Existence of cyclic $k$-cycle systems of the complete graph

Marco Buratti$^{a,*}$, Alberto Del Fra$^b$

$^a$Dipartimento di Matematica e Informatica, Università di Perugia, Via Vanvitelli 1, I-06123 Perugia, Italy
$^b$Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate, Università di Roma “La Sapienza”, Via Scarpa 16, I-00161 Roma, Italy

Received 10 May 2001; accepted 3 November 2001

Dedicated to Alex Rosa on the occasion of his sixty-fifth birthday

Abstract

Starting from earliest papers by Rosa we solve, directly and explicitly, the existence problem for cyclic $k$-cycle systems of the complete graph $K_v$ with $v \equiv 1 \pmod{2k}$, and the existence problem for cyclic $k$-cycle systems of the complete $m$-partite graph $K_{m \times k}$ with $m$ and $k$ being odd. As a particular consequence, a cyclic $p$-cycle system of $K_v$ with $p$ being a prime exists for all admissible values of $v$ but $(p, v) \neq (3, 9)$. This was previously known only for $p = 3, 5, 7$.

Keywords: (Cyclic) $k$-cycle system; (Hooked) Skolem sequence; (Hooked) Rosa sequence

1. Introduction

We use the standard notation of graph theory so that $K_v$, $K_{m \times k}$ and $C_k$ will, respectively, denote the complete graph on $v$ vertices, the complete $m$-partite graph with parts of size $k$, and the $k$-cycle. As usual, speaking of a $k$-cycle $(b_1, b_2, \ldots, b_k)$, we mean that its edges are $[b_1, b_2], [b_2, b_3], \ldots, [b_k, b_1]$.

A $k$-cycle system of a graph $G = (V, E)$ is a (multi)set $\mathcal{B}$ of $k$-cycles whose edges partition $E$. The set $\mathcal{B}$ is cyclic if $V = \mathbb{Z}_v$ and if $B = (b_1, b_2, \ldots, b_k) \in \mathcal{B}$ implies that $B + 1 = (b_1 + 1, b_2 + 1, \ldots, b_k + 1)$ is also in $\mathcal{B}$.

* Corresponding author.

E-mail address: buratti@mat.uniroma1.it (M. Buratti).
Note, in particular, that a 3-cycle system of the complete graph $K_v$, usually called a Steiner triple system, is a $2 - (v, 3, 1)$ design. For general background on $k$-cycle systems we refer to [16,19]. Quite recently it has been proved that $k$-cycle systems of $K_v$ always exist when $k$ and $v$ satisfy the necessary conditions (see [3,24]). Relevant background on STSs is provided by [9].

A $k$-cycle system of $G$ is also called a decomposition of $G$ into $k$-cycles or a $(G, C_k)$-design. More generally, given a subgraph $H$ of a graph $G$, a $(G,H)$-design is a decomposition of $G$ into copies of $H$ (see [14]).

Given a $k$-cycle $B = (b_1, b_2, \ldots, b_k)$ with vertices in $Z_v,$ the list of differences from $B$ is the multiset $\Delta B = \{ \pm (b_i - b_{i-1}) \mid i = 1, 2, \ldots, k \}$ where $b_0 = b_k.$ We call $(K_v, C_k)$-difference system (DS in short) any set $\mathcal{F} = \{B_1, B_2, \ldots, B_n\}$ of $k$-cycles (starter cycles) with vertices in $Z_v$ such that the multiset $\Delta \mathcal{F} = \bigcup_{i=1}^{n} \Delta B_i$ covers each nonzero element of $Z_v$ exactly once.

Analogously, we define a $(K_{m \times k}, C_k)$-DS to be a set $\mathcal{F} = \{B_1, B_2, \ldots, B_n\}$ of $k$-cycles with vertices in $Z_{mk}$ such that $\Delta \mathcal{F} = Z_{mk} - mZ_{mk}.$

The above terminology is justified by the fact that any $(K_v, C_k)$- or $(K_{m \times k}, C_k)$-DS generates a cyclic $(K_v, C_k)$- or $(K_{m \times k}, C_k)$-design whose cycles are all the translates of its starter cycles.

