \otimes K_k^T) + M^{z\tilde{z}}_{k|k-1} (K_k^T \otimes K_k^T) - K_k M^{z\tilde{z}}_{k|k-1} (I_{n_x} \otimes K_k^T),\quad (27)$$

where $M^{z\tilde{z}}_{k|k-1} \triangleq \mathbb{E}[\tilde{z}_{k|k-1} \tilde{z}_{k|k-1} \otimes \tilde{x}^T_{k|k-1}]$ and the remaining third moments are defined analogically.

The predictive third moment of the state is defined as

$$M^{x\tilde{z}}_{k+1|k} \triangleq \mathbb{E}[(\tilde{x}_{k+1|k})(\tilde{x}_{k+1|k})^T \otimes (\tilde{x}_{k+1|k})^T].\quad (28)$$

The relation between the predictive $M^{x\tilde{z}}_{k+1|k}$ and filtering $M^{z\tilde{z}}_{k|k}$ cannot be further specified for a general LF and will be detailed for the EKF, UKF, and the MCKF further.

Relations (27) and (28) are the recursive relations for the computation of the third moment in the LF framework. Now, the relations are detailed for three approximation techniques assuming scalar case, i.e., $n_x = n_z = 1$.

1) Third moment computation in the EKF: For the scalar case (27) reduces to

$$M^{z\tilde{z}}_{k|k} = M^{z\tilde{z}}_{k|k-1} - 3K_k M^{z\tilde{z}}_{k|k-1} + 3(K_k)^2 M^{z^2}_{k|k-1} - (K_k)^3 M^{z^3}_{k|k-1},\quad (29)$$

where $M^{z^3}_{k|k-1} = \mathbb{E}[\tilde{z}_{k|k-1}^3]$, $M^{z^2}_{k|k-1} = \mathbb{E}[\tilde{z}_{k|k-1}^2 \tilde{z}_{k|k-1}]$, etc.

Following the approximation technique of the EKF given by (11)–(13), where the nonlinear function $h_k(\cdot)$ in measurement equation (2) is approximated using TE1 (10), the required third moments equal to

$$M^{z^3}_{k|k-1} = (H_k)^3(\hat{x}_{k|k-1})M^{z^3}_{k|k-1},\quad (30)$$

$$M^{z^2}_{k|k-1} = H_k(\hat{x}_{k|k-1})M^{z^2}_{k|k-1},\quad (31)$$

$$M^{z^2}_{k|k-1} = H_k(\hat{x}_{k|k-1})M^{z^2}_{k|k-1}.$$

The TE1-based linearisation ignores the information about higher order terms caused by the nonlinear transformation. In fact, under assumption of the Gaussian initial condition (imposing $M^{z^3}_{k|k-1} = 0$), the moment $M^{z^3}_{k|k}$ is zero $\forall k$.

Thus, to get the non-zero moment approximation, let the second order Taylor expansion (TE2)

$$z_k = h_k(x_k) + v_k \approx h_k(\hat{x}_{k|k-1}) + H_k(\hat{x}_{k|k-1})\tilde{x}_{k|k-1} + \frac{1}{2}H_k(\hat{x}_{k|k-1})\tilde{x}_{k|k-1}^2 + v_k,$$

be used instead of the TE1 where $H_k = H_k(\hat{x}_{k|k-1}) = \frac{d^2h_k(x_k)}{dx_k^2}|_{x_k=\hat{x}_{k|k-1}}$. The TE2 is typically used in the second order filter design [22]. Then, after tedious calculations, the required third moments (30)–(32) can be expressed as:

$$M^{z^3}_{k|k-1} = (H_k)^3M^{z^3}_{k|k-1} + M^{z^3}_{k|k-1} - 3M^{z^3}_{k|k-1}P^{zz}_{k|k-1} + 3M^{z^2}_{k|k-1}P^{zz}_{k|k-1} + 3M^{z^2}_{k|k-1}P^{zz}_{k|k-1} + 3M^{z^2}_{k|k-1}P^{zz}_{k|k-1},\quad (33)$$

