

## Section 12.2

## Differentiation and Integration of Vector-Valued Functions

- Differentiate a vector-valued function.
- Integrate a vector-valued function.

## Differentiation of Vector-Valued Functions

In Sections 12.3–12.5, you will study several important applications involving the calculus of vector-valued functions. In preparation for that study, this section is devoted to the mechanics of differentiation and integration of vector-valued functions.

The definition of the derivative of a vector-valued function parallels that given for real-valued functions.

## Definition of the Derivative of a Vector-Valued Function

The **derivative of a vector-valued function**  $\mathbf{r}$  is defined by

$$\mathbf{r}'(t) = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

for all  $t$  for which the limit exists. If  $\mathbf{r}'(c)$  exists, then  $\mathbf{r}$  is **differentiable at  $c$** . If  $\mathbf{r}'(c)$  exists for all  $c$  in an open interval  $I$ , then  $\mathbf{r}$  is **differentiable on the interval  $I$** . Differentiability of vector-valued functions can be extended to closed intervals by considering one-sided limits.

NOTE In addition to  $\mathbf{r}'(t)$ , other notations for the derivative of a vector-valued function are

$$D_t[\mathbf{r}(t)], \quad \frac{d}{dt}[\mathbf{r}(t)], \quad \text{and} \quad \frac{d\mathbf{r}}{dt}.$$

Differentiation of vector-valued functions can be done on a *component-by-component basis*. To see why this is true, consider the function given by

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}.$$

Applying the definition of the derivative produces the following.

$$\begin{aligned} \mathbf{r}'(t) &= \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{f(t + \Delta t)\mathbf{i} + g(t + \Delta t)\mathbf{j} - f(t)\mathbf{i} - g(t)\mathbf{j}}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \left\{ \left[ \frac{f(t + \Delta t) - f(t)}{\Delta t} \right] \mathbf{i} + \left[ \frac{g(t + \Delta t) - g(t)}{\Delta t} \right] \mathbf{j} \right\} \\ &= \left\{ \lim_{\Delta t \rightarrow 0} \left[ \frac{f(t + \Delta t) - f(t)}{\Delta t} \right] \right\} \mathbf{i} + \left\{ \lim_{\Delta t \rightarrow 0} \left[ \frac{g(t + \Delta t) - g(t)}{\Delta t} \right] \right\} \mathbf{j} \\ &= f'(t)\mathbf{i} + g'(t)\mathbf{j} \end{aligned}$$

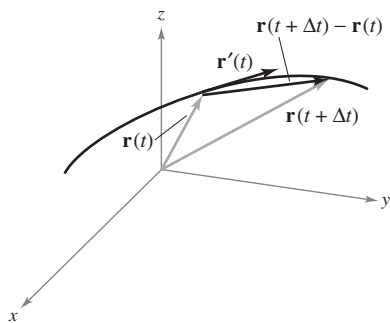


Figure 12.8

This important result is listed in the theorem on the next page. Note that the derivative of the vector-valued function  $\mathbf{r}$  is itself a vector-valued function. You can see from Figure 12.8 that  $\mathbf{r}'(t)$  is a vector tangent to the curve given by  $\mathbf{r}(t)$  and pointing in the direction of increasing  $t$ -values.

**THEOREM 12.1** Differentiation of Vector-Valued Functions

1. If  $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ , where  $f$  and  $g$  are differentiable functions of  $t$ , then

$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j}. \quad \text{Plane}$$

2. If  $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ , where  $f$ ,  $g$ , and  $h$  are differentiable functions of  $t$ , then

$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}. \quad \text{Space}$$

**EXAMPLE 1** Differentiation of Vector-Valued Functions

Find the derivative of each vector-valued function.

a.  $\mathbf{r}(t) = t^2\mathbf{i} - 4\mathbf{j}$       b.  $\mathbf{r}(t) = \frac{1}{t}\mathbf{i} + \ln t\mathbf{j} + e^{2t}\mathbf{k}$

**Solution** Differentiating on a component-by-component basis produces the following.

a.  $\mathbf{r}'(t) = 2t\mathbf{i} - 0\mathbf{j}$   
 $= 2t\mathbf{i} \quad \text{Derivative}$

b.  $\mathbf{r}'(t) = -\frac{1}{t^2}\mathbf{i} + \frac{1}{t}\mathbf{j} + 2e^{2t}\mathbf{k} \quad \text{Derivative}$

Higher-order derivatives of vector-valued functions are obtained by successive differentiation of each component function.

