

ON GROWTH TYPES OF QUOTIENTS OF COXETER GROUPS BY PARABOLIC SUBGROUPS

SANKARAN VISWANATH

ABSTRACT. The principal objects studied in this note are Coxeter groups W that are neither finite nor affine. A well known result of de la Harpe asserts that such groups have exponential growth. We consider quotients of W by its parabolic subgroups and by a certain class of reflection subgroups. We show that these quotients have exponential growth as well. To achieve this, we use a theorem of Dyer to construct a reflection subgroup of W that is isomorphic to the universal Coxeter group on three generators. The results are all proved under the restriction that the Coxeter diagram of W is simply laced, and some remarks made on how this restriction may be relaxed.

1. INTRODUCTION

Let W be a finitely generated group and S be a finite set of generators for W . The *growth function* $\gamma(m)$ is the number of elements of W expressible as a word of length m or less in $S \cup S^{-1}$. We say that W has (i) polynomial growth if $\exists C \in \mathbb{R}^{>0}$ and $d \in \mathbb{Z}^{\geq 0}$ such that $\gamma(m) \leq Cm^d \forall m \geq 0$, (ii) exponential growth if $\exists \lambda > 1$ such that $\gamma(m) \geq \lambda^m \forall m \geq 0$ and (iii) intermediate growth otherwise. For preliminaries on growth types of finitely generated groups, we refer the reader to de la Harpe's monograph [2, Chap. VI, VII] or to section 2 below.

We now specialize to the case where (W, S) is an irreducible Coxeter system. If W is a finite or affine Coxeter group, it can be easily seen to have polynomial growth. When W is an infinite, non-affine Coxeter group, it is a classical result of de la Harpe [1] that W has exponential growth. In this note, we consider the latter case. We will prove the slightly stronger result that for any proper parabolic subgroup W_J of W , the quotient W/W_J has exponential growth too. This quotient can be identified with the set W^J of minimal length left coset representatives; by the "growth type of W/W_J " we will mean the growth type of the subset W^J of W .

We remark that this assertion about the quotient W/W_J does not follow directly from the exponential growth of W given by de la Harpe's theorem. The group W could have parabolic subgroups W_J that are infinite, non-affine and thus of exponential growth themselves. For such W_J , the growth type of the quotient W/W_J is not apriori determined.

2000 *Mathematics Subject Classification.* 20F55.

Key words and phrases. Coxeter group, parabolic subgroup, growth.

The main ingredient in our approach to the growth of W/W_J is the work of Deodhar [4] and Dyer [6] on reflection subgroups of Coxeter groups. Using a criterion of Dyer, we construct a specific reflection subgroup of W ; this subgroup will turn out to have two properties of interest to us : (i) it is isomorphic to the “universal” Coxeter group on three generators, and (ii) distinct reflections in this subgroup belong to distinct cosets in W/W_J . These properties will enable us to deduce the exponential growth of W/W_J .

We apply this result on the growth of W/W_J to study the growth of more general quotients W/W' , where W' is a reflection subgroup of W . We identify a class of reflection subgroups W' of W for which the quotients W/W' have exponential growth.

We'll work throughout under the hypothesis that W is simply laced; this restriction can however be relaxed and we indicate this in the relevant places (see remark 1).

Here's a quick outline of the rest of the article: in section 2, after preliminaries on growth types, we state our main theorem. Section 3 is concerned with the construction of the special reflection subgroup mentioned above, and section 4 collects together some well known facts about universal Coxeter groups. These facts are then applied to our reflection subgroup to complete the proof of the main theorem in section 5. In the final section, we study the more general quotients W/W' , where W' is a reflection subgroup satisfying some additional hypothesis.

2. GROWTH TYPES

2.1. We follow [2, Chapter VI.C] :

Definition 1. Given a non decreasing sequence $(a_k)_{k \geq 0}$ of natural numbers, its *exponential growth rate* is defined to be $\omega := \limsup_{k \rightarrow \infty} a_k^{1/k}$.

