

# On the Probability that a Group Element Fixes a Set and its Generalized Conjugacy Class Graph

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## Abstract

In this paper,  $\Omega$  denotes the set of all subsets of commuting elements of  $G$  in the form of  $(a, b)$ , where  $a$  and  $b$  commute,  $|a| = |b| = 2$ . The probability that a group element of  $G$  fixes a set is one of the generalizations of the commutativity degree that has been recently introduced. In this paper, the probability that an element of a group fixes a set for semi-dihedral groups and quasi-dihedral groups is found. The results obtained are then applied to graph theory, more precisely to the generalized conjugacy class graph.

**Mathematics Subject Classification:** 20P05, 20B40, 97K30

**Keywords:** Commutativity degree, generalized conjugacy class graph, group actions, semi-dihedral groups, quasi-dihedral groups

## 1 Introduction

This section provides some backgrounds related to the commutativity degree and graph theory. Starting with the commutativity degree. The probability

that two elements of  $G$  randomly chosen commute is called the commutativity degree, denoted by  $P(G)$ . Gustafson [1] and MacHale [2] proved that  $P(G) \leq \frac{5}{8}$ . The commutativity degree has been generalized and extended by several authors, where various results have been obtained. Omer *et al.* [3] have extended the commutativity degree by defining the probability that a group element fixes a set. Extension has been done on the work in [3] by Mustafa *et al.* [4], where the size of the fixed set was restricted.

The following are some basic concepts related to graph theory which will be used in the later discussion.

A graph  $\Gamma$  is a mathematical structure consisting of two sets, namely vertices and edges which are denoted by  $V(\Gamma)$  and  $E(\Gamma)$ , respectively. A connected graph is a graph in which there is a partition of vertex  $V$  into non empty subsets,  $V_1, V_2, \dots, V_n$  such that two vertices  $V_1$  and  $V_2$  are connected if and only if they belong to the same set  $V_i$ . Subgraphs  $\Gamma(V_1), \Gamma(V_2), \dots, \Gamma(V_n)$  are all components of  $\Gamma$ . The graph  $\Gamma$  is connected if it has precisely one component. However, a graph is a complete graph if each ordered pair of distinct vertices is adjacent, and it is denoted by  $K_n$ , where  $n$  is the number of adjacent vertices. The graph is called empty if there is no adjacent between its vertices and it is denoted by  $K_e$ . In addition, a graph is called null if it has no vertices and in this paper we denote  $K_0$  as the null graph ([5], [6]).

Moreover, a non-empty set  $S$  of  $V(\Gamma)$  is called an *independent set* of  $\Gamma$  if there is no adjacent between two vertices of  $S$  in  $\Gamma$ . Meanwhile, the independent number is the number of vertices in maximum independent set and it is denoted by  $\alpha(\Gamma)$ . The maximum number  $c$  for which  $\Gamma$  is  $c$ -vertex colorable is known as chromatic number and is denoted by  $\chi(\Gamma)$ . The diameter is the maximum distance between any two vertices of  $\Gamma$  and  $d(\Gamma)$  is used as a notation. In addition, a complete subgraph in  $\Gamma$  is called a clique, while the clique number is the size of the largest clique in  $\Gamma$  and is denoted by  $\omega(\Gamma)$ . The *dominating set*  $X \subseteq V(\Gamma)$  is a set where for each  $v$  outside  $X$ ,  $\exists x \in X$  such that  $v$  is adjacent to  $x$ . The minimum size of  $X$  is called the dominating number, denoted by  $\gamma(\Gamma)$  ([5], [6]).

Since the groups under consideration in this paper are semi-dihedral groups and quasi-dihedral groups, the presentations of these groups are given in the following definitions.

**Definition 1.1** [7] *A finite group  $G$  is a semi-dihedral 2-group of order  $2^{n+1}$ , where  $n \geq 3$ . Thus,  $G$  has the following presentation:  $G \cong \langle a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}-1} \rangle$ , where  $n \geq 3$ .*

**Definition 1.2** [7] *A finite group  $G$  is a quasi-dihedral 2-group of order  $2^{n+1}$ , where  $n \geq 3$ . Thus,  $G$  has the following presentation:  $G \cong \langle a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}+1} \rangle$ , where  $n \geq 3$ .*

This paper is divided into three sections. The first section focuses on some background about the commutativity degree and graph theory, while the second section provides some earlier and recent publications that are related to the commutativity degree and graph theory, more specifically to generalized conjugacy class graph. The third section presents the results of this paper, which include the probability that a group element fixes a set and generalized conjugacy class graph.

## 2 Preliminary Notes

In this section, some works that are related to the probability that an element of a group fixes a set and graph theory are given. Brief information about the probability of a group element fixes a set, followed by some related work on graph theory, more specifically to graph related to conjugacy classes. In 2013, the probability that a group element fixes a set, denoted by  $P_G(\Omega)$  was firstly introduced by Omer *et al.* [3]. The probability was found for some finite non-Abelian groups including metacyclic 2-groups [8], symmetric groups and alternating groups [9]. Recently, Mustafa *et al.* [4] extended the work in [3] by restricting the order of the set  $\Omega$ . The following theorem illustrates their work.