**EXAMPLE 2** Higher-Order Differentiation

For the vector-valued function given by  $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + 2t\mathbf{k}$ , find each of the following.

a.  $\mathbf{r}'(t)$                       b.  $\mathbf{r}''(t)$   
 c.  $\mathbf{r}'(t) \cdot \mathbf{r}''(t)$           d.  $\mathbf{r}'(t) \times \mathbf{r}''(t)$

**Solution**

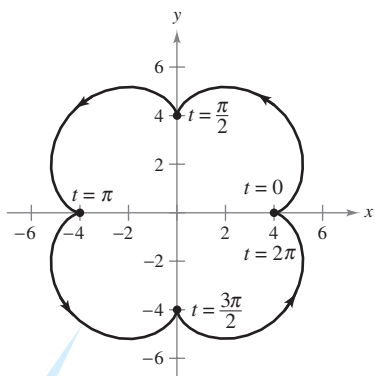
a.  $\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + 2\mathbf{k} \quad \text{First derivative}$

b.  $\mathbf{r}''(t) = -\cos t\mathbf{i} - \sin t\mathbf{j} + 0\mathbf{k}$   
 $= -\cos t\mathbf{i} - \sin t\mathbf{j} \quad \text{Second derivative}$

c.  $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = \sin t \cos t - \sin t \cos t = 0 \quad \text{Dot product}$

d.  $\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 2 \\ -\cos t & -\sin t & 0 \end{vmatrix} \quad \text{Cross product}$   
 $= \begin{vmatrix} \cos t & 2 \\ -\sin t & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -\sin t & 2 \\ -\cos t & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{vmatrix} \mathbf{k}$   
 $= 2 \sin t\mathbf{i} - 2 \cos t\mathbf{j} + \mathbf{k}$

Note that the dot product in part (c) is a *real-valued* function, not a vector-valued function.



$$\mathbf{r}(t) = (5 \cos t - \cos 5t)\mathbf{i} + (5 \sin t - \sin 5t)\mathbf{j}$$

The epicycloid is not smooth at the points where it intersects the axes.

Figure 12.9

The parametrization of the curve represented by the vector-valued function

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

is **smooth on an open interval**  $I$  if  $f'$ ,  $g'$ , and  $h'$  are continuous on  $I$  and  $\mathbf{r}'(t) \neq \mathbf{0}$  for any value of  $t$  in the interval  $I$ .

### EXAMPLE 3 Finding Intervals on Which a Curve Is Smooth

Find the intervals on which the epicycloid  $C$  given by

$$\mathbf{r}(t) = (5 \cos t - \cos 5t)\mathbf{i} + (5 \sin t - \sin 5t)\mathbf{j}, \quad 0 \leq t \leq 2\pi$$

is smooth.

**Solution** The derivative of  $\mathbf{r}$  is

$$\mathbf{r}'(t) = (-5 \sin t + 5 \sin 5t)\mathbf{i} + (5 \cos t - 5 \cos 5t)\mathbf{j}.$$

In the interval  $[0, 2\pi]$ , the only values of  $t$  for which

$$\mathbf{r}'(t) = 0\mathbf{i} + 0\mathbf{j}$$

are  $t = 0, \pi/2, \pi, 3\pi/2,$  and  $2\pi$ . Therefore, you can conclude that  $C$  is smooth in the intervals

$$\left(0, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \pi\right), \left(\pi, \frac{3\pi}{2}\right), \text{ and } \left(\frac{3\pi}{2}, 2\pi\right)$$

as shown in Figure 12.9.

**NOTE** In Figure 12.9, note that the curve is not smooth at points at which the curve makes abrupt changes in direction. Such points are called **cusps** or **nodes**.

