Endomorphism monoids of chained graphs

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Abstract

For any complete chain \( I \) whose distinct elements are separated by cover pairs, and for every family \( \{ M_i \mid i \in I \} \) of monoids, we construct a family of graphs \( \{ G_i \mid i \in I \} \) such that

\[
G_i = \bigcap \{ G_j \mid j \in I, i < j \},
\]

and

\[
G_i = \bigcup \{ G_j \mid j \in I, j < i \}.
\]

the endomorphism monoid of \( G_i \) is isomorphic to \( M_i \) for all \( i \in I \).

An analogous result is proved also for quotient graphs, and both results are applied to certain varieties of finitary algebras.

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0. Introduction

A graph is a pair \( G = (V, E) \) in which \( V \) is its vertex set, and the set \( E \) of its edges consists of some two-element subsets of \( V \). Such pairs form the object class of the category \( \text{GRA} \), and a mapping \( f : V \to V' \) is a \( \text{GRA} \)-morphism from \( G = (V, E) \) to \( G' = (V', E') \) whenever \( \{ f(v), f(w) \} \in E' \) for every \( \{ v, w \} \in E \). \( \text{GRA} \)-morphisms are sometimes called graph homomorphisms.

For any monoid \( M \), there is a graph \( G \) whose monoid \( \text{End} G \) of all \( \text{GRA} \)-endomorphisms is isomorphic to \( M \) and, in fact, there are such graphs of all sufficiently large cardinalities; see [15]. This is one of the consequences of the fact that the category \( \text{GRA} \) of graphs is algebraically universal (alg-universal), meaning that every full category of algebras is isomorphic to its full subcategory [15].

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Let $G = (V,E)$ and $G' = (V',E')$ be graphs. If $V' \subseteq V$ and if $E'$ consists of all two-element sets $\{v,w\} \subseteq V'$ for which $\{v,w\} \in E$, we say that $G'$ is an induced subgraph of $G$. An induced subgraph $G'$ of $G$ is proper if $V'$ is a proper subset of $V$. If $f : V \rightarrow V'$ is a surjective morphism for which $E' = \{ \{ f(v), f(w) \} | \{v,w\} \in E \}$, we say that $G'$ is a quotient graph of $G$. A quotient graph $G'$ of $G$ is proper if $f$ is not one-to-one.

Endomorphism monoids of graphs and their induced subgraphs are fully independent in the sense that for arbitrarily selected monoids $M$ and $M'$, there exist a graph $G$ and its induced subgraph $G'$ such that $\text{End} G \cong M$ and $\text{End} G' \cong M'$, see [6], for instance. An analogous result holds for a graph and its quotient; see [1].

This paper extends the following result by Adams et al. [2].

**Theorem (Adams et al. [2]).** Let $\alpha$ be an ordinal and let $\{ M_\gamma | \gamma \leq \alpha \}$ be a system of monoids. Then there exists a system $\{ G_\gamma | \gamma \leq \alpha \}$ of graphs such that

(a) for $\gamma, \gamma' \leq \alpha$, the graph $G_\gamma$ is a proper induced subgraph of $G_{\gamma'}$ if and only if $\gamma < \gamma'$;

(b) if $\lambda \leq \alpha$ is a limit ordinal, then $G_\lambda = \bigcup \{ G_\gamma | \gamma < \lambda \}$;

(c) $\text{End} G_\gamma$ is isomorphic to $M_\gamma$ for every $\gamma \leq \alpha$.

In particular, the endomorphism monoid of the union of a chain of graphs well-ordered by inclusion is fully independent of the endomorphism monoids of individual members of that chain. Having noted that any chain isomorphic to a successor ordinal is complete, we are led to the definition below.

**Definition.** Given a complete chain $I$, we say that a system $\{ G_i | i \in I \}$ of graphs is an exact chain of subgraphs whenever

1. $G_i$ is a proper induced subgraph of $G_i'$ if and only if $i < i'$ in $I$;
2. if $i = \inf \{ i' \in I | i' > i \}$, then $G_i = \bigcap \{ G_{i'} | i' > i \}$;
3. if $i = \sup \{ i' \in I | i' < i \}$, then $G_i = \bigcup \{ G_{i'} | i' < i \}$.

Not every complete chain $I$ indexes an exact chain of subgraphs.

A chain $I$ is nowhere dense if any two of its elements are separated by a cover pair, that is, if for any $j < j'$ in $I$ there exist $q, q' \in I$ such that $j \leq q < q' \leq j'$ and there are no elements of $I$ strictly between $q$ and $q'$. It is clear that any successor ordinal is a complete chain that is nowhere dense.

**Observation.** If $\{ G_i | i \in I \}$ is an exact chain of subgraphs, then its indexing chain $I$ is nowhere dense.

**Proof.** Suppose that the elements $j < j'$ of $I$ are not separated by a cover pair. Then, by (1), there exists a vertex $v$ of $G_{j'}$ that does not belong to $G_j$. Set $K = \{ i \in I | v \in G_i \}$. Then clearly $j' \in K$ and $j \notin K$. The element $k = \inf K$ of $I$ thus satisfies $j \leq k \leq j'$. We
claim that \( v \in G_k \). This is clear when \( k = j' \). If \( k < j' \), then the element \( k \in I \) has no cover, and hence \( k = \inf \{ i' \in I \mid i' > k \} \). From (2), it follows that \( v \in G_k \) again. Next, since \( j < k \), the element \( k \in I \) is not a cover, and hence \( k = \sup \{ i' \mid i' < k \} \). But then \( v \in G_{i'} \) for some \( i' < k \), by (3), and this contradicts the definition of \( k \).

We aim to show that the members of any nowhere dense exact chain of subgraphs, and of (dually defined) nowhere dense exact chain of quotient graphs can have arbitrarily assigned endomorphism monoids.

The paper is divided into three parts. Section 1 presents all needed graph-theoretical notions and collects useful facts. The construction of a nowhere dense exact chain of subgraphs and a nowhere dense exact chain of quotient graphs with prescribed endomorphism monoids is presented in Section 2 and summarized there by Theorem 2.6. The concluding Section 3 first replaces chains of graphs by chains of directed graphs, and then applies known categorical representation results to generalize and extend the main result of [2] to certain varieties of finitary algebras.

1. Preliminaries

Any poset \( I \) may be viewed as a small category whose objects are the elements of \( I \) and, for any \( x, y \in I \), there exists at most one \( I \)-morphism \( I_{xy} \) from \( x \) to \( y \), and this is the case exactly when \( x \leq y \).

A chain of graphs is a functor \( T \) from some chain (that is, a linearly ordered poset) \( I \) to \( \text{GRA} \). For any \( x \in I \), let \( I_x^- \) denote the subposet of \( I \) on the set \( \{ y \in I \mid y < x \} \), and let \( I_x^+ \) denote the subposet of \( I \) on the set \( \{ y \in I \mid y > x \} \).

We recall that a chain of graphs \( T : I \to \text{GRA} \) is an exact chain of subgraphs if \( I \) is a complete chain and \( T \) has these three properties:

1. for any \( x < y \) in \( I \), \( T(x) \) is a proper induced subgraph of \( T(y) \) and \( T(I_{xy}) : T(x) \to T(y) \) is the inclusion map;
2. if \( x \in I \) is such that \( \inf I_x^+ = x \), then \( T(x) = \bigcap \{ T(v) \mid v \in I_x^+ \} \);
3. if \( x \in I \) is such that \( \sup I_x^- = x \), then \( T(x) = \bigcup \{ T(u) \mid u \in I_x^- \} \).

Now we turn to a definition of the dual case of chains of quotient morphisms.

An equivalence \( \theta \) on the vertex set \( V \) of a graph \((V, E)\) is a congruence on \((V, E)\) if

\[
\text{(f)} \quad \forall \theta v' \quad \text{only when } \{v, v'\} \notin E.
\]

Denote \( W = V/\theta \) and for the surjective mapping \( f : V \to W \) corresponding to \( \theta \) define

\[
F = \{ \{ f(v), f(v') \} \subseteq W \mid \{v, v'\} \in E \}.
\]

Then \((W, F)\) is a quotient graph of \((V, E)\) and \( f \) is a quotient morphism. We shall write \((W, F) = (V, E)/\theta\).