$$M^{z^2}_{k|k-1} = H_k(\hat{x}_{k|k-1})M^{z^2}_{k|k-1} + \frac{1}{2}H_k(\hat{x}_{k|k-1})M^{z^2}_{k|k-1} + \frac{1}{4}H_k(\hat{x}_{k|k-1})M^{z^2}_{k|k-1},\quad (34)$$

$$M^{z^2}_{k|k-1} = H_k(\hat{x}_{k|k-1})M^{z^2}_{k|k-1} + \frac{1}{2}H_k(\hat{x}_{k|k-1})M^{z^2}_{k|k-1} + \frac{1}{4}H_k(\hat{x}_{k|k-1})M^{z^2}_{k|k-1} + \frac{1}{4}H_k(\hat{x}_{k|k-1})M^{z^2}_{k|k-1}.$$ (35)

Utilization of the TE2 in the higher moments computation requires conditional higher moments of the predictive state estimate, namely $M^{z^4}_{k|k-1}, M^{z^5}_{k|k-1}, \text{and } M^{z^6}_{k|k-1}$ which are not available. The proposed solution is to approximate those values by the theoretical moments computed under assumption of the Gaussian distribution [23], i.e., as

$$M^{z^4}_{k|k-1} = 3(P^{zz}_{k|k-1})^2,$$

$$M^{z^5}_{k|k-1} = 0,$$

$$M^{z^6}_{k|k-1} = 15(P^{zz}_{k|k-1})^3.$$ (38)

The predictive moment (28) is computed analogously to (33).

2) Third moment computation in the UKF: Contrary to the computations in the EKF framework, computation of the third moments $M^{z^3}_{k|k-1}, M^{z^2}_{k|k-1}, \text{and } M^{z^2}_{k|k-1}$ in (29) in the UKF is much more straightforward and basically follows the computation of the covariance matrices (19)–(20):

$$M^{z^3}_{k|k-1} = \sum_{i=0}^{2n_z} \mathbb{W}_i (Z_{i,k|k-1} - \tilde{z}_{k|k-1}^T)^3,$$ (39)

$$M^{z^2}_{k|k-1} = \sum_{i=0}^{2n_z} \mathbb{W}_i (Z_{i,k|k-1} - \tilde{z}_{k|k-1}^T)^2,$$ (40)

$$M^{z^2}_{k|k-1} = \sum_{i=0}^{2n_z} \mathbb{W}_i (Z_{i,k|k-1} - \tilde{z}_{k|k-1}^T)^2.$$ (41)

and then, used in (29). The predictive moment of the state (28) is again computed similarly to (39).

Note that the third moments can generally be computed for any value of the scaling parameter $\kappa$ except of $\kappa = 0$. In this case, all considered third moments are equal to zero irrespectively of the nonlinear functions.

3) Third moment computation in the MCKF: Considering the MC integration technique (22)–(24) used by the MCKF, the moments required by (29) are computed as

$$M^{z^3}_{k|k-1} = C \sum_{i=0}^{N_z} (\xi_{i,k|k-1} - \tilde{z}_{k|k-1}^\text{MC})^3,$$ (42)

$$M^{z^2}_{k|k-1} = C \sum_{i=0}^{N_z} (\xi_{i,k|k-1} - \tilde{z}_{k|k-1}^\text{MC})^2(\xi_{i,k|k-1} - \tilde{z}_{k|k-1}^\text{MC}),$$ (43)

$$M^{z^2}_{k|k-1} = C \sum_{i=0}^{N_z} (\xi_{i,k|k-1} - \tilde{z}_{k|k-1}^\text{MC})^2(\xi_{i,k|k-1} - \tilde{z}_{k|k-1}^\text{MC}),$$ (44)

The phenomenon is referred in the literature as the moment closure problem [2].
where \( C = \frac{N_s}{N_s^2 - 3N_s^{-2}} \) is a normalisation ensuring unbiased estimates of the third moments. Again analogously to (40), predictive moment (28) is computed.