Most of the differentiation rules in Chapter 2 have counterparts for vector-valued functions, and several are listed in the following theorem. Note that the theorem contains three versions of “product rules.” Property 3 gives the derivative of the product of a real-valued function  $f$  and a vector-valued function  $\mathbf{r}$ , Property 4 gives the derivative of the dot product of two vector-valued functions, and Property 5 gives the derivative of the cross product of two vector-valued functions (in space). Note that Property 5 applies only to three-dimensional vector-valued functions, because the cross product is not defined for two-dimensional vectors.

#### THEOREM 12.2 Properties of the Derivative

Let  $\mathbf{r}$  and  $\mathbf{u}$  be differentiable vector-valued functions of  $t$ , let  $f$  be a differentiable real-valued function of  $t$ , and let  $c$  be a scalar.

- $D_t[\mathbf{c}\mathbf{r}(t)] = \mathbf{c}\mathbf{r}'(t)$
- $D_t[\mathbf{r}(t) \pm \mathbf{u}(t)] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$
- $D_t[f(t)\mathbf{r}(t)] = f(t)\mathbf{r}'(t) + f'(t)\mathbf{r}(t)$
- $D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t)$
- $D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$
- $D_t[\mathbf{r}(f(t))] = \mathbf{r}'(f(t))f'(t)$
- If  $\mathbf{r}(t) \cdot \mathbf{r}(t) = c$ , then  $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$ .

**Proof** To prove Property 4, let

$$\mathbf{r}(t) = f_1(t)\mathbf{i} + g_1(t)\mathbf{j} \quad \text{and} \quad \mathbf{u}(t) = f_2(t)\mathbf{i} + g_2(t)\mathbf{j}$$

where  $f_1, f_2, g_1,$  and  $g_2$  are differentiable functions of  $t$ . Then,

$$\mathbf{r}(t) \cdot \mathbf{u}(t) = f_1(t)f_2(t) + g_1(t)g_2(t)$$

and it follows that

$$\begin{aligned} D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] &= f_1(t)f_2'(t) + f_1'(t)f_2(t) + g_1(t)g_2'(t) + g_1'(t)g_2(t) \\ &= [f_1(t)f_2'(t) + g_1(t)g_2'(t)] + [f_1'(t)f_2(t) + g_1'(t)g_2(t)] \\ &= \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t). \end{aligned}$$

Proofs of the other properties are left as exercises (see Exercises 73–77 and Exercise 80).

#### EXAMPLE 4 Using Properties of the Derivative

For the vector-valued functions given by

$$\mathbf{r}(t) = \frac{1}{t}\mathbf{i} - \mathbf{j} + \ln t\mathbf{k} \quad \text{and} \quad \mathbf{u}(t) = t^2\mathbf{i} - 2t\mathbf{j} + \mathbf{k}$$

find

a.  $D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)]$  and b.  $D_t[\mathbf{u}(t) \times \mathbf{u}'(t)]$ .

**Solution**

a. Because  $\mathbf{r}'(t) = -\frac{1}{t^2}\mathbf{i} + \frac{1}{t}\mathbf{k}$  and  $\mathbf{u}'(t) = 2t\mathbf{i} - 2\mathbf{j}$ , you have

$$\begin{aligned} D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] &= \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t) \\ &= \left(\frac{1}{t}\mathbf{i} - \mathbf{j} + \ln t\mathbf{k}\right) \cdot (2t\mathbf{i} - 2\mathbf{j}) \\ &\quad + \left(-\frac{1}{t^2}\mathbf{i} + \frac{1}{t}\mathbf{k}\right) \cdot (t^2\mathbf{i} - 2t\mathbf{j} + \mathbf{k}) \\ &= 2 + 2 + (-1) + \frac{1}{t} \\ &= 3 + \frac{1}{t}. \end{aligned}$$

b. Because  $\mathbf{u}'(t) = 2t\mathbf{i} - 2\mathbf{j}$  and  $\mathbf{u}''(t) = 2\mathbf{i}$ , you have