Let \( I \) be a chain. Suppose now that \( T : I \to \text{GRA} \) is a functor such that \( T(I_{xy}) : T(x) \to T(y) \) is a quotient \( \text{GRA} \)-map whenever \( x \leq y \) in \( I \), and write \( T(x) = (V_x, E_x) \).
for every $x \in I$. We say that an element $t = (t_x \mid x \in I)$ of the Cartesian product

\[ \prod \{ V_x \mid x \in I \} \]

is a $T$-string if $T(I_{xy})(t_x) = t_y$ for all $x, y \in I$ with $x \leq y$.

A graph $(V, E)$ is called a co-union of a chain $T$ of quotient graphs if $V$ is the set of all $T$-strings and $E = \{ \{t, v\} \subseteq V \mid \{t_x, v_x\} \in E_x \text{ for all } x \in I \}$. Then for every $x \in I$ a mapping $\phi_x : V \to V_x$ is a morphism from $(V, E)$ to $(V_x, E_x)$, called the $x$th co-union map. Since $T(I_{xy}) : V_x \to V_y$ is a quotient morphism whenever $x \leq y$ in $I$, a simple argument based on Zorn’s lemma shows that every $\phi_x : (V, E) \to (V_x, E_x)$ is surjective, and it is then easy to see that $\phi_x$ is a quotient morphism.

Next, on the set $V$ of all $T$-strings we define a relation $\theta$ by the requirement that $t \theta v$ if and only if $t_x = v_x$ for some $x \in I$. Since $I$ is a chain, the relation $\theta$ is an equivalence on the set $V$. For $v \in V$, let $[v]$ denote the class of $\theta$ containing $v$. Denote $W = V/\theta$ and $F = \{ \{[t], [v]\} \mid \{t_x, v_x\} \in E_x \text{ for some } x \in I \}$. Observe that $\theta$ is a congruence: if $\{t, v\} \in E$, then $\phi_x(t) = t_x = \phi_x(v)$ and $\{t_x, v_x\} \in E_x$ for all $x \in I$, and hence $[t] \neq [v]$. Since $\phi_x(t) = \phi_x(v)$ for some $x \in I$ implies $t \theta v$, for every $x \in I$ there exists a unique mapping $\psi_x : V_x \to W$ with $\phi_x(t) = \psi_x(t)$ for all $x \in I$ and $t \in V$. From the definition of $F$ it follows that $\psi_x : (V_x, E_x) \to (W, F)$ is a quotient morphism for every $x \in I$. We say that $(W, F)$ is the co-meet of $T$ and $\psi_x$ is the $x$th co-meet map.

**Definition 1.1.** Let $I$ be a complete chain. Then a chain of graphs $T : I \to \mathcal{GRA}$ is an exact chain of quotient graphs if it has these three properties:

1. For any $x < y$ in $I$, $T(y)$ is a proper quotient of $T(x)$ and $T(I_{xy}) : T(x) \to T(y)$ is the quotient morphism;
2. if $x \in I$ is such that $\inf I^+_x = x$, then $T(x)$ is a co-union of $T \upharpoonright I^+_x$ and $T(I_{xy})$ is the $y$th co-union map for each $y \in I^+_x$;
3. if $x \in I$ is such that $\sup I^-_x = x$, then $T(x)$ is a co-meet of $T \upharpoonright I^-_x$ and $T(I_{xy})$ is the $y$th co-meet map for all $y \in I^-_x$.

An argument analogous to the proof of the Observation in the introductory section shows that a complete chain $I$ indexes an exact chain of quotient graphs only when it is nowhere dense.

Next we recall the notions and results that will be used in the two construction of exact chains of graphs with prescribed endomorphism monoids.

If $\text{End } G$ is the one-element monoid then we say that the graph $G$ is rigid. Two graphs $G$ and $H$ are incomparable if there are no $\mathcal{GRA}$-morphisms between $G$ and $H$.

We say that a graph $(V, E)$ is connected if for any two vertices $v$ and $w$ of $V$ there exists a sequence of edges $\{v_i, v_{i+1}\} \mid i = 0, 1, \ldots, m - 1$ from $E$ such that $v_0 = v$ and $v_m = w$. A subset $U \subseteq V$ is called a component of $(V, E)$ if the induced subgraph $(U, E_U)$ is a maximal connected induced subgraph of $(V, E)$. The components of a graph $(V, E)$ form a decomposition of $V$, and any morphism $f : (V, E) \to (W, F)$ maps each component of $(V, E)$ into some component of $(W, F)$.
A graph \((V, E)\) is an \(m\)-clique if the cardinality of \(V\) is \(m\) and \(E\) is the set of all two-element subsets of \(V\). We say that a graph \((V, E)\) is \(k\)-clique connected, for a natural number \(k \geq 3\), if \(V\) is not a singleton and for every pair of distinct vertices \(v, w \in V\) there exists a sequence of induced subgraphs \(\{(V_i, E_i) | i = 0, 1, \ldots, p - 1\}\) of \((V, E)\) such that \(v \in V_0\), \(w \in V_{p-1}\), every \((V_i, E_i)\) is a \(k\)-clique, and the subgraph of \((V, E)\) induced on \(V_i \cap V_{i+1}\) is a \((k - 1)\)-clique for every \(i = 0, 1, \ldots, p - 2\).

For an integer \(k \geq 3\), a subset \(U \subseteq V\) is a \(k\)-clique component of the graph \(G = (V, E)\) if the induced subgraph \((U, E_0)\) of \((V, E)\) is a maximal \(k\)-clique connected induced subgraph of \((V, E)\). Let \(\text{Comp}_k(G)\) denote the set of all \(k\)-clique components of the graph \(G\), and for a \(k\)-clique connected graph \(H\), let \(\text{Comp}_k(G; H)\) denote the set of all \(k\)-clique components of \(G\) inducing a subgraph isomorphic to \(H\). Observe that \(k\)-clique components need not be disjoint and need not cover the set \(V\). Since \(E\) consists of two-element subsets of \(V\) for every graph \((V, E)\), it follows that every morphism \(f : (V, E) \rightarrow (W, F)\) maps any \(k\)-clique component of \((V, E)\) into some \(k\)-clique component of \((W, F)\).

Let \(G = (V, E)\) be a graph and let \(k \geq 3\). We say that a \(k\)-clique component \(U \in \text{Comp}_k(G)\) and a vertex \(v \in V\) are adjacent if \(\{u, v\} \in E\) for some \(u \in U\).

The following simple but useful proposition describes how endomorphisms behave on \(k\)-clique components.

**Proposition 1.2.** Let \((X, E) = \bigcup_{l \in L} (X_l, E_l)\) be the disjoint union of a family \(\{(X_l, E_l) | l \in L\}\) of \(k\)-clique connected graphs for some fixed \(k \geq 3\). Then \(f : X \rightarrow X\) is an endomorphism of \((X, E)\) if and only if for every \(l \in L\) there exists \(j \in L\) with \(f(X_l) \subseteq X_j\) and the domain-range restriction of \(f\) to \(X_l\) and \(X_j\) is a \(\text{GRA}_k\)-morphism from \((X_l, E_l)\) to \((X_j, E_j)\).

**Proof.** Let \(f : X \rightarrow X\) be an endomorphism of \((X, E)\). Since a \(\text{GRA}_k\)-morphism maps any \(k\)-clique component into a \(k\)-clique component and because \(\{X_l | l \in L\} = \text{Comp}_k(X, E)\), for every \(l \in L\) there exists some \(j \in L\) with \(f(X_l) \subseteq X_j\). Since the induced subgraph of \((X, E)\) on \(X_l\) is \((X_l, E_l)\) for every \(l \in L\), the domain-range restriction of \(f\) to \(X_l\) and \(X_j\) is a \(\text{GRA}_k\)-morphism from \((X_l, E_l)\) to \((X_j, E_j)\). Since \(E = \bigcup_{l \in L} E_l\), the converse implication is straightforward. 