### B. Self-assessment based on the third moment

Many LFs assume Gaussian distribution of the filtering and predictive PDF. However, in case of non-linearities in the system description, the PDFs deviate from being Gaussian, which may consequently lead to decreasing estimate quality. Deviation of the PDF from the Gaussian distribution can be assessed by the third filtering and predictive moments. As the theoretical third moment of a Gaussian distributed random variable is \( M_{k+1}^{xxx} = 0_{n_x \times 2n_x} \), with \( 0_{a \times b} \) is a zero matrix of marked dimensions, a criterion based on the computed and theoretical third moment of the state can be defined

\[
J_3 = \gamma(M_{k+1}^{xxx}, M_{k+1}^{xxx,G}).
\]

An example of the function \( \gamma(\cdot) \) is a sum of absolute values of differences between computed and theoretical third moments of particular states.

The value of the criterion is then compared with a threshold \( J_3,\text{thr} \) and \( J_3 \geq J_3,\text{thr} \) indicates that the corresponding PDF is far from being Gaussian and the estimate quality is possibly low. The threshold \( J_3,\text{thr} \) needs to be specified on the basis of an off-line analysis or a simulation study. An indication that a PDF is far from being Gaussian can be used for two purposes. First, it may serve as an information for the user that the filter may not provide accurate results. Second, the information may be used by the filter itself and may lead to an adaptation of the filter behavior. This adaptation is, however, out of the scope of the paper.

Note that it is possible to reduce computational complexity by selection of only several specific elements of \( M_{xxx} \) which are recursively computed.

### C. Incorporating the NGM-based self-assessment into the LF algorithm

Having the relations necessary for the recursive state third moment computation, the algorithm of the LF with self-assessment can be proposed.

#### Algorithm 2: Local Filter with Self-Assessment

**Step 1:** (initialization) as Step 1 of Algorithm 1, supplemented with specification of initial third moment estimate. If initial condition is Gaussian, then \( M_{0-k}^{xxx} = 0_{n_x \times 2n_x} \).

**Step 2:** (filtering) as Step 2 of Algorithm 1, supplemented with computation of \( M_{k|k}^{xxx} \) according to (27).

**Step 3:** (self-assessment) \( J_3 = \gamma(M_{k+1|k}^{xxx}, M_{k+1|k}^{xxx,G}) > J_3,\text{thr} \) indicates possibly low estimate quality.

**Step 4:** (prediction) as Step 3 of Algorithm 1, supplemented with computation of \( M_{k+1|k}^{xxx} \) according to (28).

**Step 5:** (self-assessment) \( J_3 = \gamma(M_{k+1|k}^{xxx}, M_{k+1|k}^{xxx,G}) > J_3,\text{thr} \) indicates possibly low estimate quality.

Let \( k = k + 1 \) and algorithm continues by Step 2.

### D. Discussion

In the EKF framework, the fourth to sixth moments of the state were computed on the basis of the estimated covariance matrix as the true values are not at disposal. These moments are thus computed under assumption of the Gaussian PDF which is an additional simplification to the nonlinear function linearisation. In the UKF and MCKF, there is also analogous simplification as the \( \sigma \)-points or the samples to be transformed are at each time instant (in both the filtering and the predictive step) generated from a symmetric (in MCKF even from Gaussian) PDF without any regard to the actual third moment. However, detailed discussion of this topic is out of the scope of this paper.