$$\begin{aligned} D_t[\mathbf{u}(t) \times \mathbf{u}'(t)] &= [\mathbf{u}(t) \times \mathbf{u}''(t)] + [\mathbf{u}'(t) \times \mathbf{u}'(t)] \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t^2 & -2t & 1 \\ 2 & 0 & 0 \end{vmatrix} + \mathbf{0} \\ &= \begin{vmatrix} -2t & 1 \\ 0 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} t^2 & 1 \\ 2 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} t^2 & -2t \\ 2 & 0 \end{vmatrix} \mathbf{k} \\ &= 0\mathbf{i} - (-2)\mathbf{j} + 4t\mathbf{k} \\ &= 2\mathbf{j} + 4t\mathbf{k}. \end{aligned}$$

**NOTE** Try reworking parts (a) and (b) in Example 4 by first forming the dot and cross products and then differentiating to see that you obtain the same results.

#### EXPLORATION

Let  $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$ . Sketch the graph of  $\mathbf{r}(t)$ . Explain why the graph is a circle of radius 1 centered at the origin. Calculate  $\mathbf{r}(\pi/4)$  and  $\mathbf{r}'(\pi/4)$ . Position the vector  $\mathbf{r}'(\pi/4)$  so that its initial point is at the terminal point of  $\mathbf{r}(\pi/4)$ . What do you observe? Show that  $\mathbf{r}(t) \cdot \mathbf{r}(t)$  is constant and that  $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$  for all  $t$ . How does this example relate to Property 7 of Theorem 12.2?

## Integration of Vector-Valued Functions

The following definition is a rational consequence of the definition of the derivative of a vector-valued function.

### Definition of Integration of Vector-Valued Functions

1. If  $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ , where  $f$  and  $g$  are continuous on  $[a, b]$ , then the **indefinite integral (antiderivative)** of  $\mathbf{r}$  is

$$\int \mathbf{r}(t) dt = \left[ \int f(t) dt \right] \mathbf{i} + \left[ \int g(t) dt \right] \mathbf{j} \quad \text{Plane}$$

and its **definite integral** over the interval  $a \leq t \leq b$  is

$$\int_a^b \mathbf{r}(t) dt = \left[ \int_a^b f(t) dt \right] \mathbf{i} + \left[ \int_a^b g(t) dt \right] \mathbf{j}.$$

2. If  $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ , where  $f$ ,  $g$ , and  $h$  are continuous on  $[a, b]$ , then the **indefinite integral (antiderivative)** of  $\mathbf{r}$  is

$$\int \mathbf{r}(t) dt = \left[ \int f(t) dt \right] \mathbf{i} + \left[ \int g(t) dt \right] \mathbf{j} + \left[ \int h(t) dt \right] \mathbf{k} \quad \text{Space}$$

and its **definite integral** over the interval  $a \leq t \leq b$  is

$$\int_a^b \mathbf{r}(t) dt = \left[ \int_a^b f(t) dt \right] \mathbf{i} + \left[ \int_a^b g(t) dt \right] \mathbf{j} + \left[ \int_a^b h(t) dt \right] \mathbf{k}.$$

The antiderivative of a vector-valued function is a family of vector-valued functions all differing by a constant vector  $\mathbf{C}$ . For instance, if  $\mathbf{r}(t)$  is a three-dimensional vector-valued function, then for the indefinite integral  $\int \mathbf{r}(t) dt$ , you obtain three constants of integration

$$\int f(t) dt = F(t) + C_1, \quad \int g(t) dt = G(t) + C_2, \quad \int h(t) dt = H(t) + C_3$$

where  $F'(t) = f(t)$ ,  $G'(t) = g(t)$ , and  $H'(t) = h(t)$ . These three *scalar* constants produce one *vector* constant of integration,

$$\begin{aligned} \int \mathbf{r}(t) dt &= [F(t) + C_1]\mathbf{i} + [G(t) + C_2]\mathbf{j} + [H(t) + C_3]\mathbf{k} \\ &= [F(t)\mathbf{i} + G(t)\mathbf{j} + H(t)\mathbf{k}] + [C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k}] \\ &= \mathbf{R}(t) + \mathbf{C} \end{aligned}$$

where  $\mathbf{R}'(t) = \mathbf{r}(t)$ .

