QUANTALOIDS DESCRIBING CAUSATION AND PROPAGATION FOR PHYSICAL PROPERTIES

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Abstract

We study some particular examples of quantaloids and corresponding morphisms, originating from primitive physical reasonings on the lattices of properties of physical systems.

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1 Introduction

The starting point for our research program is the fact, already observed in Eilenberg and Mac Lane's seminal paper [8], that preordered sets may be considered as small thin categories. One can then not only reformulate a large part of the theory of order structures in categorical terms, but also apply general categorical techniques to specific order theoretic problems. In particular, the notion of an adjunction reduces to that of a residuation [15] §4.5, whereas the notion of a monad reduces to that of a closure operator [15] §6.1-2. Now the above categorical notions have direct physical interpretations in the context of axiomatic quantum theory, the order relation in the property lattice being semantic implication and the meet being operational conjunction [1, 2, 14, 20, 21, 22]. In particular, the equivalence between suitable categories of closure spaces and complete lattices determined by the existence of monadic comparison functors manifests the primitive duality between the state and property descriptions of a physical system [16, 17, 18]. Further, this static approach can be dynamically generalized by interpreting morphisms as transition structures [3, 6, 7], thereby providing an explicit physical realization of enrichment. Finally, far from being of merely aesthetic interest, the categorical approach to operational quantum

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theory allows the recovery of concrete representations of abstract notions in the Hilbertian context via the fundamental theorems of projective geometry [10, 11, 12]. In our opinion, then, category theory is as much a tool for the theoretical physicist as for the working mathematician.

Explicitly, we shall present a common extension of the representation theories of deterministic flows [13] and compound systems [5] to the dual notions of causation and propagation, construed as a physical polarity in the property lattice, and that lifts to a quantaloidal duality — quantaloids, introduced in [24] as a generalization of quantales [19], are categories enriched in join-complete lattices [23].

Let us first recall some of the basic notions of the operational approach to physics [21, 22, 1], and fix some notation. Given a (well defined) *physical system* Ξ we define a *test* α as a real experimental procedure relative to the system where we have defined in advance the so-called positive response. We call such a test *certain* for a particular realization of Ξ iff we *would* obtain the positive response *should* we perform the experiment. For two such tests α and β relative to Ξ, we set $\alpha \prec \beta$ iff β is certain whenever α is certain. The expression $\alpha \prec \beta$ then reveals a *physical law* for Ξ, and of course $\prec$ defines a preorder on $\mathcal{Q}(\Xi)$, the collection of all possible tests for Ξ. By standard quotienting techniques we can now work with the collection $L(\Xi)$ of equivalence classes of tests, two tests α and β being equivalent iff both $\alpha \prec \beta$ and $\beta \prec \alpha$, that then comes equipped with a partial order $\leq$ derived from the preorder $\prec$. The key point of this setup is that to any such equivalence class $[\alpha]$ corresponds an "element of physical reality" [9], called a *property* of Ξ [21]. If a test α is certain for a particular realization of the physical system Ξ, then the corresponding property $a = [\alpha]$ is said to be *actual* for this realization — otherwise it is *potential*. Under the (working) hypothesis that $L(\Xi)$ constitutes a set — although in principle all the following holds also for it being a thin category — it can then be proved that $(L(\Xi), \leq)$ is a complete lattice: given a subcollection $\mathcal{A}$ of $\mathcal{Q}(\Xi)$, defining $\prod \mathcal{A}$ as "choose any α in $\mathcal{A}$ as you wish and effectuate it", provides $L(\Xi)$ with a meet induced by $\wedge\{[\alpha] | \alpha \in \mathcal{A}\} = [\prod \mathcal{A}]$. By its construction, this meet is a physical *conjunction* (an aspect to which we will refer to as [con]) — but the corresponding join has no *a priori* physical significance, so it cannot be treated as a disjunction, *e.g.* orthodox quantum mechanics. It also clearly follows that $a \leq b$ in $L(\Xi)$ can be treated as an *implication relation* (referred to as [imp]), where we say that a is *stronger* than b. Finally, for each particular realization of a system Ξ, we can write ε for the subset of $L(\Xi)$ that contains precisely all the properties that are actual for this particular realization. As any such ε is a complete co-ideal, *i.e.*, closed under meets [con] and upperbounds [imp], it can be characterized by its strongest element $p_\varepsilon = \wedge\varepsilon \in L(\Xi)$. Therefore, for each realization of Ξ there exists a strongest actual property, which is appropriately called *state* of the system [21].