### IV. Numerical Illustration

Performance of the proposed LFs with self-assessment is illustrated using a system described by

\[
x_{k+1} = 0.5x_k + \frac{25x_k}{1+x_k^2} + 20\sin(0.05k) + w_k,
\]

\[
z_k = \frac{x_k^2}{20} + v_k,
\]

with the initial condition \( x_0 = -200, P_0 = 1 \), noise variances \( \Sigma_w = 20 \) and \( \Sigma_v = 1, \forall k, \) and \( k = 0, 1, ..., K \), where \( K = 40 \). The model is a modified version of the model used in [20] where the initial condition were set to force the true state to cross the \( x \)-axis. In the neighbourhood of that crossing point, the performance of LFs is significantly degraded as the filter cannot determine whether the state is positive or negative (because of the quadratic function in (43)). Three LFs with the self-assessment are considered, namely the EKF, UKF, and the MCKF. In Fig. 1, an example of true trajectory and its filtering estimate are shown. It can be seen that if the true state switches from negative to positive values a LF gives a negative state estimate whereas the true state is positive. In such time instants, the estimated third moment \( M_{k|k}^{xxx} \) (29) reaches extreme values. The criterion function is chosen as \( J_3 = \gamma(M_{k|k}^{xxx}, M_{k|k}^{xxx,G}) = |M_{k|k}^{xxx}| \) and the threshold as \( J_3,\text{thr} = 2 \). The time instants when \( J_3 \geq J_3,\text{thr} \) is indicated are highlighted in Fig. 1 by green circles. These indications coincide with the time instants where the state changes its sign.

In Table I, average absolute state estimate errors, i.e.,

\[
\text{AAE} = \frac{1}{MK} \sum_{m=1}^{M} \sum_{k=0}^{K} |x_{k|k}^{(m)} - \hat{x}_{k|k}^{(m)}|
\]

and average absolute third moment, i.e.,

\[
\text{A3M} = \frac{1}{MK} \sum_{m=1}^{M} \sum_{k=0}^{K} |M_{k|k}^{xxx,(m)}|
\]

are given for a set of \( M = 100 \) MC simulations. The index \( (m) \) in (44) and (45) denotes the MC simulation.

Within each MC simulation, the realisation was split into two parts; time instants when \( J_3 \geq J_3,\text{thr} \), i.e., with indications, and time instants when \( J_3 < J_3,\text{thr} \), i.e., without indications. For both parts, the AAE and A3M were computed. The results reveal, significant difference between the
estimate quality at the time instant with the indication (the LF is expected to have low estimate quality) and without indication (the LF is expected to perform well). The results indicate that the estimate quality coincides with the value of NGM assessing the third moment of the state estimate.

Note that the UKF was run with $k = 2$ and the MCKF and EnKF were run with a set of $N_s = 10^5$ samples. As far as computational costs of the NGM calculation are concerned, they make 43, 11, and 56 percent of the computational costs of the EKF, UKF, and MCKF, respectively. For the MCKF, this percentage is strongly affected by the number of samples.

V. CONCLUDING REMARKS

The paper dealt with local filters in state estimation of stochastic nonlinear systems. A technique for self-assessment of the local filters was proposed based on non-Gaussian measures, with focus on filtering third moment of the state. The technique utilised the third moment to monitor deviation of the filtering PDF from the Gaussian distribution caused by nonlinearities in the system description. Non-Gaussianity of the filtering PDF often violates assumptions of the filters which in consequence tend to provide low quality estimates in such cases. In the paper a computation of the third moment in a generic local filter was shown with detailed specification for the EKF, the UKF, and the MCKF. The NGM based self-assessment of the filters was illustrated in an example.

TABLE I
VALUES OF AAE AND A3M FOR TIME INSTANTS WITH AND WITHOUT INDICATION OF $J_3 \geq J_{3,thr}$.

<table>
<thead>
<tr>
<th></th>
<th>AAE</th>
<th>A3M</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>with ind.</td>
<td>w/o ind.</td>
</tr>
<tr>
<td>EKF</td>
<td>88.95</td>
<td>5.33</td>
</tr>
<tr>
<td>UKF</td>
<td>4.52</td>
<td>0.86</td>
</tr>
<tr>
<td>MCKF</td>
<td>4.33</td>
<td>0.34</td>
</tr>
</tbody>
</table>

Fig. 1. Example of true and estimated state with indication of $J_3 \geq J_{3,thr}$ and value of the criterion $J_3$.

REFERENCES