### EXAMPLE 5 Integrating a Vector-Valued Function

Find the indefinite integral

$$\int (t\mathbf{i} + 3\mathbf{j}) dt.$$

**Solution** Integrating on a component-by-component basis produces

$$\int (t\mathbf{i} + 3\mathbf{j}) dt = \frac{t^2}{2}\mathbf{i} + 3t\mathbf{j} + \mathbf{C}.$$

Example 6 shows how to evaluate the definite integral of a vector-valued function.

### EXAMPLE 6 Definite Integral of a Vector-Valued Function

Evaluate the integral

$$\int_0^1 \mathbf{r}(t) dt = \int_0^1 \left( \sqrt[3]{t} \mathbf{i} + \frac{1}{t+1} \mathbf{j} + e^{-t} \mathbf{k} \right) dt.$$

**Solution**

$$\begin{aligned} \int_0^1 \mathbf{r}(t) dt &= \left( \int_0^1 t^{1/3} dt \right) \mathbf{i} + \left( \int_0^1 \frac{1}{t+1} dt \right) \mathbf{j} + \left( \int_0^1 e^{-t} dt \right) \mathbf{k} \\ &= \left[ \left( \frac{3}{4} \right) t^{4/3} \right]_0^1 \mathbf{i} + \left[ \ln|t+1| \right]_0^1 \mathbf{j} + \left[ -e^{-t} \right]_0^1 \mathbf{k} \\ &= \frac{3}{4} \mathbf{i} + (\ln 2) \mathbf{j} + \left( 1 - \frac{1}{e} \right) \mathbf{k} \end{aligned}$$

As with real-valued functions, you can narrow the family of antiderivatives of a vector-valued function  $\mathbf{r}'$  down to a single antiderivative by imposing an initial condition on the vector-valued function  $\mathbf{r}$ . This is demonstrated in the next example.

### EXAMPLE 7 The Antiderivative of a Vector-Valued Function

Find the antiderivative of

$$\mathbf{r}'(t) = \cos 2t \mathbf{i} - 2 \sin t \mathbf{j} + \frac{1}{1+t^2} \mathbf{k}$$

that satisfies the initial condition  $\mathbf{r}(0) = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ .

**Solution**

$$\begin{aligned} \mathbf{r}(t) &= \int \mathbf{r}'(t) dt \\ &= \left( \int \cos 2t dt \right) \mathbf{i} + \left( \int -2 \sin t dt \right) \mathbf{j} + \left( \int \frac{1}{1+t^2} dt \right) \mathbf{k} \\ &= \left( \frac{1}{2} \sin 2t + C_1 \right) \mathbf{i} + (2 \cos t + C_2) \mathbf{j} + (\arctan t + C_3) \mathbf{k} \end{aligned}$$

Letting  $t = 0$  and using the fact that  $\mathbf{r}(0) = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ , you have

$$\begin{aligned} \mathbf{r}(0) &= (0 + C_1) \mathbf{i} + (2 + C_2) \mathbf{j} + (0 + C_3) \mathbf{k} \\ &= 3\mathbf{i} + (-2)\mathbf{j} + \mathbf{k}. \end{aligned}$$

Equating corresponding components produces

$$C_1 = 3, \quad 2 + C_2 = -2, \quad \text{and} \quad C_3 = 1.$$

So, the antiderivative that satisfies the given initial condition is

$$\mathbf{r}(t) = \left( \frac{1}{2} \sin 2t + 3 \right) \mathbf{i} + (2 \cos t - 4) \mathbf{j} + (\arctan t + 1) \mathbf{k}.$$

## Exercises for Section 12.2

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–6, sketch the plane curve represented by the vector-valued function, and sketch the vectors  $\mathbf{r}(t_0)$  and  $\mathbf{r}'(t_0)$  for the given value of  $t_0$ . Position the vectors such that the initial point of  $\mathbf{r}(t_0)$  is at the origin and the initial point of  $\mathbf{r}'(t_0)$  is at the terminal point of  $\mathbf{r}(t_0)$ . What is the relationship between  $\mathbf{r}'(t_0)$  and the curve?