**Proposition 1.3.** Assume that \((X, E) = \bigcup_{l \in L} (X_l, E_l)\) is the disjoint union of a family \(\{(X_l, E_l) | l \in L\}\) of \(k\)-clique connected graphs for some fixed \(k \geq 3\). For every \(l \in L\), select a unique \(a_l \in X_l\) and denote \(Z = \{a_l | l \in L\}\). Let \((Z, F)\) be a graph and let \(0\) be an equivalence on \(X\) such that

(i) \((Z, F)\) is a disjoint union of stars;
(ii) every non-singleton class of \(0\) is contained in a component of \((Z, F)\);
(iii) the restriction of \(0\) to \(Z\) is a congruence on \((Z, F)\).
Let \([v]\) denote the class of \(0\) containing the vertex \(v \in X\). Then

1. \(0\) is a congruence on \((X, E \cup F)\);
2. \(U\) is a \(k\)-clique component of \((X, E \cup F)/0\) if and only if \(U = \{[\nu] \mid v \in X_l\}\) for some \(l \in L\); the induced subgraph of \((X, E \cup F)/0\) on \(U\) is isomorphic to \((X_l, E_l)\);
3. for every \(f \in \text{End}(X, E \cup F)/0\), there exists a unique \(g_f \in \text{End}(X, E)\) such that \([g_f(v)] = f([\nu])\) for all \(v \in X\), and the mapping \(v : \text{End}(X, E \cup F)/0 \to \text{End}(X, E)\)
given by \(v(f) = g_f\) for all \(f \in \text{End}(X, E \cup F)/0\) is an injective monoid homomorphism;
4. if \(g \in \text{End}(X, E)\), then there exists an endomorphism \(f\) of \((X, E \cup F)/0\) with \([g(v)] = f([\nu])\) for all \(v \in X\) if and only if
   \(a\) \(\{g(v), g(w)\} \in F\) for all \(\{v, w\} \in F\), and
   \(b\) \(g(v)\nu g(w)\) for all \(v, w \in X\) with \(\nu v w\).

**Proof.** From the choice of \(Z \subseteq X\) and (iii) it follows that \(0\) is a congruence on 
\((X, E \cup F)\), and hence (1) holds.

To prove (2), suppose that \(\{[u], [v], [w]\}\) is a 3-clique in \((X, E \cup F)/0\). Thus there are 
\(u_0, u_1 \in [u], v_0, v_1 \in [v]\) and \(w_0, w_1 \in [w]\) such that \(\{u_0, v_0\}, \{v_1, w_1\}, \{u_1, w_0\} \in E \cup F\). By
(i) and (ii), it is impossible to have \(\{u_0, v_0\}, \{v_1, w_1\}, \{u_1, w_0\} \in F\). Thus at least one
vertex does not belong to \(Z\). With no loss of generality we can assume that \(w_0 \notin Z\).
Then, by (ii) and the choice of \(Z\), \(w_0 = w_1 \in X_l\) for some \(l \in L\), and \(u_1, v_1 \in X_l\) follows
because \(w_0 \notin Z\). Therefore \(\{v_1, w_1\}, \{u_1, w_0\} \in E_l\) and \(u_1 \neq v_1\) because of (1).
We thus have, say, \(v_1 \notin Z\) and hence \(v_0 = v_1\) and \(\{u_0, v_0\} \in E_l\). But then \(u_0 = u_1\) because of
(ii). Whence any 3-clique of \((X, E \cup F)/0\) is a quotient of a 3-clique of \((X, E)\), and (2)
follows.

Let \(f\) be an endomorphism of \((X, E \cup F)/0\). Since any \(\text{GRA}\)-morphism maps each
\(k\)-clique component into a \(k\)-clique component, from (2) it follows that for every \(l \in L\)
there exists exactly one \(j \in L\) with \(f([v] \mid v \in X_j) \subseteq [v] \mid v \in X_l\). Hence for every
\(l \in L\) there exists exactly one mapping \(h_l : X_l \to X_l\) with \([h_l(v)] = f([v])\) for all \(v \in X_l\).
Then \(h_l\) is a \(\text{GRA}\)-morphism from \((X_l, E_l)\) to \((X_l, E_l)\), by (2), and thus the mapping
\(g_f : X \to X\) defined by

\[ g_f(v) = h_l(v) \quad \text{for all } v \in X_l \text{ with } l \in L \]

is, by Proposition 1.2, an endomorphism of \((X, E)\). Clearly, \([g_f(v)] = f([\nu])\) for all
\(v \in X\). Straightforward calculation shows that \(g_{f_1} \circ g_{f_2} = g_{f_1 \circ f_2}\) for any \(f_1, f_2 \in \text{End}\)
\((X, E \cup F)/0\). If \(f_1\) and \(f_2\) are distinct, then \(f_1([v]) \neq f_2([v])\) for some \(v \in X_l\) with
\(l \in L\). Hence \(g_{f_1}(v) \neq g_{f_2}(v)\), and (3) is proved.

The verification of (4) is straightforward. 

The claim below is immediate.

**Proposition 1.4.** Let \(\mathcal{X}\) be a set of incomparable \(k\)-clique connected graphs for some
fixed \(k \geq 3\), and let \(G\) be a graph with \(\text{Comp}_k(G) = \bigcup_{H \in \mathcal{X}} \text{Comp}_k(G; H)\). Then, for
any $H \in \mathcal{K}$, any vertex $v$ of $G$, and any $f \in \text{End } G$,

(1) if $v$ is adjacent to a member of $\text{Comp}_k(G; H)$, then $f(v)$ is;
(2) if $v$ belongs to a member of $\text{Comp}_k(G; H)$, then $f(v)$ does.

Next we recall a folklore result.

**Statement 1.5** (Pultr and Hedrlín [14] or Pultr and Trnková [15] or Adams et al. [2]).

For any monoid $(M; \cdot)$, there is a graph $(V; E)$ such that $M \subseteq X$, and an isomorphism $\varphi : (M; \cdot) \to \text{End}(V; E)$ such that

$$\varphi(m)(x) = m \cdot x \quad \text{for all } m \in M \text{ and } x \in M \subseteq V.$$ 

Therefore the left translation of the monoid $M$ by its element $m$ uniquely extends to an endomorphism $\varphi(m)$ of the graph $(V; E)$, and $(V; E)$ has no other endomorphisms.

Proposition 1.6 below is a combination of results due to Hedrlín and Sichler [8] and Hell and Nešetřil [9]. First we recall several categorical notions.

Let $\mathcal{K}$ and $\mathcal{L}$ be categories. Recall that a functor $\Psi : \mathcal{K} \to \mathcal{L}$ is called a **full embedding** if it is faithful and full. Let $(\mathcal{K}, U)$ and $(\mathcal{L}, V)$ be concrete categories, that is, let $U : \mathcal{K} \to \text{Set}$ and $V : \mathcal{L} \to \text{Set}$ be faithful functors. A full embedding $\Psi : \mathcal{K} \to \mathcal{L}$ is called an **extension** from $(\mathcal{K}, U)$ to $(\mathcal{L}, V)$ if there is a monotransformation $\mu : U \to V \circ \Psi$.

Endowed by the faithful functor $U : \mathcal{K} \to \text{Set}$ that assigns the underlying map of the vertex sets to any $\mathcal{K}$-morphism, any category $\mathcal{K}$ of graphs is concrete. Let $\text{GRA}_k$ denote the full subcategory of $\text{GRA}$ formed by all $k$-clique connected graphs for a given $k \geq 3$.

**Proposition 1.6** (Hedrlín and Sichler [8] and Hell and Nešetřil [9]). For any natural number $k \geq 3$ and for any set $J$ there exist extensions

$$\Phi_j : \text{GRA} \to \text{GRA}_k \quad \text{for } j \in J$$

such that for any graphs $G$ and $G'$ the graphs $\Phi_j G$ and $\Phi_j G'$ are incomparable whenever $j, j' \in J$ are distinct.

Combining these two results, we obtain

**Corollary 1.7.** Let $\{ (M_j, \cdot) \mid j \in J \}$ be a family of monoids indexed by a set $J$ and let $k \geq 3$ be an integer. Then for every $j \in J$, there exists a graph $G_j = (V_j, E_j)$ such that

(1) $G_j$ is $k$-clique connected;
(2) $M_j \subseteq V_j$ and there is an isomorphism $\varphi_j : (M_j; \cdot) \to \text{End}(V_j, E_j)$ such that

$$\varphi_j(m)(m') = m \cdot m' \quad \text{for all } m, m' \in M_j;$$
(3) the graphs $G_j$ and $G_{j'}$ are incomparable whenever $j, j' \in J$ are distinct.
In Section 2, the graphs from Corollary 1.7 are used to construct graphs forming exact chains, and Propositions 1.2–1.4 applied to show that the constructed graphs have the desired endomorphism monoids.