We point out, however, that not every cyclic $k$-cycle system of $K_v$ or $K_{m \times k}$ is generated by a difference system. For instance, it is obvious that a cyclic $(K_k, C_k)$-design (whose existence will be shown for $k$ prime in the last section) cannot be generated by a $(K_k, C_k)$-DS since any $k$-cycle produces too many (exactly $2k$) differences.

A description in terms of differences of any cyclic $k$-cycle system may be found in [8].

Note the analogy between the notions of DSs given above and difference families. A $(v,k,1)$ difference family generates a cyclic $2 - (v,k,1)$ design, i.e., a cyclic $(K_v, K_k)$-design while a $(mk,k,k,1)$ difference family generates a cyclic $(k,1)$-group divisible design of type $k^m$, i.e., a cyclic $(K_{m \times k}, K_k)$-design (see [1,5,6] for more information).

In a series of early papers [20,21,22,23], Rosa studied the existence problem for cyclic $k$-cycle systems of the complete $(m$-partite) graph. Rosa gave cyclic $k$-cycle systems for $k \equiv 2 \pmod{4}$ and Kotzig provided a solution for $k \equiv 0 \pmod{4}$. For cyclic $k$-cycle systems of $K_v$ with $k$ even see also [11].

We briefly illustrate the strategy used by Rosa. He proves the existence of an $n \times k$ matrix $A = (a_{ih})$ with entries in $Z$ satisfying the following conditions:

$$\{ \{a_{ih}\mid 1 \leq i \leq n; \ 1 \leq h \leq k\} = \{1, 2, \ldots, kn\},$$
$$\sum_{h=1}^{k} a_{ih} \equiv 0 \pmod{2kn + 1}. \quad (2)$$

Then he associates with $A$ a set $\mathcal{F}(A) = \{B_1, \ldots, B_n\}$ of closed $k$-trails in the complete graph on $Z_{2kn+1}$ where $B_i = (b_1, b_2, \ldots, b_k)$ is defined by $b_{ih} = \sum_{h=1}^{t} a_{ih}.$

The set $\mathcal{F}(A)$ will fail to be a $(K_{2kn+1}, C_k)$-DS only if some $B_i$ is not a cycle. In this case, $\mathcal{F}(A)$ generates a decomposition of $K_{2kn+1}$ into closed $k$-trials.
Starting from $A$, one may generate $(k!)^n$ matrices satisfying (1) and (2). These matrices are all of the form $A^\sigma = (a_{i,\sigma(i)})$ where $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ is an arbitrary $n$-tuple of permutations on the set $\{1, 2, \ldots, k\}$.

In view of this, there are good chances that $\mathcal{F}(A^\sigma)$ is actually a $(K_{2kn+1}, C_k)$-DS for a suitable $\sigma$. Hence this method is a strong indication about the existence of a cyclic $(K_{2kn+1}, C_k)$-design for any $(k, n)$. In spite of this, for $k$ odd, Rosa was able to prove its existence for $k = 3, 5, 7$.

In a similar way, Rosa gave a strong indication about the existence of a cyclic $(K_{m \times k}, C_k)$-design with both $m$ and $k$ being odd but, also here, he leaves the problem open for $k \geq 7$.

In this paper we prove, directly and explicitly, the existence of a cyclic $(K_{2kn+1}, C_k)$-design for any pair of odd integers $m$ and $k$.

In our constructions we need the crucial help of Skolem sequences and Rosa sequences.

**Definition 1.1.** A Skolem sequence of order $n$ is a sequence of $n$ integers $(s_1, \ldots, s_n)$ such that $\bigcup_{i=1}^n \{s_i, s_i + i\} = \{1, 2, \ldots, 2n + 1\} - \{k\}$ where

$$k = \begin{cases} 2n + 1 & \text{for } n \equiv 0 \text{ or } 1 \pmod{4}, \\ 2n & \text{otherwise}. \end{cases}$$

In the case of $n \equiv 2$ or $3 \pmod{4}$ one usually speaks of a hooked Skolem sequence.