Now suppose (W, S) is a Coxeter system and $F \subset W$ is a subset such that $1 \in F$. For $k \geq 0$, set $\gamma(F, k) := \#\{f \in F : \ell(f) \leq k\}$ and $\omega(F) := \limsup_{k \rightarrow \infty} \gamma(F, k)^{1/k}$. Since $1 \leq \gamma(F, k) \leq (\#S + 1)^k \forall k$, we have $1 \leq \omega(F) \leq (\#S + 1)$.

Definition 2. We say that F has exponential growth if $\omega(F) > 1$ and subexponential growth otherwise.

A special case of subexponential growth is *polynomial growth*, which occurs if $\exists C \in \mathbb{R}^{>0}$ and $d \in \mathbb{Z}^{\geq 0}$ such that $\gamma(F, k) \leq Ck^d$ for all $k \geq 0$. If F is of subexponential growth and not of polynomial growth, we say it has *intermediate growth*.

When $F = W$, the function $\gamma(W, k)$ is submultiplicative i.e, satisfies $\gamma(W, k+l) \leq \gamma(W, k) \cdot \gamma(W, l)$. This implies (see [2, VI.56]) that $\lim_{k \rightarrow \infty} \gamma(W, k)^{1/k}$ exists and equals $\inf_{k \geq 0} \gamma(W, k)^{1/k}$. Thus we get an equivalent formulation: W has exponential growth iff $\exists \lambda > 1$ such that $\gamma(W, k) \geq \lambda^k$ for all $k \geq 0$.

If F is a proper subset of W , then submultiplicativity need not hold, and we will be content with definition 2 for our notion of exponential growth.

2.2. Rational generating functions and growth. Given $\{1\} \subset F \subset W$ as above, let $\gamma_F(q) \in \mathbb{C}[[q]]$ be the generating function:

$$\gamma_F(q) := \sum_{k=0}^{\infty} \gamma(F, k) q^k$$

Observe that $\omega(F)^{-1}$ is the radius of convergence of this power series. For a Coxeter group W , there are many natural choices of F (e.g parabolic subgroups, their minimal coset representatives) for which $\gamma_F(q)$ is a rational function. When this happens, one clearly also has:

Proposition 1. *Suppose $\gamma_F(q)$ is a rational function. Then F has exponential growth iff $\gamma_F(q)$ has a pole ξ with $0 < |\xi| < 1$.*

See [9, proposition 3.3] for the situation when F has polynomial growth.

2.3. Let (W, S) be an irreducible Coxeter system. Let $W_J, J \subsetneq S$ be a parabolic subgroup of W , with W^J being the set of minimal length elements in left cosets of W_J . Recall that each $w \in W$ can be uniquely written as $w = \sigma\tau, \sigma \in W^J, \tau \in W_J$ with $\ell(w) = \ell(\sigma) + \ell(\tau)$. Our objective is to study the growth type of the subset W^J .

When W is finite or affine, it is easy to see that the set W^J has the same growth type as W . We consider the case where W is an infinite, non-affine Coxeter group. We will further assume that the Coxeter diagram of W is connected and simply laced i.e for each pair $s \neq s' \in S, ss'$ has order 2 or 3 in W . Our main theorem is the following:

Theorem 1. *Let (W, S) be an irreducible Coxeter system. Suppose W is infinite, non-affine and has a simply laced Coxeter diagram. Then for all $J \subsetneq S, W^J$ has exponential growth.*

Corollary 1. *Under the assumptions of theorem 1, the Poincaré series (length generating function) of W^J has a pole ξ with $0 < |\xi| < 1$.*

Proof of Corollary 1: If $W^J(q) = \sum_{\sigma \in W^J} q^{\ell(\sigma)}$ is the Poincaré series of W^J , we have $\gamma_{W^J}(q) = W^J(q)/(1 - q)$. The corollary now follows from theorem 1 and proposition 1. \square

Observe that if W_J is a finite group, the assertion of Theorem 1 is trivial. So we may as well assume that W_J is infinite. To show that W/W_J has exponential growth, we will do two things: (A) construct a large (exponential in m) number of elements in W of length $\leq m$ and (B) show that these elements lie in distinct left cosets of W_J .