2. Two constructions

Let \( I \) be a complete nowhere dense chain, let \( k \geq 3 \), and let \( \{ M_j \mid j \in I \} \) be a family of monoids. Denote
\[
Q = \{ q \in I \mid q \neq \sup I_q \}, \quad \text{and}
\]
\[
M = \bigcup \{ M_j \mid j \in I \},
\]
where the latter union is disjoint. Then the mapping \( \gamma : M \to I \) given by
\[
\gamma(m) = i \text{ exactly when } m \in M_i.
\]
is correctly defined.

The claim below easily follows from Corollary 1.7.

Lemma 2.1. For any \( I, Q, M \) as above, there exists a system
\[
\mathcal{G} = \{ G_j = (V_j, E_j) \mid j \in I \}
\]
\[
\cup \{ K_{q,m} = (W_{q,m}, F_{q,m}), L_{q,m} = (T_{q,m}, G_{q,m}) \mid (q, m) \in Q \times M \}
\]
of \( k \)-connected pairwise disjoint and pairwise incomparable graphs such that for every \( j \in I \), we have \( M_j \subseteq V_j \) and a monoid isomorphism
\[
x_j : (M_j, \cdot) \to \text{End } G_j
\]
such that
\[
x_j(m)(m') = m \cdot m' \quad \text{for all } m, m' \in M_j
\]
and for every \( (q, m) \in Q \times M \) the graphs \( K_{q,m} \) and \( L_{q,m} \) are rigid.

Lemma 2.1 will be used to build graphs with prescribed endomorphism monoids forming exact chains. Observe that no graph from \( \mathcal{G} \) is a singleton graph.

To begin, for \( i \in I \) and \( m \in M \) we define sets
\[
S_{i,m} = \{ (q, r) \in Q \times M \mid (q < \gamma(m) \text{ and } r \neq m) \text{ or } (q < i \text{ and } r = m) \}\)
\[
S_m = \{ (q, r) \in Q \times M \mid (q < \gamma(m) \text{ and } r \neq m) \text{ or } r = m \}.
\]

Lemma 2.2. The sets \( S_{i,m} \) have these properties:

1. if \( i, i' \in I \) and \( m \in M \), then \( S_{i,m} \subseteq S_{i',m} \) if and only if \( i < i' \);
2. if \( m, m' \in M \) and \( i \in I \) are such that \( \gamma(m) = \gamma(m') \neq i \), then \( S_{i,m} \subseteq S_{i,m'} \) only when \( m = m' \).
Proof. For (1), it is clear that $i \leq i'$ implies that $S_{i,m} \subseteq S_{i',m}$. If $i \not\leq i'$ then $i' < i$ because $I$ is a chain. Since $I$ is nowhere dense, there is a $q \in Q$ such that $i' \leq q < i$. But then $(q,m) \in S_{i,m} \setminus S_{i',m}$. This proves (1).

To prove (2), suppose that $m \neq m'$ and $\gamma(m) = \gamma(m') \neq i$. If $i < \gamma(m)$, then $i < q < \gamma(m)$ for some $q \in Q$ because the chain $I$ is nowhere dense. But then $(q,m') \in S_{i,m} \setminus S_{i,m'}$. If $\gamma(m) < i$, then there exists some $q \in Q$ with $\gamma(m) = \gamma(m') \leq q < i$, and hence $(q,m) \in S_{i,m} \setminus S_{i,m'}$ this time.

For (3), it suffices to note that $S_{i,m} = \{(q,r) \in Q \times M \mid q < i\} = S_{i,m'}$ for any $m,m' \in M_i$.

Suppose that $i = \inf I_i^+$. Then $S_{i,m} \subseteq \cap \{S_{i',m} \mid i' \in I_i^+\}$ because of (1). If $(q,r) \in S_{i',m}$ for all $i' > i$ and $(q,r) \not\in S_{i,m}$ then $r = m$ and $i \leq q < i'$ for all $i' > i$. Thus either $i < q$ or $i = q < q^* \leq i'$ for all $i' \in I_i^+$, contradicting $i = \inf I_i^+$. This proves (4).

To prove (5), suppose that $i = \sup I_i^-$. Then $S_{i,m} \supseteq \cup \{S_{i',m} \mid i' \in I_i^-\}$ by (1). If $(q,r) \in S_{i,m}$ and $(q,r) \not\in S_{i',m}$ for every $i' < i$, then $r = m$ and $i' \leq q < i$ for every $i' < i$. But then $q$ is an upper bound of $I_i^-$, contradicting the hypothesis. \(\square\)

For any $(q,r) \in Q \times M$, select and fix vertices $k_{q,r} \in W_{q,r}$ and $l_{q,r} \in T_{q,r}$.

Now we are prepared to construct an exact chain of subgraphs with prescribed endomorphism monoids.

Select and fix an $i \in I$. For every $m \in M$ choose a copy of $K_{q,r} = (W_{q,r},F_{q,r})$ for every $(q,r) \in S_{i,m}$ and denote it $K_{q,r} \times \{m\}$. Any vertex $w \in W_{q,r}$ in the copy $K_{q,r} \times \{m\}$ of $K_{q,r}$ is denoted by $(w,m)$, and we write $W_{q,r} \times \{m\}$ for the set $\{(w,m) \mid w \in W_{q,r}\}$.

Let the family $\mathcal{A}_i = \{G_j \mid j \in J\} \cup \{K_{q,r} \times \{m\} \mid m \in M, (q,r) \in S_{i,m}\}$ consist of disjoint graphs. By Lemma 2.1, any graph from $\mathcal{A}_i$ is $k$-clique connected. Define

$$U_i = \{V_j \mid j \in J\} \cup \{(w,m) \mid m \in M, w \in W_{q,r} \text{ for } (q,r) \in S_{i,m}\},$$

$$A_i = \{m,(k_{q,r},m)\} \mid m \in M, (q,r) \in S_{i,m}\},$$

$$H'_i = \{E_j \mid j \in J\} \cup \{(w,m),(w',m)\} \mid m \in M, (w,w') \in F_{q,r} \text{ for } (q,r) \in S_{i,m}\},$$

$$H_i = H'_i \cup A_i.$$ Denote $A'_i = (U_i,H'_i)$ and $A_i = (U_i,H_i)$. Thus $A'_i$ is the disjoint union of all graphs from $\mathcal{A}_i$, and the graph $A_i$ is obtained by adding the set $A_i$ of edges to $A'_i$.

Lemma 2.3. For every $i \in I$,

(1) the family $\mathcal{A}_i$, the set $A_i$ and the diagonal congruence on $U_i$ satisfy the hypothesis of Proposition 1.3;

(2) $\operatorname{End} A_i$ is isomorphic to $M_i$;
(3) if \( i < i' \) in \( I \), then \( A_i \) is a proper induced subgraph of \( A_{i'} \);
(4) if \( i = \inf I_i^+ \) then \( A_i \) is a meet of \( \{ A_{i'} \mid i' \in I_i^+ \} \);
(5) if \( i = \sup I_i^- \) then \( A_i \) is a union of \( \{ A_{i'} \mid i' \in I_i^- \} \).

**Proof.** The verification of (1) is straightforward.

Let \( f \in \text{End} A_i \) be given. By Proposition 1.3, there exists an injective monoid homomorphism \( v : \text{End} A_i \rightarrow \text{End} A_i' \) such that \( f(u) = v(f)(u) \) for all \( u \in U_i \). Set \( v(f) = g \).

By Lemma 2.1, the graphs from \( \mathcal{B} \) are pairwise incomparable and, by Proposition 1.3, \( \text{Comp}_k(A_i'; G_j) = \{ V_j \} \) for all \( j \in I \). Hence \( g(V_j) \subseteq V_j \) and thus also \( f(V_j) \subseteq V_j \) for all \( j \in I \). By the definition of \( A_i \), for every \( m \in M \) and for every \( (q, r) \in S_{i,m} \) there exists exactly one \( k \)-clique component from \( \text{Comp}_k(A_i; K_{q,r}) \) adjacent to \( m \), namely \( W_{q,r} \times \{ m \} \).