**Definition 1.2.** A Rosa sequence of order $n$ is a sequence of $n$ integers $(r_1, \ldots, r_n)$ such that $\bigcup_{i=1}^n \{r_i, r_i + i\} = \{1, 2, \ldots, 2n + 2\} - \{n + 1, k\}$ where

$$k = \begin{cases} 2n + 2 & \text{for } n \equiv 0 \text{ or } 3 \pmod{4}, \\ 2n + 1 & \text{otherwise}. \end{cases}$$

In the case of $n \equiv 1$ or $2 \pmod{4}$ one usually speaks of a hooked Rosa sequence.

**Theorem 1.3.** There exists a Skolem sequence of order $n$ for any $n \geq 1$ [25].

There exists a Rosa sequence of order $n$ for any $n > 1$ [20].

We point out that Skolem sequences and their generalizations (see also [4]) have revealed to be very useful in the construction of several kinds of combinatorial designs (see, e.g., [7,12,18]).

**2. Cyclic $k$-cycle systems of $K_{2kn+1}$**

In this section, we explicitly construct a $(K_{2kn+1}, C_k)$-design for any pair of positive integers $k$ and $n$. First, we consider the case of $n = 1$ where we have the following nice easy solution.
Lemma 2.1. There exists a cyclic $k$-cycle system of $K_{2k+1}$ for any $k$.

Proof. Let $B = (b_1, b_2, \ldots, b_k)$ be the $k$-cycle defined by

$$b_i = \begin{cases} 
  i(-1)^{i+1} & \text{for } i < \frac{k}{2}, \\
  i(-1)^i & \text{for } i \geq \frac{k}{2}.
\end{cases}$$

It is immediate to see that the $b_i$'s are pairwise distinct so that $B$ is actually a $k$-cycle.
Also, it is easy to check that $\Delta B = Z_{2k+1} - \{0\}$ so that $B$ generates the required $k$-cycle system. \qed

Figs. 1 and 2 give examples of the $k$-cycle $B$ when $k = 10$ and 11.
In all the figures, representing a cycle with vertices in $Z_v$, we only indicate the differences from that cycle not exceeding $(v - 1)/2$.

Theorem 2.2. There exists a cyclic $(K_{2kn+1}, C_k)$-design for any pair of positive integers $k$ and $n$.

Proof. We divide the proof into 7 cases.

Case 1: $k$ is even.
As pointed out in the introduction, this case was already solved by Rosa and Kotzig. Here we propose another easy solution.
For \( i = 1, \ldots, n \) consider the \( k \)-cycle \( B_i = (b_1, b_2, \ldots, b_k) \) defined by

\[
b_j = \begin{cases} 
(1 - j)n/2 & \text{for } j \text{ odd,} \\
 i + (j - 2)n/2 & \text{for } j = \frac{k + 2}{2}, \\
 i + (k + j + \epsilon)n/2 & \text{for } j > \frac{k + 2}{2},
\end{cases}
\]

where \( \epsilon = 0 \) or \(-2\) according to whether \( k \equiv 0 \) or \( 2 \) (mod 4), respectively, i.e., \( \epsilon = (-1)^{k/2} - 1 \).

We note, first, that in each \( B_i \) the \( b_j \)'s with \( j \) even form a decreasing sequence \( b_2 > b_4 > \cdots > b_{k-1} \) in the interval \( I = [0, (k + 1)n] \) while the \( b_j \)'s with \( j \) odd form an increasing sequence \( b_1 < b_3 < \cdots < b_k \) in the complement of \( I \) in \( \mathbb{Z}_{2kn+1} \). Thus, the \( b_j \)'s in \( B_i \) are pairwise distinct and hence \( B_i \) is actually a \( k \)-cycle.

Now, we want to prove that \( \mathcal{F} = \{B_1, B_2, \ldots, B_n\} \) is a \( (K_{2kn+1}, C_k) \)-DS. To do this it is enough to show that \( \Delta \mathcal{F} \) covers the set \( \{1, 2, \ldots, kn\} \).