To achieve step (B), we will employ the following nice result due to Deodhar [3]:

Theorem 2. (Deodhar) Let (W, S) be a Coxeter system, $T := \bigcup_{w \in W} wSw^{-1}$ be the set of reflections and $J \subset S$. If $t_1, t_2 \in T \setminus W_J$ with $t_1 \neq t_2$, then $t_1W_J \neq t_2W_J$; i.e distinct elements of $T \setminus W_J$ are in distinct left cosets of W_J .

The next proposition makes step (A) above more precise:

Proposition 2. Assume notation as in the statement of theorem 1. Suppose also that W_J is an infinite group. Then there exists a natural number M such that for all $k \geq 0$, \exists at least 2^k elements $t \in T \setminus W_J$ with $\ell(t) \leq M(2k + 1)$.

Given the truth of this proposition, we now have:

Proof of theorem 1: For $t \in T$, let $[t] \in W^J$ denote the unique minimal length element in tW_J . For $t \in T \setminus W_J$ as in proposition 2, $\ell([t]) \leq \ell(t) \leq M(2k + 1)$. Invoking theorem 2, we conclude that there exist at least 2^k elements $w \in W^J$ with $\ell(w) \leq M(2k + 1)$ i.e $\gamma(W^J, M(2k + 1)) \geq 2^k$. This gives for $k \geq 1$:

$$\gamma(W^J, M(2k + 1))^{1/M(2k+1)} \geq 2^{k/M(2k+1)} \geq 2^{1/3M}$$

Thus $\omega(W^J) = \limsup_{k \rightarrow \infty} \gamma(W^J, k)^{1/k} \geq 2^{1/3M} > 1$. So W^J has exponential growth. This completes the proof of theorem 1 \square

The next three sections will be devoted to a proof of proposition 2.

3. A REFLECTION SUBGROUP ISOMORPHIC TO $W_{(3)}$

3.1. As a first step toward proving proposition 2, we will construct a reflection subgroup of W that is isomorphic to the universal Coxeter group $W_{(3)} = \langle s_1, s_2, s_3 : s_i^2 = 1 \forall i \rangle$ [5] with Coxeter diagram



We collect together the relevant facts about reflection subgroups from Deodhar [4] and Dyer [6]. We recall that the elements of the set $T := \bigcup_{w \in W} wSw^{-1}$ are called *reflections*.

Definition 3. Let (W, S) be a Coxeter system. A subgroup W' of W generated by reflections is called a reflection subgroup.

Reflection subgroups of Coxeter groups turn out to be Coxeter groups in their own right. Specifically:

Theorem 3. (Deodhar [4], Dyer [6]) Let W' be a reflection subgroup of W . Then $\exists S' \subset W' \cap T$ such that S' forms a set of Coxeter generators for W' .

3.2. Let (W, S) be a Coxeter system with simply laced Coxeter diagram X . We assume that the nodes of X are labelled by the elements of S . We let V denote the geometric representation of W [7, §5.3]; V has a basis $\{\alpha_s : s \in S\}$ and a symmetric bilinear form $(,)$ determined by the conditions: (i) $(\alpha_s, \alpha_s) = 1 \forall s \in S$; (ii) $(\alpha_p, \alpha_q) = -1/2$ when $p \neq q \in S$ and the nodes

p and q are connected by an edge in X and $(\alpha_p, \alpha_q) = 0$ otherwise. The W action on V preserves the form $(,)$ and is determined by $s(v) = v - 2(v, \alpha_s)\alpha_s$ where $s \in S, v \in V$.

Let $\Phi(W) = W \cdot \{\alpha_s : s \in S\} \subset V$ be the root system of W [7, §5.4] and let $\Phi^+(W)$ (resp. $\Phi^-(W)$) denote the set of positive (resp. negative) roots of W . The set $\Phi^+(W)$ is in 1-1 correspondence with the set T via $\alpha \mapsto s_\alpha$ defined by $s_\alpha(v) := v - 2(v, \alpha)\alpha$. The following theorem of Dyer [6, Theorem 4.4] will be important in what follows:

Theorem 4. (*Dyer*) *Let $F \subset \Phi^+(W)$, $S' := \{s_\alpha : \alpha \in F\}$ and W' be the reflection subgroup of W generated by S' . Suppose for all $\alpha \neq \beta \in F$, $(\alpha, \beta) \in \{-\cos(\pi/n) : n \in \mathbb{N}, n \geq 2\} \cup (-\infty, -1]$, then S' is precisely the set of Coxeter generators of W' given by theorem 3.*