By Proposition 1.4(1), \( f(w, m) = (w, f(m)) \) for all \( m \in M \) and \( w \in W_{q,r} \) with \( (q, r) \in S_{i,m} \) because, by Lemma 2.1, every \( K_{q,r} \) is rigid. By the definition of \( A_i \), a vertex \( m \in M \) is adjacent to a \( k \)-clique component \( C \) of \( A_i \) exactly when either \( C \in \text{Comp}_k(A_i; G_{(m)}) \) or \( C \in \text{Comp}_k(A_i; K_{q,r}) \) for some \( (q, r) \in S_{i,m} \). Hence \( S_{i,m} \subseteq S_{i,f(m)} \) by Proposition 1.4, and Lemma 2.2(2) implies that \( f(m) = m \) for all \( m \in M \setminus M_i \). But then, by Lemma 2.1, \( f(x) = x \) for all

\[
x \in \bigcup \{ V_j \mid j \in I \setminus \{ i \} \} \cup \bigcup \{ W_{q,r} \times \{ m \} \mid m \in M \setminus M_i \text{ and } (q, r) \in S_{i,m} \}.
\]

Let \( \mu(f) \) be the domain-range restriction of \( f \) to \( V_i \). Then, by Proposition 1.2, \( \mu(f) \in \text{End} G_i \). Summarizing these facts, we conclude that \( \mu : \text{End} A_i \rightarrow \text{End} G_i \) is an injective monoid homomorphism. To show that \( \mu \) is onto, for \( h \in \text{End} G_i \) define \( g : U_i \rightarrow U_i \) as follows:

\[
g(x) = \begin{cases} h(x) & \text{if } x \in V_i, \\ (w, h(m)) & \text{if } x = (w, m) \in W_{q,r} \times \{ m \} \text{ with } m \in M_i \text{ and } (q, r) \in S_{i,m}, \\ x & \text{for all other } x. \end{cases}
\]

By Proposition 1.2, \( g \) is an endomorphism of \( A_i' \). By Lemma 2.2(3), \( g \) satisfies (4a) and (4b) of Proposition 1.3, and hence there exists \( f \in \text{End} A_i \) with \( v(f) = g \). But then necessarily \( \mu(f) = h \). Thus \( \mu \) is an isomorphism, and Lemma 2.1 completes the proof of (2).

By Lemma 2.2(1), \( A_i \subseteq A_{i'} \) for \( i < i' \) and (3) follows. Properties (4) and (5) follow from Lemma 2.2(4) and (5).

Now we turn to the construction of an exact chain of quotient graphs with prescribed endomorphism monoids.

Analogously to the case of subgraphs, for every \( m \in M \) we choose a copy of \( K_{q,r} \) and a copy of \( L_{q,r} \) for every \( (q, r) \in S_m \) and denote them \( K_{q,r} \times \{ m \} \) and \( L_{q,r} \times \{ m \} \). Thus the vertex \( w \in W_{q,r} \) (or \( t \in T_{q,r} \)) in the copy \( K_{q,r} \times \{ m \} \) (or \( L_{q,r} \times \{ m \} \)) is denoted by \( (w, m) \) (or \( (t, m) \), respectively). We write \( W_{q,r} \times \{ m \} \) for the set \( \{(w, m) \mid w \in W_{q,r}\} \) and \( T_{q,r} \times \{ m \} \) for the set \( \{(t, m) \mid t \in T_{q,r}\} \).

Denote \( \mathcal{B} = \{ G_j \mid j \in I \} \cup \{ K_{q,r} \times \{ m \}, L_{q,r} \times \{ m \} \mid m \in M, (q, r) \in S_m \} \) and assume that the members of \( \mathcal{B} \) are pairwise disjoint. By Lemma 2.1, any graph from \( \mathcal{B} \) is
$k$-clique connected. Define

$$X = \bigcup \{ V_j \mid j \in I \} \cup \{(z,m) \mid m \in M, z \in T_{q,r} \cup W_{q,r} \text{ for } (q,r) \in S_m\},$$

$$B = \{ \{m,(k_{q,r},m)\}, \{m,(l_{q,r},m)\} \mid m \in M, (q,r) \in S_m\},$$

$$D = \bigcup \{ E_j \mid j \in I \} \cup \{(z,m),(z',m) \mid m \in M, \{z,z'\} \in F_{q,r} \cup G_{q,r} \text{ for } (q,r) \in S_m\}.$$  

Denote $B = (X,D)$. Then $B$ is the disjoint union of the graphs from $\mathcal{B}$. The graph $B$ and the set $B$ of edges satisfy the hypothesis (i) of Proposition 1.3. For every $i \in I$, let $\theta_i$ be the least equivalence on $X$ such that

$$(k_{q,r},m)\theta_i(l_{q,r},m) \quad \text{for all } m \in M \text{ and } (q,r) \in S_{i,m}.$$  

Since $S_{i,m} \subseteq S_m$ for all $m \in M$ and all $i \in I$, this definition is correct. The lemma below follows by a direct calculation.

**Lemma 2.4.** The graph $B$, the set of edges $B$ and the equivalence $\theta_i$ satisfy the hypothesis of Proposition 1.3 for all $i \in I$. Any class of the congruence $\theta_i$ on $B$ has at most two vertices, and the class of $\theta_i$ containing any $v \in \bigcup \{V_j \mid j \in I\}$ is a singleton for all $i \in I$.

Define $B_i = (X,D \cup B)/\theta_i = (X_i,B_i)$. Let $[x]_i$ denote the class of $\theta_i$ containing $x \in X$. When the index is clear from the context, we omit it. By Lemma 2.2(1), $S_{i,m} \subseteq S_{i',m}$ for $m \in M$ if and only if $i < i'$ in $I$. Hence $\theta_i$ is properly coarser than $\theta_i$ whenever $i < i'$ in $I$, and thus there exists a unique mapping $\psi_{i,i'} : X/\theta_i \to X/\theta_i'$ satisfying $\psi_{i,i'}([x]_i) = [x]_{i'}$ for all $x \in X$. For the sake of simplicity, we identify $[v]_i$ and $v$ for all $v \in \bigcup \{V_j \mid j \in I\}$, see Lemma 2.4.

**Lemma 2.5.** The graphs $B_i$ and the mappings $\psi_{i,i'}$ have these properties:

1. End $B_i$ is isomorphic to $M_i$ for every $i \in I$;
2. $\psi_{i,i'} : B_i \to B_{i'}$ is a proper quotient morphism for $i < i'$ in $I$;
3. if $i < i' < i''$ in $I$ then $\psi_{i,i''} \circ \psi_{i,i'} = \psi_{i,i''}$;
4. if $i = \inf I_i^+$ then $B_i$ is a co-union of $\{B_{i'} \mid i' \in I_i^+\}$;
5. if $i = \sup I_i^-$ then $B_i$ is a co-meet of $\{B_{i'} \mid i' \in I_i^-\}$.

**Proof.** Let $f \in \text{End } B_i$. By Lemma 2.4 and Proposition 1.3, there exists $v(f) \in \text{End } B$ such that $[v(f)(x)] = f([x])$ for all $x \in X$. Denote $g = v(f)$. By Lemma 2.1, the graphs from $\mathcal{G}$ are pairwise incomparable and, by Proposition 1.3, $\text{Comp}_q(B;G_j) = \{V_j\}$ for every $j \in I$. Hence $g(V_j) \subseteq V_j$ and thus also $f(V_j) \subseteq V_j$ for every $j \in I$. By the definition of $B$, for every $m \in M$ and for every $(q,r) \in S_m$ there exists exactly one $k$-clique component from $\text{Comp}_q(B;K_{q,r})$ and exactly one $k$-clique component from $\text{Comp}_q(B;L_{q,r})$ adjacent to $m$, namely $\{(w,m)\}_w \in W_{q,r}$ and $\{(t,m)\}_t \in T_{q,r}$.