Let \( z \in \{1, 2, \ldots, kn\} \). Then \( z = x + yn \) with \( 1 \leq x \leq n \) and \( 0 \leq y \leq k - 1 \). One may easily check that

\[
x + yn = \begin{cases} 
(1 - y)(b_{x,y+2} - b_{x,y+1}) & \text{for } y \leq \frac{k - \epsilon - 2}{2}, \\
(1 - y)(b_{i,j+(-1)^{\beta+1}} - b_{ij}) & \text{for } \frac{k - \epsilon - 2}{2} < y < k - 1,
\end{cases}
\]

where \( \alpha = y + k/2, i = n + 1 - x, j = (3k - 2y)/2 \) and \( \beta = k/2 \).

We also have

\[
x + (k - 1)n = \begin{cases} 
b_{n+1-x,1} - b_{n+1-x,k} & \text{for } k \equiv 0 \text{ (mod 4)}, \\
b_{xk} - b_{x1} & \text{for } k \equiv 2 \text{ (mod 4)}.
\end{cases}
\]

Thus, in any case, we have \( z \in \Delta \mathcal{F} \).

Figs. 3 and 4 give examples of \( (K_{2kn+1}, C_k) \)-DSs when \( k = 10 \) and 12.

Case 2: \( n = 1 \).

Here the solution is given by Lemma 2.1.

Case 3: \( k = 3, n > 1 \).
Fix a Skolem sequence \((s_1, \ldots, s_n)\) of order \(n\). Then the required design is generated by the starter cycles \(B_1, \ldots, B_n\) where \(B_i = (0, i, s_i + i + n)\).

Case 4: \(k = 5\), \(n > 1\).

Fix a Skolem sequence \((s_1, \ldots, s_n)\) of order \(n\). Here the required design is generated by the starter cycles \(B_1, \ldots, B_n\) where \(B_i = (0, s_i + i, i, -2n, i + 3n)\) with \(B_1\) replaced by \(B'_1 = (0, s_1 + 1, 1, 5n + 1, 2n)\) in the case of \(n \equiv 2\) or \(3\) (mod 4).

Case 5: \(k > 5\) is odd and \(1 < n \equiv 0\) or \(1\) (mod 4).

Fix a Skolem sequence \((s_1, \ldots, s_n)\) of order \(n\).

Then, for \(i = 1, \ldots, n\), consider the \(k\)-cycle \(B_i = (b_{i1}, b_{i2}, \ldots, b_{ik})\) defined as follows:

\[
b_{i1} = 0, \quad b_{i2} = -s_i, \quad b_{ij} = \begin{cases} 
-\frac{jn}{2} & \text{for } j \text{ even, } j \neq 2, \\
i + \frac{(j-3)n}{2} & \text{for } j \text{ odd, } j \leq \frac{k+1}{2}, \\
i + \frac{(k+j+\varepsilon)n}{2} & \text{for } j \text{ odd, } j > \frac{k+1}{2}, 
\end{cases}
\]

where \(\varepsilon = -2\) or 0 according to whether \(k \equiv 1\) or \(3\) (mod 4), respectively, i.e., \(\varepsilon = (-1)^{(k+1)/2} - 1\).

Reasoning as in the 1st case one may see that each \(B_i\) is actually a \(k\)-cycle.

Let us show that \(\mathcal{F} = \{B_1, \ldots, B_n\}\) is a \((K_{2kn+1}, C_k)\)-DS.

First of all, by Definition 1.1 we have

\[
\bigcup_{i=1}^{n} \{\pm(b_{i1} - b_{i2}), \pm(b_{i2} - b_{i3})\} = \bigcup_{i=1}^{n} \{\pm(s_i, \pm(s_i + i)\} = \pm \{1, 2, \ldots, 2n\},
\]

so that each element in \(\{1, 2, \ldots, 2n\}\) is covered by \(\Delta\mathcal{F}\).