Corollary 2. *If $\beta_i (i = 1, 2, 3) \in \Phi^+(W)$ are such that $(\beta_i, \beta_j) \leq -1$ for all $i \neq j$, then the reflection subgroup $W' = \langle s_{\beta_1}, s_{\beta_2}, s_{\beta_3} \rangle \subset W$ is isomorphic to $W_{(3)}$.*

Proof: By Theorem 4 the s_{β_i} are the Coxeter generators of W' . It is an easy fact (see for e.g [7, §5.3]) that if $(\beta_i, \beta_j) \leq -1$, then $\text{order}(s_{\beta_i}s_{\beta_j}) = \infty$. Thus $W' \cong W_{(3)}$. \square

3.3. We now assume the notation as in the statement of Proposition 2. So (i) (W, S) is an irreducible Coxeter system, (ii) W is infinite, non-affine with simply laced Coxeter diagram X and (iii) $J \subsetneq S$ with W_J infinite. We will use corollary 2 to construct a reflection subgroup of W isomorphic to $W_{(3)}$.

In what follows, we will identify (without explicit mention) subsets K of S with the corresponding subdiagram of X formed by taking only the nodes labelled by K , together with all edges between these nodes.

First, we decompose $J = \bigsqcup J_j$ where the J_j are the *connected components* of J . Since $\prod_j W_{J_j} \cong W_J$ is infinite, $\exists i$ such that W_{J_i} is infinite. Let $Z := J_i$; Z is thus not a diagram of *finite type*. It is a classical result (verifiable by hand) that any connected, simply laced diagram either contains or is contained in one of the affine simply laced diagrams $\tilde{A}_n, n \geq 2, \tilde{D}_n, n \geq 4, \tilde{E}_n, n = 6, 7, 8$ (this result can in fact be used to quickly classify the finite simply laced Coxeter groups). Applying this to Z , one concludes that Z must *contain* an affine diagram Y ; this is because if Z were properly contained in an affine diagram, then Z would end up being of finite type.

Now pick $p \in S \setminus J$; clearly $p \notin Y$. Since the Coxeter diagram X is connected, we can pick a shortest path in X between p and Y ; i.e, $\exists s_0, s_1, \dots, s_r \in S$ such that

- (1) $s_0 = p$.
- (2) $s_r \in Y$.
- (3) s_i and s_{i+1} are connected by an edge in $X \forall i$.
- (4) r is the smallest such natural number.

The minimality of r is easily seen to imply the following:

- (1) The $s_i, 0 \leq i \leq r$ are pairwise distinct.
- (2) $s_i \notin Y$ for $i < r$.
- (3) s_i and s_j are not connected by an edge in X if $|i - j| > 1$.

Thus the subdiagram of X formed by the nodes labelled s_i is just the classical diagram A_{r+1} .

Now, if K is a subset of S , we will naturally identify $\Phi(W_K)$ with the subset $W_K \cdot \{\alpha_k : k \in K\}$ of $\Phi(W)$. We recall that since Y is an affine diagram, there exists $\delta_Y \in \Phi^+(W_Y)$ such that $(\delta_Y, \alpha_q) = 0 \forall$ nodes $q \in Y$. Further, if $\delta_Y = \sum_{q \in Y} c_q \alpha_q$, then we have $c_q \geq 1$ for all $q \in Y$.

Now, define positive roots $\beta_i (i = 1, 2, 3) \in \Phi^+(W)$ as follows:

$$\begin{aligned}\beta_1 &:= \sum_{i=0}^{r-1} \alpha_{s_i} = s_{r-1} \cdots s_2 s_1(\alpha_p) \\ \beta_2 &:= \alpha_{s_r} + \delta_Y \\ \beta_3 &:= -\alpha_{s_r} + 3\delta_Y\end{aligned}$$