2 A categorical duality induced by causality

In this section we aim to give a common extension of the operational theory of on the one hand deterministic flows [13] and on the other compound systems [5]; the result of our analysis will be that the deeper structural ingredient in both situations is that "causation is adjoint to propagation".

Considering an evolving physical system Ξ, any test α relative to the system at time t_2 defines a test $\phi(\alpha)$ relative to the system at an earlier time t_1 as "evolve Ξ from t_1 to t_2 and effectuate α". The property $[\phi(\alpha)]$ has a clear interpretation, namely "guaranteeing actuality of $[\alpha]$". The assignment $L(\Xi) \to L(\Xi) : [\alpha] \mapsto [\phi(\alpha)]$, as we will see below, describes the evolution of Ξ. On the other hand, considering two interacting physical systems Ξ_1 and Ξ_2, any test α_2 on Ξ_2 defines a test $\phi(\alpha_2)$ on Ξ_1 by "let the systems interact and effectuate α_2". The assignment $L(\Xi_2) \to L(\Xi_1) : [\alpha_2] \mapsto [\phi(\alpha_2)]$ now encodes the interaction of Ξ_1 on Ξ_2.

Keeping these two cases in mind, for any two property lattices L_1 and L_2 we dispose of a *causal relation* $\rightsquigarrow \subseteq L_1 \times L_2$ where:

$$a_1 \rightsquigarrow a_2 \Leftrightarrow \text{"actuality of } a_1 \text{ guarantees actuality of } a_2\text{"}. \tag{1}$$

Lemma 1 *By the operational significance of* $\rightsquigarrow$ *the following holds:*

$$b_1 \le a_1, a_1 \rightsquigarrow a_2, a_2 \le b_2 \quad \Rightarrow \quad b_1 \rightsquigarrow b_2 \tag{2}$$

$$\forall a_2 \in A_2 : a_1 \rightsquigarrow a_2 \quad \Rightarrow \quad a_1 \rightsquigarrow \wedge A_2 \tag{3}$$

where A_2 *is a non-empty subset of* L_2.

Proof: The proof of eq.(2) relies on [imp], eq.(3) follows by [con]. □

(From an axiomatic point of view the conditions in the previous lemma are axioms for a causal relation.) Next, consider the following map prescription:

$$f : L_1 \setminus K \to L_2 : a_1 \mapsto \wedge\{a_2 \in L_2 \mid a_1 \rightsquigarrow a_2\}. \tag{4}$$

with $K = \{a_1 \in L_1 | \nexists a_2 \in L_2 : a_1 \rightsquigarrow a_2\}$, as such avoiding non-empty meets.

Lemma 2 *By lemma 1 and the explicit definition of* f *we have:*

$$a_1 \le a_1' \quad \Rightarrow \quad f(a_1) \le f(a_1') \tag{5}$$

$$a_1 \rightsquigarrow a_2 \quad \Leftrightarrow \quad f(a_1) \le a_2 \tag{6}$$

where it is understood that $a_1' \notin K$.