For \(1 \leq x \leq n\) we have

\[
x + yn = \begin{cases} 
(\cdot 1)^y(b_{x,y+1} - b_{x,y+2}) & \text{for } 2 \leq y \leq \frac{k - \varepsilon - 3}{2}, \\
(\cdot 1)^y(b_{x,y+(-1)^{y+1} - b_{ij}}) & \text{for } \frac{k - \varepsilon - 3}{2} < y < k - 1,
\end{cases}
\]

where \(x = y + (k - 1)/2\), \(i = n + 1 - x\), \(j = (3k - 2y - 1)/2\) and \(\beta = (k + 1)/2\).
Finally, for $1 \leq x \leq n$ we have

$$x + (k - 1)n = \begin{cases} 
  b_{n+1-x,k} - b_{n+1-x,k} & \text{for } k \equiv 3 \pmod{4}, \\
  b_{x} - b_{x} & \text{for } k \equiv 1 \pmod{4}.
\end{cases}$$

In this way we see that each element in the set $\{1, 2, \ldots, kn\}$, namely of the form $x + yn$ with $1 \leq x \leq n$ and $0 \leq y \leq k - 1$, appears in $\Delta \mathcal{F}$. This assures that $\mathcal{F}$ is a $(K_{2kn+1}, C_k)$-DS.

Figs. 5 and 6 give examples of $(K_{2kn+1}, C_k)$-DSs for $k = 11$ and $9$ when $n \equiv 0$ or $1 \pmod{4}$.

Case 6: $5 < k \equiv 1 \pmod{4}$ and $n \equiv 2$ or $3 \pmod{4}$.

Consider the set of $k$-cycles $\mathcal{F}$ defined as in case 5. Here we have $\bigcup_{i=1}^{n} \{s_i, s_i + i\} = \{1, 2, \ldots, 2kn - 1, 2n + 1\}$ so that $\mathcal{F}$ fails to be a $(K_{2kn+1}, C_k)$-DS only because $2n + 1$ appears twice in $\Delta \mathcal{F}$ while $2n$ does not appear there. Consider the $k$-cycle $A = (a_1, a_2, \ldots, a_k)$ defined as follows:

$$a_1 = 1, \quad a_3 = 0, \quad a_{k-2} = 5n, \quad a_{k-1} = (4 - k)n, \quad a_k = (k + 3)n/2 + 1$$

and $a_j = b_{ij}$ for all $j$'s $\notin \{1, 3, k - 2, k - 1, k\}$.

It is straightforward to check that $A$ is actually a $k$-cycle (its vertices are pairwise distinct). Let us calculate its list of differences (mod $2kn + 1$). We have

$$a_1 - a_2 = b_{13} - b_{12} = s_1 + 1, \quad a_2 - a_3 = b_{12} - b_{11} = -s_1,$$

$$a_{k-2} - a_{k-3} = b_{1,k-3} - b_{1,k-2} = \frac{(k + 7)n}{2},$$
\[ a_{k-1} - a_{k-2} = b_{1k} - b_{11} = (k - 1)n + 1, \]
\[ a_{k-1} - a_{k} = b_{1,k-1} - b_{1,k-2} = \frac{(k + 5)n}{2}, \quad a_{k} - a_{1} = b_{1,k-1} - b_{1k} = \frac{(k + 3)n}{2}. \]

The above identities imply that
\[ \pm\{a_{j} - a_{j+1} \mid j = 1, 2, k - 3, k - 2, k - 1, k\} \]
\[ = \pm\{b_{1j} - b_{1,j+1} \mid j = 1, 2, k - 3, k - 2, k - 1, k\}. \]

We also have
\[ b_{13} - b_{14} = 2n + 1, \quad a_3 - a_4 = 2n, \]
\[ b_{1j} - b_{1,j+1} = a_{j} - a_{j+1} \quad \text{for} \ 4 \leq j \leq k - 4. \]

Thus, we have \( \Delta A = (\Delta B_1 - \{\pm(2n + 1)\}) \cup \{\pm 2n\} \) so that, in view of the previous observation on \( \Delta \mathcal{F} \), we may claim that \( \mathcal{F}' = \{A, B_2, \ldots, B_n\} \) is a \((K_{2kn+1}, C_k)\)-DS.

Case 7: \( 3 < k \equiv 3 \pmod{4} \) and \( n \equiv 2 \) or \( 3 \pmod{4} \).

The set \( \mathcal{F} \) defined as in the 5th case fails to be a \((K_{2kn+1}, C_k)\)-DS for the same reason for which it fails in the 6th one. Also here we overcome this inconvenience by replacing the cycle \( B_1 \) with a cycle \( A \) such that \( \Delta A = (\Delta B_1 - \{\pm(2n + 1)\}) \cup \{\pm 2n\} \).