Observe that the β_i are linearly independent. By the well known characterization of positive roots of affine Coxeter groups, we have $\beta_2, \beta_3 \in \Phi^+(W_Y) \subset \Phi^+(W)$. Further β_1 has been explicitly demonstrated to be an element of $\Phi^+(W)$. For $i = 2, 3$, if we write $\beta_i = \sum_{q \in Y} c_q^{(i)} \alpha_q$, then we have $c_{s_r}^{(i)} \geq 2$ for both values of i . So

$$\begin{aligned}(\beta_1, \beta_2) &= \left(\sum_{i=0}^{r-1} \alpha_{s_i}, \alpha_{s_r} + \delta_Y \right) = \left(\sum_{i=0}^{r-1} \alpha_{s_i}, c_{s_r}^{(2)} \alpha_{s_r} + \sum_{q \in Y, q \neq s_r} c_q^{(2)} \alpha_q \right) \\ &\leq (\alpha_{s_{r-1}}, 2\alpha_{s_r}) = -1\end{aligned}$$

Similarly $(\beta_1, \beta_3) \leq -1$ too. Finally $(\beta_2, \beta_3) = (\alpha_{s_r} + \delta_Y, -\alpha_{s_r} + 3\delta_Y) = (\alpha_{s_r}, -\alpha_{s_r}) = -1$. Corollary 2 can now be applied to deduce:

Proposition 3. $\langle s_{\beta_1}, s_{\beta_2}, s_{\beta_3} \rangle \cong W_{(3)}$

Remark 1. Suppose the Coxeter diagram of W is not simply laced, but J contains an affine subdiagram Y , then it is clear that the above construction can still be carried out with some simple modifications. In particular if for all pairs $s \neq s' \in S, (ss')^{m_{ss'}} = 1$ with $m_{ss'} = 2, 3$ or ∞ and $\text{rank } W \geq 3$, the above construction works for all $J \subsetneq S$ with $\#W_J = \infty$.

We also note the following interesting corollary to the above construction:

Corollary 3. Let (W, S) be an irreducible Coxeter system. If W is infinite, non-affine and simply laced, then W contains a reflection subgroup isomorphic to the universal Coxeter group $W_{(3)}$.

Proof: Let X be the Coxeter diagram of W . We take Y to be an affine subdiagram of X , p to be a node in $X \setminus Y$, and repeat the argument that proves proposition 3 above. \square

Remark 2. 1. In view of remark 1, corollary 3 also holds for non simply laced, irreducible Coxeter groups W whose Coxeter diagrams contain a proper affine subdiagram.

2. It is easy to see that (cf §4 below) that $W_{(3)}$ has exponential growth. Thus this proposition gives another proof (in the simply laced case) of the result mentioned in the introduction: an irreducible Coxeter group W which is infinite and non-affine has exponential growth.

4. PROPERTIES OF $W_{(3)}$

To complete the proof of proposition 2, we must study the reflection subgroup constructed in proposition 3 more closely. We collect together some useful properties of the Coxeter group $W_{(3)}$. Note that $W_{(3)}$ is just the free product of three groups of order 2. The following facts are all fairly standard, and we omit proofs:

Proposition 4. (1) The Poincaré series $W_{(3)}(q) = \frac{1+q}{1-2q}$.

(2) Each $w \in W_{(3)}$ has a unique reduced expression as a product of Coxeter generators.

(3) If $S_{(3)} := \{s_1, s_2, s_3\}$ are the Coxeter generators of $W_{(3)}$, the conjugacy classes of the s_i are pairwise disjoint. Further $w \in W_{(3)}$, $ws_iw^{-1} = s_i \Leftrightarrow w \in \{1, s_i\}$.

We let $T_{(3)} := \bigcup_{w \in W_{(3)}} wS_{(3)}w^{-1}$ be the set of reflections in $W_{(3)}$. Proposition 4 implies that $T_{(3)}$ is a disjoint union of the orbits of the s_i , ($i = 1, 2, 3$) under the conjugation action of $W_{(3)}$; further, the stabilizer of s_1 is $\{1, s_1\}$. Let $\mathcal{O}_1 \subset T_{(3)}$ be the orbit of s_1 ; if we let $K := \{s_1\} \subset S_{(3)}$, then $\mathcal{O}_1 = \{\sigma s_1 \sigma^{-1} : \sigma \in W_{(3)}^K\}$, with $\sigma s_1 \sigma^{-1} \neq \tau s_1 \tau^{-1}$ for $\sigma \neq \tau \in W_{(3)}^K$; here $W_{(3)}^K$ is the set of minimal left coset representatives of the parabolic subgroup $(W_{(3)})_K = \{1, s_1\}$.