By Proposition 1.4(1), $f([(z,m)]) = [(z,f(m))]$ for all $m \in M$ and $(z,m) \in (W_{q,r} \cup T_{q,r}) \times \{m\}$ with $(q,r) \in S_m$ because, by Lemma 2.1, the graphs $K_{q,r}$ and $L_{q,r}$ are rigid and, by Proposition 1.3(2), the induced subgraph of $B_i$ on the set $\{(w,m)\}_w \in W_{q,r}$
is isomorphic to $K_{q,r}$ and the induced subgraph of $B_i$ on the set $\{(i,m)\} | t \in T_{q,r}\}$ is isomorphic to $L_{q,r}$.

By the definition of $\theta_i$, each its class $[(k_{q,r},m)]_i$ with $m \in M$ and $(q,r) \in S_m$ belongs to exactly two distinct $k$-clique components of $B_i$ if and only if $(q,r) \in S_{i,m}$ (in which case it belongs to $\{[(w,m)]_i | w \in W_{q,r}\}$ and $\{(i,m)\} | t \in T_{q,r}\})$. By Proposition 1.4(2) it follows that $S_{i,m} \subseteq S_{i,f(m)}$, and Lemma 2.2(2) implies that $f(m) = m$ for all $m \in M \setminus M_i$. Then, by Lemma 2.1, $f([x_i]) = [x_i]$ for all $x \in \bigcup V_j | j \in I \setminus \{i\} \cup \bigcup (W_{q,r} \cup T_{q,r}) \times \{m\} | m \in M \setminus M_i$ and $(q,r) \in S_m$.

Let $\mu(f)$ be the domain-range restriction of $f$ to $V_i$. Then $\mu(f) \in \text{End} B_i$, by Proposition 1.2. Summarizing these facts, we conclude that $\mu : \text{End} B_i \rightarrow \text{End} G_i$ is an injective monoid homomorphism. To show that $\mu$ is onto, for any given $h \in \text{End} G_i$ we define $g : X \rightarrow X$ by

$$g(x) = \begin{cases} h(x) & \text{if } x \in V_i, \\ (z,h(m)) & \text{if } x = (z,m) \in (W_{q,r} \cup T_{q,r}) \times M_i \text{ and } (q,r) \in S_m, \\ x & \text{for all other } x. \end{cases}$$

Then $g$ is an endomorphism of $B_i$, by Proposition 1.2. From Lemma 2.2(3) it follows that $g$ satisfies conditions (4a) and (4b) of Proposition 1.3. Thus $v(f) = g$ for some $f \in \text{End} B_i$. But then necessarily $\mu(f) = h$. Thus $\mu$ is an isomorphism, and Lemma 2.1 completes the proof of (1).

Since $\theta_i$ is properly coarser that $\theta_{i'}$, for $i,i' \in I$ with $i < i'$ we deduce that $\psi_{i,i'}$ is a proper quotient morphism, and (2) is proved. The verification of (3) is straightforward.

Before turning to (4) and (5), we note that (2) and (3) imply that setting $T(i) = B_i$ for all $i \in I$ and $T(I_{i,i'}) = \psi_{i,i'}$ for $i < i'$ in $I$ defines a functor $T$ satisfying (1) of Definition 1.1.

To prove (4), suppose that $i = \inf I_i^+$. From Lemma 2.2(4), for any vertices $u,v$ of $B_i$ we have

$$(c) \quad \psi_{i,i'}(u) = \psi_{i,i'}(v) \text{ for all } i' \in I^+_i \text{ only when } u = v.$$ 

If $C$ is the co-union of $\{B_{i'} | i' \in I^+_i\}$ and $\phi_{i'} : C \rightarrow B_{i'}$ is the $i'$th co-union morphism, then $\psi_{i,i'} \circ \phi_{i'} = \phi_{i'}$ whenever $i < i' < i''$. By (3), any vertex $v$ of $B_i$ gives rise to a unique $(T \upharpoonright I^+_i)$-string $\sigma(v) = (\psi_{i,i'}(v) | i' > i)$. The mapping $\sigma : B_i \rightarrow C$ thus satisfies $\sigma \circ \phi_{i'} = \psi_{i,i'}$ for all $i' > i$ and, because of (c), the definition of $C$ and the fact that every $\psi_{i,i'}$ is a quotient morphism, the mapping $\sigma$ is one-to-one and such that $\{\sigma(u), \sigma(v)\}$ is an edge of $C$ exactly when $\{u,v\}$ is an edge of $B_i$. To show that $\sigma$ is surjective, let $(c_{i'} | i' \in I^+_i)$ be a $(T \upharpoonright I^+_i)$-string. Fix $l \in I^+_i$ arbitrarily. By Lemma 2.4, each class of the congruence $\theta_l$ has at most two elements. Since $\psi_{l,l}$ is a quotient morphism and $\psi_{l,l}([x_l]) = [x_l]$ for every vertex $x$ of $(X,D \cup B)$, the set $\psi_{l,l}^{-1}\{c_l\}$ is non-void and has at most two elements. If $\psi_{l,l}^{-1}\{c_l\} = \{v\}$, then $\psi_{l,l}(v) = c_{i'}$ for all $i' \in I^+_i$ because of (3), and hence $(c_{i'} | i' \in I^+_l) = \sigma(v)$. If $\psi_{l,l}^{-1}\{c_l\} = \{v,w\}$, then (c) implies the existence of some $j \in I^+_l$ such that $\psi_{l,l}^{-1}\{c_j\}$ is a singleton, and the previous argument applies again, this time to $j$ instead of $l$. Therefore $\sigma$ is surjective, and (4) holds.
To prove (5), let \( i = \sup I_i^- \) and let \( D \) be a co-meet of \( \{ B_{i'} \mid i' \in I_i^- \} \) with the \( i' \)th co-meet map \( \delta_{i'} : B_{i'} \to D \) for each \( i' \in I_i^- \). From the definition of co-meet it follows that \( \delta_{i'}([x],') = \delta_{i''}([x],') \) for any \( i', i'' \in I_i^- \) and any vertex \( x \) of \((X, B \cup D)\). The mapping \( \kappa \) defined by \( \kappa(x) = \delta_i([x],i) \) with an arbitrarily chosen \( i \in I_i^- \) is then a GRA-morphism from \((X, B \cup D)\) to \( D \), and satisfies \( \kappa(x) = \kappa(y) \) if and only if \([x]_j = [y]_j \) for some \( j \in I_i^- \). Therefore, the equivalence \( \theta_i \) and the kernel of \( \kappa \) coincide. Since both \( \kappa \) and the map \( x \mapsto [x]_i \) are quotient morphisms, it follows that \( B_i \) is isomorphic to \( D \).

The results of this section can be summarized as follows.

**Theorem 2.6.** Let \( I \) be a complete nowhere dense chain and let \( \{ M_i \mid i \in I \} \) be a family of monoids. Then

1. there exists an exact chain \( A : I \to \text{GRA} \) of subgraphs such that \( \text{End} A(i) \) is isomorphic to \( M_i \) for every \( i \in I \);
2. there exists an exact chain \( B : I \to \text{GRA} \) of quotient graphs such that \( \text{End} B(i) \) is isomorphic to \( M_i \) for every \( i \in I \).

**Proof.** This is an immediate consequence of Lemmas 2.3 and 2.5.

### 3. Applications

First we prove a simple consequence of Theorem 2.6.

Let \( I \) be a complete chain. A functor \( T : I \to \text{GRA} \) is a sup-exact chain of subgraphs if it satisfies

1. for any \( x < y \) in \( I \), the graph \( T(x) \) is a proper induced subgraph of \( T(y) \) and \( T(I_{x,y}) : T(x) \to T(y) \) is the inclusion map;
2. if \( x \in I \) is such that \( \sup I_x^- = x \), then \( T(x) = \bigcup \{ T(u) \mid u \in I_x^- \} \);

a sup-exact chain of quotient graphs if it satisfies

1. for any \( x < y \) in \( I \), \( T(y) \) is a proper quotient of \( T(x) \) and \( T(I_{x,y}) : T(x) \to T(y) \) is the quotient morphism;
2. if \( x \in I \) is such that \( \sup I_x^- = x \), then \( T(x) \) is a co-meet of \( T \upharpoonright I_x^- \) and \( T(I_{y}) \) is the \( y \)th co-meet map for all \( y \in I_x^- \).