One may easily check that a cycle \( A \) satisfying this condition is the one defined by the following rules:
\[ a_1 = 1, \quad a_3 = 0, \quad a_{j} = b_{1j} \quad \text{for} \ 1 \neq j \neq 3. \]

It is worthwhile to note that when \( k \) is odd and \( 2kn + 1 \) is a prime, a nice solution of the problem treated in this section may be obtained as follows. Fix a primitive \( k \)-th root of unity \( e \in \mathbb{Z}_{2kn+1} \) and construct, for \( i = 1, \ldots, n \), the \( k \)-cycle
\[ B_i = (e^{i+1}, e^{i+2}, \ldots, e^{i+k}). \]

It is straightforward to check that \( \{B_1, \ldots, B_n\} \) is a \((K_{2kn+1}, C_k)\)-DS.

3. Cyclic \( k \)-cycle systems of \( K_m \times k \)

First we consider the case of \( m = 3 \).

**Lemma 3.1.** There exists a cyclic \((K_{3 \times k}, C_k)\)-design for any odd \( k \) but \( k = 3 \).

**Proof.** It is well known that no cyclic \((K_{3 \times 3}, C_3)\)-design exists. So, assume \( k \geq 5 \). Let\( B = (b_1, b_2, \ldots, b_k) \) be the \( k \)-cycle defined as follows:
\[ b_i = (3i - 2)(-1)^i \quad \text{for} \ 1 \leq i \leq k - 4, \]
\[ b_{k-3} = 1, \quad b_{k-2} = -6, \quad b_{k-1} = 19, \quad b_k = 0. \]
It is easy to see that the $b_j$'s are pairwise distinct. Let us show that $\{B\}$ is a $(K_3 \times k, C_k)$-DS. Observing that $Z_{3k} - 3Z_{3k} = \{6x + 1 | 0 \leq x < k\}$, it suffices to check that $6x + 1 \in \Delta B$ for each $x \in \{0, 1, \ldots, k - 1\}$:

\[
\begin{align*}
6 \times 0 + 1 &= b_k - b_1, \\
6 \times 1 + 1 &= b_k - 3 - b_{k-2}, \\
6 \times 2 + 1 &= b_k - 4 - b_{k-3}, \\
6 \times 3 + 1 &= b_k - b_k, \\
6 \times 4 + 1 &= b_k - b_{k-2}, \\
6x + 1 &= (-1)^{\gamma}(b_{k-x} - b_{k-x+1}) & \text{for } 5 \leq x \leq k - 1.
\end{align*}
\]

Figure 7 gives a $(K_3 \times 11, C_{11})$-DS.

**Theorem 3.2.** For any pair of odd integers $(m,k)$, but $(m,k) \neq (3,3)$, there exists a cyclic $k$-cycle system of $K_{m\times k}$.

**Proof.** Set $k = 2h + 1$ and $m = 2n + 1$.

Case 1: $n = 1$.

Here the solution is given by Lemma 3.1.

Case 2: $n > 1$.

For $i = 1, 2, \ldots, n$, consider the $k$-cycle $B_i = (b_{i1}, b_{i2}, \ldots, b_{ik})$ defined by the following rules:

\[
b_{ij} = \begin{cases} 
\frac{m(j - 1)}{2} & \text{for } j \text{ odd}, \ j \neq k, \\
\frac{m\left(h - \frac{j}{2}\right)}{2} - i & \text{for } j \text{ even},
\end{cases}
\]

where $(r_1, r_2, \ldots, r_n)$ is a fixed Rosa sequence of order $n$.

For our purpose it is enough to prove that $\mathcal{F} = \{B_1, \ldots, B_n\}$ is a $(K_{m\times k}, C_k)$-DS or, equivalently, that the set $Z = \{1, 2, \ldots, hm + n\} - \{0, m, 2m, \ldots, hm\}$ is covered by $\Delta \mathcal{F}$.

Each $z \in Z$ may be written in the form $z = mx \pm y$ with $0 \leq x \leq h$ and $1 \leq y \leq n$.