**Theorem 2.1** [4] *Let  $G$  be a finite group. Let  $S$  be a set of elements of  $G$  of size two in the form of  $(a, b)$  where  $a$  and  $b$  commute,  $|a| = |b| = 2$ . Let  $\Omega$  be the set of all subsets of commuting elements of  $G$  of size two and  $G$  acts on  $\Omega$ . Then the probability that an element of a group fixes a set is given by:*

$$P_G(\Omega) = \frac{K(\Omega)}{|\Omega|},$$

where  $K(\Omega)$  is the number of orbits of  $\Omega$  in  $G$ .

The work in [4] has then been extended by finding the probability for some finite non-abelian groups such as the metacyclic 2-groups of negative type of nilpotency class two and class at least three [10]. In addition, Mustafa *et al.* computed the probability that a group element fixes a set for metacyclic 2-groups of positive type of nilpotency class at least three [11]. In this paper, the results obtained from the probability that a group element fixes a set are then applied to graph theory. Next, some related works on conjugacy class graph are presented in the following.

Bianchi *et al.* [12] studied the regularity of the graph related to conjugacy classes and provided some results. In 2005, Moreto *et al.* [13] classified the groups in which conjugacy classes sizes are not coprime for any five distinct classes. You *et al.* [14] classified the groups in which conjugacy classes are not

set-wise relatively prime for any four distinct classes. In 2013, Moradipour *et al.* [15] used the graph related to conjugacy classes to find some graph properties of some finite metacyclic 2-groups. The graph related to conjugacy classes was also extended by Omer *et al.* [16], in which the vertices are orbits under the group action on a set. The following is the definition of the generalized conjugacy class graph.

**Definition 2.2** [16] *Let  $G$  be a finite non-abelian group and  $\Omega$  is a set of  $G$ . If  $G$  acts on  $\Omega$ , then the number of vertices of generalized conjugacy classes graph is  $|V(\Gamma_G^{\Omega_c})| = K(\Omega) - |A|$ , where  $A$  is  $\{g\omega = \omega g : \omega \in \Omega\}$ . Two vertices  $\omega_1$  and  $\omega_2$  in  $\Gamma_G^{\Omega_c}$  are adjacent if their cardinalities are not coprime.*

### 3 Main Results

In this section, the probability that an element of a group fixes a set is computed. The results that are obtained from the probability are then applied to graph theory, more precisely to the generalized conjugacy class graph.

#### 3.1 The probability that an element of a group fixes a set

We begin this section with the first result on the presentation of semi-dihedral groups. In the following theorems, let  $S$  be a set of elements of  $G$  of size two in the form of  $(a, b)$  where  $a$  and  $b$  commute and  $|a|, |b| = 2$ . Let  $\Omega$  be the set of all subsets of commuting elements of  $G$  of size two.

**Theorem 3.1** *Let  $G$  be a semi-dihedral group,  $G \cong \langle a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}-1} \rangle$ , where  $n \geq 3$ . If  $G$  acts on  $\Omega$  by conjugation, then  $P_G(\Omega) = \frac{2}{|\Omega|}$ .*

**proof 3.2** *The elements of  $G$  of order two are  $a^{2^{n-1}}$  and  $a^i b, 0 \leq i \leq 2^n$  and  $i$  is even. Thus, the elements of  $\Omega$  of size two are described as follows: There are  $2^{n-1}$  elements in the form of  $(a^{2^{n-1}}, a^i b), 0 \leq i \leq 2^n$  where  $i$  is even and there are  $2^{n-2}$  elements in the form of  $(a^i b, a^{i+2^{n-1}}), 0 \leq i \leq 2^n$  where  $i$  is even. It follows that,  $|\Omega| = 2^{n-2} + 2^{n-1} = 3(2^{n-2})$ . If  $G$  acts on  $\Omega$  by conjugation, then  $cl(\omega) = g\omega g^{-1}, g \in G$  and  $\omega \in \Omega$ . Then, the orbits under group action on  $\Omega$  can be described as follows: There is one orbit in the form of  $(a^{2^{n-1}}, a^i b), 0 \leq i \leq 2^n$  where  $i$  is even and one orbit is in the form of  $(a^i b, a^{i+2^{n-1}}), 0 \leq i \leq 2^n$  where  $i$  is even. Using Theorem 2.1,  $P_G(\Omega) = \frac{2}{|\Omega|}$ , as desired.*

Next,  $P_G(\Omega)$  of quasi-dihedral groups is computed.