Proof: For eq.(5), remark that $a_1' \notin K \Rightarrow a_1 \notin K$ by lemma 1, so both $f(a_1)$ and $f(a_1')$ are defined; then computation shows that indeed $f(a_1) \leq f(a_1')$. In eq.(6), the sufficiency is trivial; to prove necessity is, by [imp], to prove that $a_1 \rightsquigarrow f(a_1)$, which is true by [con]. □

Now it is clear that $f(a_1)$ is the strongest property of L_2 the actuality of which is guaranteed by the actuality of a_1, i.e., f describes the *propagation of (strongest actual) properties.* Next, set:

$$f^* : L_2 \to L_1 : a_2 \mapsto \vee\{a_1 \in L_1 \mid a_1 \rightsquigarrow a_2\}. \tag{7}$$

Lemma 3 *By eq.(2) and the explicit definition of f^* we have:*

$$a_2 \leq a_2' \quad \Rightarrow \quad f^*(a_2) \leq f^*(a_2') \tag{8}$$

$$a_1 \rightsquigarrow a_2 \quad \Rightarrow \quad a_1 \leq f^*(a_2) \tag{9}$$

Proof: By computation. □

If moreover the condition $1_1 \rightsquigarrow 1_2$ can be derived from the physical particularity of the system under consideration (or formally, if it is an 'axiom' on $\rightsquigarrow$), then $K = \emptyset$ in eq.(4), and thus:

$$f(a_1) \leq a_2 \Leftrightarrow a_1 \rightsquigarrow a_2 \Rightarrow a_1 \leq f^*(a_2) \tag{10}$$

so it remains to show that eq.(11) is valid to obtain adjointness of f and f^*.

Lemma 4 *By the operational significance of f^* (via that of $\rightsquigarrow$) we have:*

$$a_1 \leq f^*(a_2) \quad \Rightarrow \quad a_1 \rightsquigarrow a_2. \tag{11}$$

Proof: Since $[\phi(\alpha_2)] \rightsquigarrow [\alpha_2]$ by the definition of ϕ, and since $a_1 \rightsquigarrow a_2$ implies that $a_1 \leq [\phi(\alpha_2)]$ we obtain that $f^*([\alpha_2]) = [\phi(\alpha_2)]$. Since eq.(11) is equivalent to $f^*(a_2) \rightsquigarrow a_2$ this completes the proof. □

(Note that formally eq.(11) is an additional 'axiom' on $\rightsquigarrow$.) Physically, lemma 4 states that there exists a well defined "weakest cause" $f^*(a_2)$ in L_1 of any a_2 in L_2, so f^* describes the *assignment of (weakest) causes (for actuality).* We can now read that f is adjoint to f^*.

In case that $1_1 \not\rightsquigarrow 1_2$, one can always extend the domain and codomain of f and f^* to the upper pointed extensions $L_1\dot{\cup}\underline{1}$ and $L_2\dot{\cup}\underline{1}$ of L_1 and L_2, such that $f^*(\underline{1}) = \underline{1}$, $f(\underline{1}) = \underline{1}$ and $\forall a_1 \in K : f(a_1) = \underline{1}$. Then again we obtain that $f \dashv f^*$, which fully justifies our slogan in the opening paragraph of this section. Physically, an interpretation of $\underline{1}$ follows from that of 1_1 and 1_2, respectively being existence of Ξ_1 and Ξ_2.

When considering L_1, L_2 and L_3 — or if necessary $L_i\dot{\cup}\underline{1}$ — denote the corresponding propagation of properties by $f_{i,j} : L_i \to L_j$.

Lemma 5 *By the operational significance of* $f^*_{i,j}$ *we have* $f^*_{1,3} = f^*_{1,2} \circ f^*_{2,3}$.