For $0 \leq x \leq h - 1$ and $1 \leq y \leq n$ we have

\[
mx \pm y = \pm (-1)^{k+x}(y, h, h+1) - b_{y, h+x} \in \Delta \mathcal{F}.
\]

Now consider the elements $z \in Z$ of the form $z = mh \pm y$ with $1 \leq y \leq n$. 
First, assume that \( z \neq hm + n \), i.e., \( y \neq n \). In this case, we have \( z = t + (hm - n - 1) \) with \( t \in \{1, 2, \ldots, 2n\} - \{n + 1\} \). Thus, by Definition 1.2, we have \( t = r_i \) or \( t = r_i + i \) for some \( i \) so that \( z = r_i + (hm - n - 1) \) or \( z = r_i + i + (hm - n - 1) \) for some \( i \). Then \( z \in \Delta \mathcal{F} \) since for each \( i = 1, \ldots, (m - 1)/2 \) we have

\[
b_{hj} = r_i + i + (hm - n - 1) \quad \text{and} \quad b_{hj} - b_{hj-1} = r_i + (hm - n - 1).
\]

It remains to show that \( hm + n \) also appears in \( \Delta \mathcal{F} \). Observe that

\[
hm + n = (2n + 1) + (hm - n - 1) \quad \text{and} \quad -(hm + n) = (2n + 2) + (hm - n - 1).
\]

By Definition 1.2 there is a suitable \( i \) for which one of the following identities holds:

\[
2n + 1 = r_i \quad \text{or} \quad 2n + 1 = r_i + i \quad \text{or} \quad 2n + 2 = r_i \quad \text{or} \quad 2n + 2 = r_i + i.
\]

Thus, in view of (3) and (4), we obtain that \( hm + n \in \Delta \mathcal{F} \). □

Figure 8 gives an example of \((K_{m \times 11}, C_{11})\)-DS when \( m \) is odd.

Now, we are going to show that in the case where \( m \) and \( k \) are coprime, there is an elegant solution of the problem considered in this section.

Note that in this case we may identify \( Z_{mk} \) and \( mZ_{mk} \) with the groups \( Z_m \oplus Z_k \) and \( \{0\} \oplus Z_k \), respectively.

First, recall that a starter [10] in \( Z_m \), \( m \) odd, is a set \( \{(x_i, y_i) | 1 \leq i \leq (m-1)/2\} \) of pairs of elements of \( Z_m \) such that

\[
\bigcup_{i=1}^{(m-1)/2} \{x_i, y_i\} = \bigcup_{i=1}^{(m-1)/2} \{x_i - y_i, y_i - x_i\} = Z_m - \{0\}.
\]

Theorem 3.3. Let \( m, k \) be coprime odd integers and let \( \mathcal{F} = \{(x_i, y_i) | 1 \leq i \leq (m-1)/2\} \) be a starter in \( Z_m \). For \( i = 1, \ldots, (m-1)/2 \), define \( B_i = (b_{i0}, b_{i1}, \ldots, b_{ik-1}) \) by

\[
b_{i0} = (0, 0), \quad b_{ij} = \begin{cases} (x_i, j) & \text{for } j \text{ even}, \\ (y_i, -j) & \text{for } j \text{ odd}. \end{cases}
\]

Then, identifying \( Z_{mk} \) with \( Z_m \oplus Z_k \), we have that \( \{B_1, \ldots, B_n\} \) is a \((K_{m \times k}, C_k)\)-DS.
Proof. It is not difficult to see that for $i = 1, \ldots, n$ we have

$$\Delta B_i = \pm \{(x_i, y_i) \times \{-1\}\} \cup \bigcup_{h \in Z_n - \{\pm 1\}} \{x_i - y_i, y_i - x_i\} \times \{h\}.$$ 

So, in view of (5) we have

$$\Delta \mathcal{F} = \bigcup_{i=1}^{n} \Delta B_i = \bigcup_{h=0}^{k-1} (Z_m - \{0\}) \times \{h\} = (Z_m \oplus Z_k) - (\{0\} \oplus Z_k).$$