The Poincaré series of $W_{(3)}^K$ is $W_{(3)}(q)/(1+q) = 1/(1-2q)$; so for each $k \geq 0$, there are 2^k elements $\sigma \in W_{(3)}^K$ such that $\ell(\sigma) = k$. For these σ , $\ell(\sigma s_1 \sigma^{-1}) \leq 2\ell(\sigma) + 1 = 2k + 1$. So,

Proposition 5. For each $k \geq 0$, there are $\geq 2^k$ elements $t \in \mathcal{O}_1$ such that $\ell(t) \leq 2k + 1$.

Let $V_{(3)}$ be the geometric representation of $W_{(3)}$ with basis $\{\alpha_1, \alpha_2, \alpha_3\}$ and invariant bilinear form $(,)$. We remark that there are many choices for the $W_{(3)}$ invariant form $(,)$ on $V_{(3)}$. It only needs to satisfy $(\alpha_i, \alpha_i) = 1 \forall i$ and $(\alpha_i, \alpha_j) \in \mathbb{Z}^{\leq -1}$ $i \neq j$. Let $\Phi(W_{(3)}) \subset V_{(3)}$ be the root system of $W_{(3)}$. We then have:

Proposition 6. Let $\alpha \in \Phi^+(W_{(3)})$ such that $s_\alpha \in \mathcal{O}_1$. If $\alpha = \sum_{i=1}^3 c_i \alpha_i$ ($c_i \in \mathbb{Z}^{\geq 0}$), then $c_1 > 0$.

Proof: Given $\gamma_1, \gamma_2 \in \Phi(W_{(3)})$ write $\gamma_1 > \gamma_2$ if $\gamma_1 - \gamma_2$ is a nonnegative integer linear combination of the α_i . It is a well known fact (see for e.g the argument used in [8, proposition 5.1(e)]) that given a positive root α , there exists a sequence $\gamma_0 > \gamma_1 > \dots > \gamma_r$ such that (i) $\gamma_0 = \alpha$, $\gamma_r \in \{\alpha_i : i = 1, 2, 3\}$, $\gamma_j \in \Phi^+(W_{(3)}) \forall j$ (ii) For each p , $\gamma_{p+1} = s_{i_p}(\gamma_p)$ for some $i_p \in \{1, 2, 3\}$.

Thus each $\gamma_p \in W_{(3)} \cdot \alpha$ or equivalently s_{γ_p} is $W_{(3)}$ conjugate to s_α . The disjointness of the orbits of the s_i mentioned before and the hypothesis that $s_\alpha \in \mathcal{O}_1$ imply that $\gamma_r = \alpha_1$. So $\alpha = \alpha_1 + \sum_{p=0}^{r-1} (\gamma_p - \gamma_{p+1})$. \square

5. PROOF OF PROPOSITION 2

We now put together the results of the previous two sections. Let W, S, J, β_i be as in §3.3. Let $W' = \langle s_{\beta_i} : i = 1, 2, 3 \rangle$ be the reflection subgroup isomorphic to $W_{(3)}$ constructed in Proposition 3. Let $S' := \{s_{\beta_i} : i = 1, 2, 3\}$. Define $\Phi(W') := W' \cdot \{\beta_i\}_{i=1}^3 \subset \Phi(W)$. We identify $\Phi(W_{(3)})$ with $\Phi(W')$ by sending $\alpha_i \mapsto \beta_i$ and requiring that this map commute with the $W_{(3)}$ action (for this identification to be a linear map of the underlying vector spaces, we will need to use the form on $V_{(3)}$ that satisfies $(\alpha_i, \alpha_j) = (\beta_i, \beta_j) \forall i, j$). Now, applying propositions 5 and 6 to $W' \cong W_{(3)}$, we deduce that for each $k \geq 0$, there are $\geq 2^k$ elements $\beta \in \Phi^+(W') \subset \Phi^+(W)$ such that $\ell_{S'}(s_\beta) \leq 2k + 1$ and $\beta = \sum_{i=1}^3 c_i \beta_i$ with $c_1 > 0$. Here $\ell_{S'}(\cdot)$ denotes the length function of W' w.r.t S' . Now let $M := \max\{\ell(s_{\beta_i}) : i = 1, 2, 3\}$, where $\ell(\cdot)$ is the usual length function on W w.r.t S . We clearly have $\ell(w) \leq M \ell_{S'}(w) \forall w \in W'$. Thus for the β 's above, $\ell(s_\beta) \leq M(2k + 1)$. We now *Claim:* for each of the above β 's, $s_\beta \in T \setminus W_J$. Referring back to the statement of Proposition 2, we see that this claim together with what we have shown thus far would complete the proof of that proposition.