**Theorem 3.3** *Let  $G$  be a quasi-dihedral group,  $G \cong \langle a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}+1} \rangle$ , where  $n \geq 3$ . If  $G$  acts on  $\Omega$  by conjugation, then  $P_G(\Omega) = \frac{2}{|\Omega|}$ .*

**proof 3.4** *The elements of  $G$  of order two are  $a^{2^{n-1}}, b$  and  $a^{2^{n-1}}b$ . Then, the elements of  $\Omega$  are described as follows: There are two elements in the form of  $(a^{2^{n-1}}, a^{2^{n-1}i}b), 0 \leq i \leq 2^n$ , and there is only one element in the form of  $(a^{2^{n-1}}b, b)$ , from which it follows that,  $|\Omega| = 3$ . If  $G$  acts on  $\Omega$  by conjugation, then  $cl(\omega) = g\omega g^{-1}, g \in G$  and  $\omega \in \Omega$ . Then, the orbits under group action on  $\Omega$  can be described as follows: There is one orbit in the form of  $(a^{2^{n-1}}, a^{2^{n-1}i}b), 0 \leq i \leq 2^n$  and one orbit is in the form of  $(a^{2^{n-1}}b, b)$ . Hence, there are two orbits under group action of  $G$  on  $\Omega$ . Using Theorem 2.1,  $P_G(\Omega) = \frac{2}{|\Omega|}$ , as desired.*

In the next section, the obtained results are associated to graph theory, more precisely to generalized conjugacy class graph.

### 3.2 The Generalized Conjugacy Class Graph

In this section, the results obtained are applied to generalized conjugacy class graph. First, the following theorem illustrates the case when the generalized conjugacy class graph is a null graph.

**Theorem 3.5** *Let  $G$  be a finite non-abelian group and let  $\Omega$  be the set of all subsets of commuting elements of  $G$  of size two. Let  $G$  act on  $\Omega$  by conjugation. Then  $P_G(\Omega) = 1$  if and only if  $\Gamma_G^{\Omega_c}$  is a null graph.*

**proof 3.6** *First, if  $P_G(\Omega) = 1$ , then by Theorem 3.1 in [11] all commuting elements in  $G$  are in the center of  $G$ . Thus  $|\Omega| = |A| = K(\Omega)$ , therefore  $|V(\Gamma_G^{\Omega_c})| = 0$ . Now, if  $|V(\Gamma_G^{\Omega_c})| = 0$ , then  $K(\Omega) = |A|$  which means  $|\Omega| = |A|$ . Thus,  $P_G(\Omega) = 1$ , as claimed.*

Next, the generalized conjugacy class graph of semi-dihedral groups and quasi-dihedral groups is determined, starting with semi-dihedral groups.

**Theorem 3.7** *Let  $G$  be a finite non-abelian semi-dihedral group,  $G \cong \langle a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}-1} \rangle$ . If  $G$  acts on  $\Omega$  by conjugation, then  $\Gamma_G^{\Omega_c} = K_2$ .*

**proof 3.8** *Using Theorem 3.1, there are two orbits, for the group  $G$  in which one of them has size  $2^{n-2}$  and one of them has size  $2^{n-1}$ . Therefore, the number of vertices in  $\Gamma_G^{\Omega_c}$  is  $|V(\Gamma_G^{\Omega_c})| = 2$ . Since two vertices are joined by an edge if their cardinalities are set-wise relatively prime, hence  $\gcd(2^{n-1}, 2^{n-2}) \neq 1$ . Therefore,  $\Gamma_G^{\Omega_c}$  consists of one complete graph of  $K_2$ . The proof then follows.*

**Theorem 3.9** *Let  $G$  be a finite non-abelian quasi-dihedral group,  $G \cong \langle a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}+1} \rangle$ . If  $G$  acts on  $\Omega$  by conjugation, then  $\Gamma_G^{\Omega_c} = K_e$ .*

**proof 3.10** *In accordance with Theorem 3.3 and Definition 2.2, there are two orbits. One of them has size two and another one has size one. By Definition 2.2, the number of vertices are two. Based on the adjacency of vertices  $\gcd(\omega_i, \omega_j) = 1$ , where  $\omega_i$  and  $\omega_j \in \Gamma_G^{\Omega_c}$ . Thus  $\Gamma_G^{\Omega_c}$  consists of two isolated vertices. Hence,  $\Gamma_G^{\Omega_c}$  is an empty graph.*

## 4 Conclusion

In this paper, the probability that an element of a group fixes a set is found for semi-dihedral groups and quasi-dihedral groups. The results obtained are applied to graph theory, specifically to generalized conjugacy class graph. Moreover, we provide the necessary and sufficient condition for the generalized conjugacy class graph to be null.

**Acknowledgements.** The first author would like to acknowledge the Ministry of Higher Education in Libya for the scholarship. The authors would also like to acknowledge Ministry of Education (MOE) Malaysia and Universiti Teknologi Malaysia for the financial funding through the Research University Grant (GUP) Vote No 08H07.

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**Received: November 11, 2014; Published: January 12, 2015**