The assertion follows. \hfill \Box

Corollary 3.4. Let $m, k$ be coprime odd integers. Then, identifying $Z_{mk}$ with $Z_m \oplus Z_k$, we have that a $(K_{m \times k}, C_k)$-DS is given by the cycles $B_1, \ldots, B_{(m-1)/2}$ where $B_i = (b_{i0}, b_{i1}, \ldots, b_{ik-1})$ is defined by

$$b_{i0} = (0, 0), \quad b_{ij} = (-1)^{i-j}(i,j) \quad \text{for} \ j > 0.$$ 

Proof. It suffices to apply Theorem 3.3 using as $\mathcal{S}$ the so-called patterned starter $\{(i, -i) \mid 1 \leq i \leq (m - 1)/2\}$. \hfill \Box

4. Cyclic $p$-cycle systems of the complete graph with $p$ a prime

The existence of a cyclic $(K_{m \times k}, C_k)$-design may be helpful to get a cyclic $(K_{mk}, C_k)$-design. In fact we have

Theorem 4.1. If there exists a cyclic $(K_{m \times k}, C_k)$-design and a cyclic $(K_k, C_k)$-design, then there also exists a $(K_{mk}, C_k)$-design.

Proof. Let $\mathcal{A}$ and $\mathcal{B}$, respectively, be a cyclic $(K_{m \times k}, C_k)$-design and a cyclic $(K_k, C_k)$-design. For any $B = (b_1, \ldots, b_i) \in \mathcal{B}$ and any $i \in \{1, 2, \ldots, m\}$, set $mb + i = (mb_1 + i, \ldots, mb_k + i) \pmod{mk}$. It is straightforward to check that $\mathcal{A} \cup \{mb + i \mid B \in \mathcal{B}; 1 \leq i \leq m\}$ is a cyclic $(K_{mk}, C_k)$-design. \hfill \Box

Because of Theorem 4.1, the existence problem for cyclic $(K_k, C_k)$-designs is remarkable. For $k$ a prime the easy answer is given by the following theorem.

Theorem 4.2. There exists a cyclic $(K_k, C_k)$-design for any odd prime $k$.

Proof. It is straightforward to check that a cyclic $(K_k, C_k)$-design is given by $\mathcal{B} = \{B_1, \ldots, B_{(k-1)/2}\}$ where $B_i = (b_{i1}, \ldots, b_{ik})$ is the $k$-cycle defined by $b_{ij} = i \cdot j$. \hfill \Box

The result established in the previous sections and the above two theorems allow us to state:

Theorem 4.3. If $k$ is an odd prime, then there exists a cyclic $(K_v, C_k)$-design for any admissible value of $v$ but $(v,k) \neq (9,3)$. 

Proof. The admissible values of $v$ for which there exists a $(K_v, C_k)$-design are those satisfying the following conditions:

$$v(v - 1) \equiv 0 \pmod{2k} \quad \text{and} \quad v \equiv 1 \pmod{2}.$$  \hspace{1cm} (6)

This is because in a $(K_v, C_k)$-design we have exactly $v(v - 1)/2k$ cycles and the number of cycles through any given vertex is $(v - 1)/2$. On the other hand, if $k$ is a prime, condition (6) is equivalent to the following:

$$v \equiv 1 \quad \text{or} \quad k \pmod{2k}.$$  

For $v \equiv 1 \pmod{2k}$ the existence of a cyclic $(K_v, C_k)$-design is guaranteed by Theorem 2.2. For $v \equiv k \pmod{2k}$, i.e., $v = mk$ with $m$ odd, we get a cyclic $(K_v, C_k)$-design by applying Theorem 3.2 in conjunction with Theorems 4.1 and 4.2.  

The existence problem for cyclic $(K_k, C_k)$-designs for general $k$ will be considered in a forthcoming paper.

Added in Proof. In a very recent paper [13] solving a problem posed by Alspach [2], Fu and Wu also got the existence of a cyclic $(K_{2kn+1}, C_k)$-design for any $(k, n)$.

References

[22] A. Rosa, On cyclic decompositions of the complete graph into polygons with odd number of edges (Slovak), Časopis Pěst. Mat. 91 (1966) 53–63.