Proof of Claim: Recall from §3.3 that p was chosen to be an element of $S \setminus J$, and that $\beta_1 = \sum_{q \in S} d_q \alpha_q$ with $d_p = 1$. Now each of the β 's of the above paragraph can be written as $\beta = \sum_{i=1}^3 c_i \beta_i$ with $c_1 > 0$. It follows then that we can write $\beta = \sum_{q \in S} e_q \alpha_q$ with $e_p > 0$. Since $p \notin J$, this means that β is not a linear combination of the simple roots $\alpha_q, q \in J$. It is an easy fact that this implies $s_\beta \notin W_J$ (sketch of proof: If $w \in W_J, w(\beta) = \beta - (\text{a linear combination of } \alpha_q, q \in J) = \sum_{u \in S} k_u \alpha_u$ with $k_p = e_p > 0$. Thus $w(\beta) \in \Phi^+(W), \forall w \in W_J$. But $s_\beta(\beta) = -\beta \in \Phi^-(W)$; this gives the desired contradiction).

Thus, putting everything back together, our main theorem 1 is proved. \square

6. QUOTIENTS BY REFLECTION SUBGROUPS

We assume (W, S) to be a simply laced, irreducible Coxeter system which is neither finite nor affine. As usual, we let V be its geometrical realization, (\cdot, \cdot) the invariant bilinear form etc. Let W' be a finitely generated reflection subgroup of W and $S' = \{s_{\beta_i}\}_{i=1}^k$ be its Coxeter generators as in theorem 3, with $\beta_i \in \Phi^+(W)$. Let $\Phi(W') := W' \cdot \{\beta_i\}_{i=1}^k$ be its root system.

It was shown by Dyer [6, (3.4)] that the left cosets of W' in W have *unique* elements of minimal length; these elements w are determined by the condition that $\ell(ws_{\beta_i}) > \ell(w) \forall i$ or equivalently by the condition $w(\beta_i) \in \Phi^+(W) \forall i$. We let $[W']$ denote the set of these minimal coset representatives. We remark that while each $w \in W$ can be uniquely written as $w = \sigma\tau$ with $\sigma \in [W'], \tau \in W'$, it may no longer be true that $\ell(w) = \ell(\sigma) + \ell(\tau)$. The natural question now is to study the growth of W/W' or more precisely, the growth of $[W']$.

To make our arguments simpler, we assume further that W' is irreducible as a Coxeter group, leaving the details of the reducible case to the reader. We first consider an additional hypothesis on the W' :

Lemma 1. *Let notation be as above. TFAE:*

- (1) $\exists \alpha \in \Phi^+(W) \setminus \Phi^+(W')$ such that its W' orbit satisfies $\{\alpha\} \subsetneq W' \cdot \alpha \subset \Phi^+(W)$.
- (2) $\exists \gamma \in \Phi^+(W) \setminus \Phi^+(W')$ such that $(\gamma, \beta_i) \leq 0 \forall i = 1 \cdots k$, with not all (γ, β_i) equal to zero.

Proof: (1) \Rightarrow (2): For a positive root $\beta = \sum_{s \in S} c_s \alpha_s \in \Phi^+(W)$, define its *height* to be $\sum c_s (> 0)$. Given α as in (1), let γ be an element in the W' orbit $W' \cdot \alpha$ of minimal height. It is clear that γ satisfies (2). For (2) \Rightarrow (1): given such γ , it is a standard argument that $\forall \sigma \in W', \gamma - \sigma\gamma = \sum_{i=1}^k d_i \beta_i$ with $d_i \leq 0 \forall i$. Thus $\sigma\gamma \in \Phi^+(W) \forall \sigma \in W'$. So $\alpha := \gamma$ satisfies (1). \square

The next proposition explains the usefulness of the above conditions:

Proposition 7. *If W' satisfies the equivalent conditions of lemma 1, then $[W']$ has exponential growth.*

Proof: First observe that if W' is finite or affine, the truth of the proposition follows from the exponential growth of W coupled with the polynomial growth of W' . So, assume W' is neither finite nor affine.