Using Theorem 2.6, we obtain the following generalization of the result from [2] quoted in the introduction.

**Theorem 3.1.** For any complete chain \( I \) and for any family \( \{ M_i \mid i \in I \} \) of monoids

1. there is a sup-exact chain of subgraphs \( T : I \to \text{GRA} \) with \( \text{End} T(i) \cong M_i \) for every \( i \in I \);
(2) there is a sup-exact chain of quotient graphs $T : I \to \mathcal{GRA}$ with $\text{End} T(i) \cong M_i$ for every $i \in I$.

**Proof.** Let $J$ be the filter completion of $I$. Then $J$ is a nowhere dense complete chain such that the embedding $\mu : I \to J$ sending $i \in I$ to the principal filter $[i)$ generated by $i$ preserves all suprema. Let $\{S_j \mid j \in J\}$ be any family of monoids such that $S_{\mu(i)} = M_i$ for all $i \in I$. Applying Theorem 2.6 to $J$ and the family $\{S_j \mid j \in J\}$, we obtain a functor $T' : J \to \mathcal{GRA}$ whose restriction $T = T' \circ \mu : I \to \mathcal{GRA}$ to $I$ has the required properties. 

To apply our results to other structures (and specifically to algebras of a finitary type), we first introduce several general categorical notions and then translate our results to the category $\mathcal{DG}$ of directed graphs. The translation to $\mathcal{DG}$ is needed for subsequent algebraic applications.

**Definition.** Let $\mathcal{H}$ be a category and let $\mathcal{C}$ be a class of $\mathcal{H}$-morphisms. A functor $T : I \to \mathcal{H}$ from a complete chain $I$ is an exact $\mathcal{C}$-chain in $\mathcal{H}$ if

1. $T(i, j) \in \mathcal{C}$ for all $i, j \in I$ with $i \leq j$ in $I$, and $T(i, j)$ is an isomorphism only when $i = j$;
2. if $i = \text{sup} I_0^-$ in $I$, then $(T(i), \{T(i, j) : T(j) \to T(i) \mid j \in I_0^-\})$ is the colimit $\text{colim} \ T \uparrow I_0^-$;
3. if $i = \text{inf} I_1^+$ in $I$, then $(T(i), \{T(i, j) : T(i) \to T(j) \mid j \in I_1^+\})$ is the limit $\text{lim} \ T \downarrow I_1^+$.

When $T : I \to \mathcal{H}$ satisfies only the first two conditions, we say that $T$ is a sup-exact $\mathcal{C}$-chain.

Finally, let $\mathcal{M}_I = \{M_i \mid i \in I\}$ be a family of monoids indexed by $I$. We say that a functor $T : I \to \mathcal{H}$ represents $\mathcal{M}_I$ if $\text{End} T(i) \cong M_i$ for every $i \in I$.

The following four classes of $\mathcal{H}$-morphisms will be of interest:
- $\text{Mono}_{\mathcal{H}}$ — the class of all monomorphisms of $\mathcal{H}$ ($f$ is a monomorphism if it is left cancellative);
- $\text{Epi}_{\mathcal{H}}$ — the class of all epimorphisms of $\mathcal{H}$ ($f$ is an epimorphism if it is right cancellative);
- $\text{ExtMono}_{\mathcal{H}}$ — the class of all extremal monomorphisms of $\mathcal{H}$ ($f$ is an extremal monomorphism if $f$ is a monomorphism, and $f = g \circ h$ with an epimorphism $h$ only when $h$ is an isomorphism);
- $\text{ExtEpi}_{\mathcal{H}}$ — the class of all extremal epimorphisms of $\mathcal{H}$ ($f$ is an extremal epimorphism if $f$ is an epimorphism, and $f = g \circ h$ with a monomorphism $g$ only when $g$ is an isomorphism).

It is a folklore fact that any inclusion from the induced subgraph belongs to $\text{ExtMono}_{\mathcal{GRA}}$ and any quotient morphism belongs to $\text{ExtEpi}_{\mathcal{GRA}}$. Since the union of a chain of
induced subgraphs and the co-meet of a chain of quotient graphs are colimits, and the
meet of a chain of induced subgraphs and the co-union of a chain of quotient graphs
are limits, we can reformulate Theorems 2.6 and 3.1 as follows:

**Corollary 3.2.** Let \( \mathcal{C} = \text{ExtMono}_{\text{GRA}} \) or \( \mathcal{C} = \text{ExtEpi}_{\text{GRA}} \). Let \( I \) be a complete chain
and let \( \mathcal{M}_I = \{ M_i \mid i \in I \} \) be a family of monoids. Then

1. if \( I \) is nowhere dense, then there exists an exact \( \mathcal{C} \)-chain in \( \text{GRA} \) representing \( \mathcal{M}_I \);
2. there is a sup-exact \( \mathcal{C} \)-chain in \( \text{GRA} \) representing \( \mathcal{M}_I \).

Let \( \mathcal{DG} \) denote the category of all directed graphs (or \( d \)-graphs) and all their
morphisms: the objects of \( \mathcal{DG} \) are all pairs \( (X,R) \) where \( X \) is a set and \( R \subseteq X \times X \), and
morphisms from \( (X,R) \) to \( (Y,S) \) are all mappings \( f:X \to Y \) such that \( (f(x), f(y)) \in S \)
for every \((x, y) \in R\).

Analogously to undirected graphs, we now define the notions of induced \( d \)-subgraph,
quotient \( d \)-graph, etc.

A \( d \)-graph \( (X,R) \) is an **induced \( d \)-subgraph** of \( (Y,S) \) if \( X \subseteq Y \) and \( R = \{(x,y) \in S \mid x,y \in X\} \). Then the inclusion is a morphism of the induced \( d \)-subgraph
into the original graph. Any such inclusion belongs to \( \text{ExtMono}_{\mathcal{DG}} \).

A \( d \)-graph \( (X,R) \) is a **quotient \( d \)-graph** of \( (Y,S) \) if there exists a surjective map
\( f:Y \to X \) such that \( R = \{(f(x), f(y)) \mid (x,y) \in S\} \). Then \( f:(Y,S) \to (X,R) \) is a morphism,
and we say that it is a **quotient morphism**. Any quotient morphism belongs to
\( \text{ExtEpi}_{\mathcal{DG}} \).

Let \( T:I \to \mathcal{DG} \) be a chain such that \( T(I_{x,y}) \) is an inclusion and \( T(x) \) is an induced
\( d \)-subgraph of \( T(y) \) for all \( x \preceq y \). We write \( T(x) = (X_x, R_x) \) for every \( x \in I \). Then a
\( d \)-graph \( (X,R) \) is called a union of \( T \) (a meet of \( T \), resp.) if \( X = \bigcup_{x \in I} X_x \) and \( R = \bigcup_{x \in I} R_x \)
(or \( X = \bigcap_{x \in I} X_x \) and \( R = \bigcap_{x \in I} R_x \), resp.). Observe that the union with inclusions is a
colimit of \( T \) in \( \mathcal{DG} \), and the meet with inclusions is a limit of \( T \) in \( \mathcal{DG} \).

For a chain \( T:I \to \mathcal{DG} \) of non-void quotient \( d \)-graphs, any \( T(I_{x,y}) \) with \( x \preceq y \) is
a quotient morphism. We write \( T(x) = (X_x, R_x) \) for every \( x \in I \). A \( d \)-graph \( (X,R) \)
is called a co-union of a chain \( T \) of quotient \( d \)-graphs if \( X \subseteq I \) is the set of all \( T \)-strings
and \( R = \{(t,v) \in X \times X \mid (t_x, v_x) \in R_x \text{ for all } x \in I\} \). Then for every \( x \in I \) a mapping
\( \phi_x:X_x \to X \) given by \( \phi_x(t) = t_x \) is a morphism from \( (X,R) \) to \( (X_x, R_x) \), called the
\( x \)th co-union map. Since \( T(I_x) \):\( X_x \to X_y \) is a quotient morphism whenever \( x \preceq y \) in \( I \),
every \( \phi_x:(X,R) \to (X_x, R_x) \) is a quotient morphism as well. It is well known that
\( ((X,R), \{\phi_x:(X,R) \to (X_x, R_x) \mid x \in I\}) = \text{lim } T \) in \( \mathcal{DG} \).