Proof: Denoting the corresponding tests for $f^*_{i,j}(a_j)$ by $\phi_{i,j}(\alpha_j)$ we clearly have $\phi_{1,3}(\alpha_3) = \phi_{1,2}(\phi_{2,3}(\alpha_3))$, so $[\phi_{1,3}(\alpha_3)] = [\phi_{1,2}(\phi_{2,3}(\alpha_3))]$ and thus it follows that $f^*_{1,3}(a_3) = f^*_{1,2}([\phi_{2,3}(\alpha_3)]) = f^*_{1,2}(f^*_{2,3}(a_3))$ for all $a_1 \in L_1$. □

By $f \circ g \dashv g^* \circ f^*$ for $f \dashv f^*$ and $g \dashv g^*$ we also obtain $f_{1,3} = f_{2,3} \circ f_{1,2}$ and thus, when equipping the maps f^* and f with a composition operation $\circ$ that respectively stands for "chaining" causal assignments and "consecutive" propagation of properties, we obtain categories of meet-complete and join-complete lattices — again, upper pointed if necessary — equipped with the duality $L \mapsto L; f^* \mapsto f$. Since $\underline{\mathrm{JCLat}}$ (and $\underline{\mathrm{JCLat}}_{\dot\cup\underline{1}}$) are quantaloids for pointwise order and the assignment $\underline{\mathrm{MCLat}}(L_2, L_1) \to \underline{\mathrm{JCLat}}(L_1, L_2) : f^*_{1,2} \mapsto f_{1,2}$ is a local duality with respect to pointwisely computed ordering we have a representation of our setting in $\underline{\mathrm{QUANT}}$:

$$\begin{array}{ccc} \underline{\mathrm{MCLat}}^{coop}_{\dot\cup\underline{1}} & \hookrightarrow & \underline{\mathrm{MCLat}}^{coop} \\ \cong\updownarrow & & \cong\updownarrow \\ \underline{\mathrm{JCLat}}_{\dot\cup\underline{1}} & \hookrightarrow & \underline{\mathrm{JCLat}} \end{array}$$

By the pointwisely computability of the enrichment $\bigvee$ of the $\underline{\mathrm{JCLat}}$ Hom-sets via the underlying $\vee$, conjunction $\wedge$ for properties is dually "lifted" to "conjunction $\bigwedge$ for causal assignments" in the Hom-sets of $\underline{\mathrm{MCLat}}$ (and implicitly $\underline{\mathrm{MCLat}}_{\dot\cup\underline{1}}$). We can conclude all this by:

Theorem 1 *Causal assignment and propagation of properties are dualized by a quantaloidal morphism* $F : \underline{\mathrm{MCLat}}^{coop} \to \underline{\mathrm{JCLat}}$.

We will now briefly discuss some examples of this general setting. The adjunction $[f^* : 1_2 \mapsto 1_1,\ \mathrm{rest} \mapsto 0_1] \vdash [f : 0_1 \mapsto 0_2,\ \mathrm{rest} \mapsto 1_2]$ describes 'separation' of the systems described by L_1 and L_2, a situation that previously could not be described in a consistent way within quantum theory [1, 5]. By way of contrast, for L_1 and L_2 atomistic the maps that send atoms to atoms or the bottom represent the strongest types of interaction, or analogously, maximally deterministic evolution. When considering lattices of closed subspaces of Hilbert spaces this setting yields representational theorems for the description of compound quantum systems by the Hilbert space tensor product [5] and description of evolution by Schrödinger flows [13], so it is exactly the enrichment that allows a joint consideration of the types of entanglement encountered in classical and quantum physics.

3 A physical origin of categorical enrichment

In order to introduce aspects of 'uncertainty' and 'arbitrary choice' we extend a property lattice $(L, \wedge)$ by introducing 'propositions' that represent disjunctions of properties [dis]. This is realized within $PL = 2^{L\setminus\{0\}}$ when equipped with the so called *operational resolution*:

$$\vee : PL \to L : A \mapsto \vee A \tag{12}$$

which, recalling the operational construction of L, physically stands for *verifiability* of collections of properties [3, 6]. Its right adjoint:

$$\vee^* : L \to PL : 0 \mapsto \emptyset; a \mapsto]0, a] \tag{13}$$

preserves *inf*'s such that [con] and thus also [imp] are preserved by this embedding of the properties L in the propositions PL. Given ε, call $A \in PL$ with $A \cap \varepsilon \neq \emptyset$ an *actuality set*, i.e., a set in which at least one property is actual. In absence of specification on ε, by $\vee A = \wedge\{b \in L | \forall a \in A : b \geq a\}$, [con] and [imp], we have that $\vee A$ is the strongest property whose actuality is guaranteed for an actuality set A. Thus, taking into account the previous section, propagation of actuality sets is described by maps $g : PL_1 \to PL_2$ for which there exists a $\wedge$-preserving map $f^* : L_2 \to L_1$, or equivalently, a $\vee$-preserving map $f : L_1 \to L_2$, such that:

$$\begin{array}{ccc} L_1 & \xleftarrow{\quad f^* \quad} & L_2 \\ & \top & \\ L_1 & \xrightarrow{\quad f \quad} & L_2 \\ \uparrow \vee & & \uparrow \vee \\ PL_1 & \xrightarrow{\quad g \quad} & PL_2 \end{array}$$

which in addition by [dis] should be $\cup$-preserving. It can easily be verified that whenever there exists a map $f : L_1 \to L_2$ such that $f \circ \vee = \vee \circ g$ when g is $\cup$-preserving then f is automatically $\vee$-preserving. As such, it remains to require that f is well defined, resulting in the continuity condition:

$$\vee A = \vee B \Rightarrow \vee g(A) = \vee g(B). \tag{14}$$

Next observe that all the following diagrams commute:

$$\begin{array}{ccc} L_1 & = & L_2 \\ \uparrow \vee & & \uparrow \vee \\ PL_1 & = & PL_2 \end{array} \qquad \begin{array}{ccccc} L_1 & \xrightarrow{f} & L_2 & \xrightarrow{f'} & L_3 \\ \uparrow \vee & & \uparrow \vee & & \uparrow \vee \\ PL_1 & \xrightarrow{g} & PL_2 & \xrightarrow{g'} & PL_3 \end{array}$$

$$\begin{array}{ccc} L_1 & \xrightarrow{\bigvee_i f_i} & L_2 \\ \uparrow \vee & & \uparrow \vee \\ PL_1 & \xrightarrow{\bigcup_i g_i} & PL_2 \end{array} \qquad \begin{array}{ccc} L_1 & \xrightarrow{f} & L_2 \\ \uparrow \vee & & \uparrow \vee \\ PL_1 & \xrightarrow{Pf} & PL_2 \end{array}$$

where by $\bigcup_i g_i$ we denoted the pointwise union of a family of such maps g_i, and where $Pf : PL_1 \to PL_2 : A \mapsto \{f(a) \mid a \in A\}$ is the direct image mapping for $f : L_1 \to L_2$.

Defining the bicategory $Q^{\#}\underline{\text{JCLat}}$ with objects the $PL = 2^{L\setminus\{0\}}$ for all complete lattices L, as morphisms the above by g denoted maps, and the local structure being the evident pointwise order of such maps, the above proves that this bicategory is a quantaloid. Furthermore, the action $F_{\#} : Q^{\#}\underline{\text{JCLat}} \to \underline{\text{JCLat}} : PL \mapsto L; g \mapsto f$ proves to be a full quantaloidal morphism, and we have the following scheme in $\underline{\text{QUANT}}$:

$$Q^{\#}\underline{\text{JCLat}} \stackrel{F_{\#}}{\longrightarrow} \underline{\text{JCLat}} \stackrel{\cong}{\longleftrightarrow} \underline{\text{MCLat}}^{coop}$$

that express how the propagation of actuality sets is related to causal assignments — with an obvious analogous in the case of $(\dot{\cup}\underline{1})$. The bifunctorial nature of $F_{\#}$ then reveals that the enrichment of the collection of causal assignments $\underline{\text{MCLat}}^{coop}$ originates — physically — from the presence of an underlying uncertainty encoded in the local structure of $Q^{\#}\underline{\text{JCLat}}$. Note that $Q^{\#}\underline{\text{JCLat}}$ neither coincides with the categories with the same objects and on the one hand all union preserving maps, and on the other hand pointwise unions of direct image maps of $\vee$-preserving maps — a precise characterization can be found in [7]. However, the considerations in this section can be refined, *e.g.* in case that L is a complete Boolean algebra then the extension of $L \mapsto PL$ is redundant in the sense that the join in L is already to be understood as a disjunction.