Let $\{s_{\beta_i}\}_{i=1}^k$ be the Coxeter generators of W' . Pick γ as in condition (2) of the lemma and define \widetilde{W}' to be the reflection subgroup generated by s_γ and W' . By Dyer's criterion (theorem 4), $\{s_\gamma, s_{\beta_i} (i = 1 \cdots k)\}$ are the Coxeter generators of \widetilde{W}' . Now, \widetilde{W}' is an irreducible Coxeter group which is neither finite nor affine and this contains W' as a parabolic subgroup. By our main theorem 1, the set of minimal coset representatives of W' in \widetilde{W}' has exponential growth (wrt the length function on \widetilde{W}'). Thus, if

$$a_m := \#\{\sigma \in [W'] \cap \widetilde{W}' : \ell_{\widetilde{W}'}(\sigma) \leq m\}$$

then $\limsup_{m \rightarrow \infty} a_m^{1/m} > 1$. If $K := \max\{\ell(s_\gamma), \ell(s_{\beta_i}) (i = 1 \cdots k)\}$, we have $\ell(\sigma) \leq K \ell_{\widetilde{W}'}(\sigma) \forall \sigma \in \widetilde{W}'$. So, if $b_m := \#\{\sigma \in [W'] : \ell(\sigma) \leq m\}$, then $b_{Km} \geq a_m$. Thus

$$\limsup_{m \rightarrow \infty} b_m^{\frac{1}{m}} \geq \limsup_{m \rightarrow \infty} (b_{Km})^{1/Km} \geq \left(\limsup_{m \rightarrow \infty} a_m^{\frac{1}{m}} \right)^{\frac{1}{K}} > 1$$

Thus b_m , and hence $[W']$, has exponential growth. \square

Final Remarks: It is easily seen if $W' = W_J \subsetneq W$ or $W' = \sigma W_J \sigma^{-1}$ (parabolic subgroups and their conjugates), then the equivalent conditions of lemma 1 are satisfied. In practice, when W has small rank and given the Coxeter generators s_{β_i} of W' explicitly, it is often easy to show that this lemma is satisfied by explicitly producing a positive root γ such that $(\gamma, \beta_i) \leq 0 \forall i$. From such examples worked out by hand, it appears that this lemma is satisfied for a large number of reflection subgroups of W (when W is non-finite, non-affine). It would be useful to be able to completely characterize such reflection subgroups.

REFERENCES

1. P. de la Harpe, *Groupes de Coxeter infinis non affines*, Exposition. Math **5** (1987), no. 1, 91–96.
2. ———, *Topics in geometric group theory*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 2000.
3. V. V. Deodhar, *On the root system of a Coxeter group*, Comm. Algebra **10** (1982), no. 6, 611–630.
4. ———, *A note on subgroups generated by reflections in Coxeter groups*, Arch. Math. (Basel) **53** (1989), no. 6, 543–546.
5. M. Dyer, *On some generalisations of the Kazhdan-Lusztig polynomials for "universal" Coxeter systems*, J. Algebra **16** (1988), no. 2, 353–371.
6. ———, *Reflection subgroups of Coxeter systems.*, J. Algebra **135** (1990), no. 1, 57–73.
7. J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, no. 29, Cambridge University Press, Cambridge, 1990.
8. V. G. Kac, *Infinite-dimensional Lie algebras*, third ed., Cambridge University Press, Cambridge, 1990.
9. P. Wagreich, *The growth function of a discrete group*, Group actions and vector fields (Vancouver, B.C., 1981), Lecture Notes in Math., vol. 956, Springer, Berlin, 1982, pp. 125–144.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, DAVIS, CA 95616, USA

E-mail address: svis@math.ucdavis.edu