As for undirected graphs, on the set \( X \) of all \( T \)-strings we define a relation \( \theta \) by the
requirement that \( thv \) if and only if \( t_x = v_x \) for some \( x \in I \). Since \( I \) is a chain, the relation
\( \theta \) is an equivalence on the set \( X \). For \( v \in X \), let \([v] \) denote the class of \( \theta \) containing \( v \).
Let \( Y = X/\theta \) and \( S = \{[\{x\},\{y\}]) \mid (t_x, v_x) \in R_x \text{ for some } x \in I\} \). Analogously as for undirected
graphs, for every \( x \in I \) there exists a unique mapping \( \psi_x:X_x \to Y \) with \( \psi_x \circ \phi_x(t) = [t] \)
for all \( x \in I \) and \( t \in X_x \). From the definition of \( S \) it follows that \( \psi_x:(X_x, R_x) \to (Y, S) \) is a
quotient morphism for every \( x \in I \). We say that \((Y,S)\) is the co-meet of \( T \) and \( \psi_x \) is the \( x \)th co-meet map. Again, it is well-known that \( ((Y,S), \{ \psi_x : (X_x, R_x) \to (Y, S) \} \mid x \in I) = \text{colim} \ T \) in the category \( \mathcal{D}G \).

For an undirected graph \((V,E)\), define \( R_E = \{(v,w) \mid \{v,w\} \in E\} \), and then identify the undirected graph \((V,E)\) with the \( d \)-graph \((V,R_E)\). The category \( \mathcal{G}RA \) then becomes a full subcategory of \( \mathcal{D}G \).

The statement below is folklore and its verification is straightforward.

**Statement 3.3.** If \( \mathcal{G}RA \) is regarded as a full subcategory of \( \mathcal{D}G \) and if \( I \) is a complete chain, then

1. a (sup-)exact chain \( T : I \to \mathcal{G}RA \) of subgraphs is a (sup-)exact \( \text{ExtMono}_{\mathcal{D}G} \)-chain in \( \mathcal{DG} \);
2. a (sup-)exact chain \( T : I \to \mathcal{G}RA \) of quotient graphs is a (sup-)exact \( \text{ExtEpi}_{\mathcal{D}G} \)-chain in \( \mathcal{DG} \).

Statement 3.3 thus translates Corollary 3.2 to the following result for \( d \)-graphs.

**Corollary 3.4.** Let \( \mathcal{C} = \text{ExtMono}_{\mathcal{D}G} \) or \( \mathcal{C} = \text{ExtEpi}_{\mathcal{D}G} \). Let \( I \) be a complete chain and let \( \mathcal{M}_I = \{ M_i \mid i \in I \} \) be a family of monoids. Then

1. for nowhere dense \( I \), there is an exact \( \mathcal{C} \)-chain \( T : I \to \mathcal{D}G \) representing \( \mathcal{M}_I \);
2. there is a sup-exact \( \mathcal{C} \)-chain \( T : I \to \mathcal{D}G \) representing \( \mathcal{M}_I \).

Next we apply Corollary 3.4(2) to varieties of algebras.

Let \( \mathcal{V} \) be a variety of algebras of a finitary similarity type. We say that \( \mathcal{V} \) is finitarily universal if there exists a full embedding of the full subcategory \( \mathcal{D}G_{\text{fin}} \) of \( \mathcal{D}G \) determined by all finite \( d \)-graphs into the full subcategory \( \mathcal{V}^{[\text{lin}]} \) of \( \mathcal{V} \) determined by all finitely generated algebras in \( \mathcal{V} \). Let \( \text{Sur}_\mathcal{V} \) denote the class of all surjective \( \mathcal{V} \)-homomorphisms, and let \( \text{Inj}_\mathcal{V} \) be the class of all injective \( \mathcal{V} \)-homomorphisms.

Corollary 5.10(1) in [12] implies the following claim.

**Statement 3.5** (Koubek and Sichler [12]). For any finitarily universal variety \( \mathcal{V} \) of algebras of a finitary type, there exists a full embedding \( F_\mathcal{V} : \mathcal{D}G \to \mathcal{V} \) that preserves colimits of directed diagrams, and such that \( F_\mathcal{V}(\text{ExtEpi}_{\mathcal{D}G}) \subseteq \text{Sur}_\mathcal{V} \) and \( F_\mathcal{V}(\text{Mono}_{\mathcal{D}G}) \subseteq \text{Inj}_\mathcal{V} \).

Since \( \text{ExtMono}_{\mathcal{D}G} \subseteq \text{Mono}_{\mathcal{D}G} \), from Corollary 3.4(2) and Statement 3.5 we immediately obtain

**Corollary 3.6.** Let \( \mathcal{V} \) be any finitarily universal variety of a finitary type, and let \( \mathcal{C} = \text{Inj}_\mathcal{V} \) or \( \mathcal{C} = \text{Sur}_\mathcal{V} \). Let \( I \) be a complete chain, let \( \mathcal{M}_I = \{ M_i \mid i \in I \} \) be a family of monoids. Then there exists a sup-exact \( \mathcal{C} \)-chain \( T : I \to \mathcal{V} \) representing \( \mathcal{M}_I \).
Since the class \( V_{\text{fin}} \subseteq V \) of all finite algebras is contained in \( V^{[\text{fin}]} \), Statement 3.5 applies to any variety \( V \) of algebras of a finitary similarity type for which there is a full embedding \( F_{\text{fin}} : \mathbb{D}G_{\text{fin}} \to V_{\text{fin}} \) and, in particular, to any finite-to-finite universal variety, that is, a variety for which there is a full embedding \( G : \mathbb{D}G \to V \) with \( G(\mathbb{D}G_{\text{fin}}) \subseteq V_{\text{fin}} \).

Varieties listed below are finitarily universal. The conclusion of Corollary 3.6 thus holds for:

- any variety \( \mathbb{A}(A) \) of all algebras of a similarity type \( A \) with \( \Sigma A \geq 2 \), see [15];
- the variety of semigroups, see [7] or [15];
- the variety of rings with 1, see [4] or [15];
- the variety of De Morgan algebras, see [3];
- the variety of totally symmetric quasigroups, see [10];
- all finite-to-finite universal finitely generated varieties of distributive double \( p \)-algebras — these varieties were fully characterized in [11];
- all alg-universal varieties of \((0,1)\)-lattices — characterized in [5] as the varieties containing a simple non-distributive lattice.

It is not known whether or not every alg-universal variety of a finitary similarity type is finitarily universal, and hence Corollary 3.5 does not directly apply to such varieties. In some cases, however, other suitable full embeddings can be used.

A full embedding \( F : \mathcal{K} \to \mathcal{L} \) between concrete categories \( (\mathcal{K}, V) \) and \( (\mathcal{L}, U) \) is **strong** if there exists a set functor \( G : \text{Set} \to \text{Set} \) — called the **carrier** of \( F \) — such that \( U \circ F = G \circ V \). Since \( F(\text{Sur}_\mathcal{K}) \subseteq \text{Sur}_\mathcal{L} \) and \( F(\text{Inj}_\mathcal{K}) \subseteq \text{Inj}_\mathcal{L} \) for any strong embedding \( F \), and because \( F \) preserves colimits of directed diagrams whenever its carrier \( G \) does, we have

**Theorem 3.7.** Let \( V \) be a variety of algebras such that there exists a strong embedding \( F : \mathbb{A}(1,1) \to V \) whose carrier \( G \) is a quotient of a disjoint union of hom-functors \( \text{hom}(A, -) \) with finite \( A \). Let \( \mathcal{C} = \text{Inj}_V \) or \( \mathcal{C} = \text{Sur}_V \). Then, for any complete chain \( I \) and any family \( \mathcal{M}_I = \{M_i \mid i \in I\} \) of monoids, there is a sup-exact \( \mathcal{C} \)-chain \( T : I \to V \) representing \( \mathcal{M}_I \).

Theorem 3.7 applies to

- all alg-universal varieties of unary algebras, see [16];
- all alg-universal varieties of semigroups, see [13] — these are the varieties that contain all commutative semigroups and fail the identity \((xy)^n = x^ny^n\) for every \( n \geq 2 \).

**References**