References

[1] D. Aerts, Description of many separated physical entities without the paradoxes encountered in quantum mechanics, *Found. Phys.* **12** (1982) 1131-1170.

[2] D. Aerts, Quantum structures, separated physical entities and probability, *Found. Phys.* **24** (1994) 1227-1259.

[3] H. Amira, B. Coecke, I. Stubbe, How quantales emerge by introducing induction within the operational approach, *Helv. Phys. Acta* **71** (1998) 554-572.

[4] G. Birkhoff, J. von Neumann, The logic of quantum mechanics, *Ann. Math.* **37** (1936) 823-843.

[5] B. Coecke, Structural characterization of compoundness, *Int J. Theor. Phys.* **39** (2000) 581–590.

[6] B. Coecke, I. Stubbe, Operational resolutions and state transitions in a categorical setting, *Found. Phys. Lett.* **12** (1999) 29-49.

[7] B. Coecke, I. Stubbe, State transitions as morphisms of complete lattices, *Int J. Theor. Phys.* **39** (2000) 601–610.

[8] S. Eilenberg, S. Mac Lane, General theory of natural equivalences, *Trans. AMS* **58** (1945) 231-294.

[9] A. Einstein, B. Podolsky, N. Rosen, Can quantum-mechanical description of physical reality be considered complete, *Phys. Rev.* **47** (1935) 777-780.

[10] Cl.-A. Faure, A. Frölicher, Morphisms of projective geometries and of corresponding lattices, *Geom. Ded.* **47** (1993) 25-40.

[11] Cl.-A. Faure, A. Frölicher, Morphisms of projective geometries and semilinear maps, *Geom. Ded.* **53** (1994) 237-269.

[12] Cl.-A. Faure, A. Frölicher, Categorical aspects in projective geometry, *Appl. Cat. Struct* **6** (1996) 87-103.

[13] Cl.-A. Faure, D.J. Moore, C. Piron, Deterministic evolutions and Schrödinger flows, *Helv. Phys. Acta* **68** (1995) 150-157.

[14] J.M. Jauch, C. Piron, On the structure of quantal proposition systems, *Helv. Phys. Acta* **42** (1969) 842-848.

[15] S. Mac Lane, *Categories for the Working Mathematician*, Springer-Verlag, Berlin, 1971.

[16] D.J. Moore, Categories of representations of physical systems, *Helv. Phys. Acta* **68** (1995) 658-678.

[17] D.J. Moore, Closure categories, *Int. J. Theor. Phys.* **36** (1997) 2707-2723.

[18] D.J. Moore, On state spaces and property lattices, *Stud. Hist. Phil. Mod. Phys.* **30** (1999) 61-83.

[19] C.J. Mulvey, &, *Rend. Circ. Math. Palermo* **12** (1986) 99-104.

[20] C. Piron, Axiomatique quantique, *Helv. Phys. Acta* **37** (1964) 439-468.

[21] C. Piron, *Foundations of Quantum Physics*, W.A. Benjamin, Reading, 1976.

[22] C. Piron, *Mécanique Quantique Bases et Applications*, Presses polytechniques et universitaires romandes, Lausanne, 1990 & 1998.

[23] A.M. Pitts, Applications of sup-lattice enriched category theory to sheaf theory, *Proc. London Math. Soc.* **57** (1988) 433-480.

[24] K.I. Rosenthal, Free quantaloids, *J. Pure. Appl. Alg.* **77** (1991) 67-82.