1.  $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j}$ ,  $t_0 = 2$       2.  $\mathbf{r}(t) = t\mathbf{i} + t^3\mathbf{j}$ ,  $t_0 = 1$

3.  $\mathbf{r}(t) = t^2\mathbf{i} + \frac{1}{t}\mathbf{j}$ ,  $t_0 = 2$

4.  $\mathbf{r}(t) = (1+t)\mathbf{i} + t^3\mathbf{j}$ ,  $t_0 = 1$

5.  $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$ ,  $t_0 = \frac{\pi}{2}$

6.  $\mathbf{r}(t) = e^t\mathbf{i} + e^{2t}\mathbf{j}$ ,  $t_0 = 0$

7. **Investigation** Consider the vector-valued function

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}.$$

(a) Sketch the graph of  $\mathbf{r}(t)$ . Use a graphing utility to verify your graph.

(b) Sketch the vectors  $\mathbf{r}(1/4)$ ,  $\mathbf{r}(1/2)$ , and  $\mathbf{r}(1/2) - \mathbf{r}(1/4)$  on the graph in part (a).

(c) Compare the vector  $\mathbf{r}'(1/4)$  with the vector

$$\frac{\mathbf{r}(1/2) - \mathbf{r}(1/4)}{1/2 - 1/4}.$$

8. **Investigation** Consider the vector-valued function

$$\mathbf{r}(t) = t\mathbf{i} + (4 - t^2)\mathbf{j}.$$

(a) Sketch the graph of  $\mathbf{r}(t)$ . Use a graphing utility to verify your graph.

(b) Sketch the vectors  $\mathbf{r}(1)$ ,  $\mathbf{r}(1.25)$ , and  $\mathbf{r}(1.25) - \mathbf{r}(1)$  on the graph in part (a).

(c) Compare the vector  $\mathbf{r}'(1)$  with the vector  $\frac{\mathbf{r}(1.25) - \mathbf{r}(1)}{1.25 - 1}$ .

In Exercises 9 and 10, (a) sketch the space curve represented by the vector-valued function, and (b) sketch the vectors  $\mathbf{r}(t_0)$  and  $\mathbf{r}'(t_0)$  for the given value of  $t_0$ .

9.  $\mathbf{r}(t) = 2 \cos t\mathbf{i} + 2 \sin t\mathbf{j} + t\mathbf{k}$ ,  $t_0 = \frac{3\pi}{2}$

10.  $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{3}{2}\mathbf{k}$ ,  $t_0 = 2$

In Exercises 11–18, find  $\mathbf{r}'(t)$ .

11.  $\mathbf{r}(t) = 6t\mathbf{i} - 7t^2\mathbf{j} + t^3\mathbf{k}$       12.  $\mathbf{r}(t) = \frac{1}{t}\mathbf{i} + 16t\mathbf{j} + \frac{t^2}{2}\mathbf{k}$

13.  $\mathbf{r}(t) = a \cos^3 t\mathbf{i} + a \sin^3 t\mathbf{j} + \mathbf{k}$

14.  $\mathbf{r}(t) = 4\sqrt{t}\mathbf{i} + t^2\sqrt{t}\mathbf{j} + \ln t^2\mathbf{k}$

15.  $\mathbf{r}(t) = e^{-t}\mathbf{i} + 4\mathbf{j}$

16.  $\mathbf{r}(t) = \langle \sin t - t \cos t, \cos t + t \sin t, t^2 \rangle$

17.  $\mathbf{r}(t) = \langle t \sin t, t \cos t, t \rangle$       18.  $\mathbf{r}(t) = \langle \arcsin t, \arccos t, 0 \rangle$

In Exercises 19–26, find (a)  $\mathbf{r}''(t)$  and (b)  $\mathbf{r}'(t) \cdot \mathbf{r}''(t)$ .

19.  $\mathbf{r}(t) = t^3\mathbf{i} + \frac{1}{2}t^2\mathbf{j}$

20.  $\mathbf{r}(t) = (t^2 + t)\mathbf{i} + (t^2 - t)\mathbf{j}$

21.  $\mathbf{r}(t) = 4 \cos t\mathbf{i} + 4 \sin t\mathbf{j}$       22.  $\mathbf{r}(t) = 8 \cos t\mathbf{i} + 3 \sin t\mathbf{j}$

23.  $\mathbf{r}(t) = \frac{1}{2}t^2\mathbf{i} - t\mathbf{j} + \frac{1}{6}t^3\mathbf{k}$

24.  $\mathbf{r}(t) = t\mathbf{i} + (2t + 3)\mathbf{j} + (3t - 5)\mathbf{k}$

25.  $\mathbf{r}(t) = \langle \cos t + t \sin t, \sin t - t \cos t, t \rangle$

26.  $\mathbf{r}(t) = \langle e^{-t}, t^2, \tan t \rangle$

In Exercises 27 and 28, a vector-valued function and its graph are given. The graph also shows the unit vectors  $\frac{\mathbf{r}'(t_0)}{\|\mathbf{r}'(t_0)\|}$  and  $\frac{\mathbf{r}''(t_0)}{\|\mathbf{r}''(t_0)\|}$ . Find these two unit vectors and identify them on the graph.

27.  $\mathbf{r}(t) = \cos(\pi t)\mathbf{i} + \sin(\pi t)\mathbf{j} + t^2\mathbf{k}$ ,  $t_0 = -\frac{1}{4}$

28.  $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + e^{0.75t}\mathbf{k}$ ,  $t_0 = \frac{1}{4}$

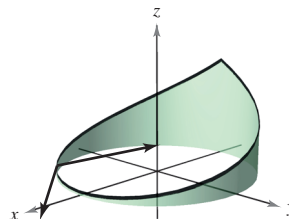


Figure for 27

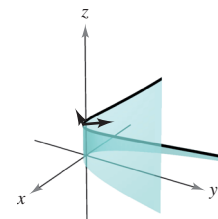


Figure for 28

In Exercises 29–38, find the open interval(s) on which the curve given by the vector-valued function is smooth.

29.  $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$

30.  $\mathbf{r}(t) = \frac{1}{t-1}\mathbf{i} + 3t\mathbf{j}$

31.  $\mathbf{r}(\theta) = 2 \cos^3 \theta\mathbf{i} + 3 \sin^3 \theta\mathbf{j}$

32.  $\mathbf{r}(\theta) = (\theta + \sin \theta)\mathbf{i} + (1 - \cos \theta)\mathbf{j}$

33.  $\mathbf{r}(\theta) = (\theta - 2 \sin \theta)\mathbf{i} + (1 - 2 \cos \theta)\mathbf{j}$

34.  $\mathbf{r}(t) = \frac{2t}{8+t^3}\mathbf{i} + \frac{2t^2}{8+t^3}\mathbf{j}$

35.  $\mathbf{r}(t) = (t-1)\mathbf{i} + \frac{1}{t}\mathbf{j} - t^2\mathbf{k}$       36.  $\mathbf{r}(t) = e^t\mathbf{i} - e^{-t}\mathbf{j} + 3t\mathbf{k}$

37.  $\mathbf{r}(t) = t\mathbf{i} - 3t\mathbf{j} + \tan t\mathbf{k}$

38.  $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + (t^2 - 1)\mathbf{j} + \frac{1}{4}t\mathbf{k}$

In Exercises 39 and 40, use the properties of the derivative to find the following.

(a)  $\mathbf{r}'(t)$       (b)  $\mathbf{r}''(t)$       (c)  $D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)]$

(d)  $D_t[3\mathbf{r}(t) - \mathbf{u}(t)]$       (e)  $D_t[\mathbf{r}(t) \times \mathbf{u}(t)]$       (f)  $D_t[\|\mathbf{r}(t)\|]$ ,  $t > 0$

39.  $\mathbf{r}(t) = t\mathbf{i} + 3t\mathbf{j} + t^2\mathbf{k}$ ,  $\mathbf{u}(t) = 4t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$


40.  $\mathbf{r}(t) = t\mathbf{i} + 2 \sin t\mathbf{j} + 2 \cos t\mathbf{k}$ ,

$$\mathbf{u}(t) = \frac{1}{t}\mathbf{i} + 2 \sin t\mathbf{j} + 2 \cos t\mathbf{k}$$

In Exercises 41 and 42, find (a)  $D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)]$  and (b)  $D_t[\mathbf{r}(t) \times \mathbf{u}(t)]$  by differentiating the product, then applying the properties of Theorem 12.2.

41.  $\mathbf{r}(t) = t\mathbf{i} + 2t^2\mathbf{j} + t^3\mathbf{k}, \quad \mathbf{u}(t) = t^4\mathbf{k}$

42.  $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}, \quad \mathbf{u}(t) = \mathbf{j} + t\mathbf{k}$

 In Exercises 43 and 44, find the angle  $\theta$  between  $\mathbf{r}(t)$  and  $\mathbf{r}'(t)$  as a function of  $t$ . Use a graphing utility to graph  $\theta(t)$ . Use the graph to find any extrema of the function. Find any values of  $t$  at which the vectors are orthogonal.

43.  $\mathbf{r}(t) = 3 \sin t\mathbf{i} + 4 \cos t\mathbf{j}$       44.  $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j}$

In Exercises 45–48, use the definition of the derivative to find  $\mathbf{r}'(t)$ .

45.  $\mathbf{r}(t) = (3t + 2)\mathbf{i} + (1 - t^2)\mathbf{j}$       46.  $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + \frac{3}{t}\mathbf{j} - 2t\mathbf{k}$

47.  $\mathbf{r}(t) = \langle t^2, 0, 2t \rangle$       48.  $\mathbf{r}(t) = \langle 0, \sin t, 4t \rangle$

In Exercises 49–56, find the indefinite integral.

49.  $\int (2t\mathbf{i} + \mathbf{j} + \mathbf{k}) dt$       50.  $\int (4t^3\mathbf{i} + 6t\mathbf{j} - 4\sqrt{t}\mathbf{k}) dt$

51.  $\int \left(\frac{1}{t}\mathbf{i} + \mathbf{j} - t^{3/2}\mathbf{k}\right) dt$       52.  $\int \left(\ln t\mathbf{i} + \frac{1}{t}\mathbf{j} + \mathbf{k}\right) dt$

53.  $\int \left[(2t - 1)\mathbf{i} + 4t^3\mathbf{j} + 3\sqrt{t}\mathbf{k}\right] dt$

54.  $\int (e^t\mathbf{i} + \sin t\mathbf{j} + \cos t\mathbf{k}) dt$

55.  $\int \left(\sec^2 t\mathbf{i} + \frac{1}{1 + t^2}\mathbf{j}\right) dt$

56.  $\int (e^{-t}\sin t\mathbf{i} + e^{-t}\cos t\mathbf{j}) dt$

In Exercises 57–62, evaluate the definite integral.

57.  $\int_0^1 (8t\mathbf{i} + t\mathbf{j} - \mathbf{k}) dt$       58.  $\int_{-1}^1 (t\mathbf{i} + t^3\mathbf{j} + \sqrt[3]{t}\mathbf{k}) dt$

59.  $\int_0^{\pi/2} [(a \cos t)\mathbf{i} + (a \sin t)\mathbf{j} + \mathbf{k}] dt$

60.  $\int_0^{\pi/4} [(\sec t \tan t)\mathbf{i} + (\tan t)\mathbf{j} + (2 \sin t \cos t)\mathbf{k}] dt$

61.  $\int_0^2 (t\mathbf{i} + e^t\mathbf{j} - te^t\mathbf{k}) dt$       62.  $\int_0^3 \|t\mathbf{i} + t^2\mathbf{j}\| dt$

In Exercises 63–68, find  $\mathbf{r}(t)$  for the given conditions.

63.  $\mathbf{r}'(t) = 4e^{2t}\mathbf{i} + 3e^t\mathbf{j}, \quad \mathbf{r}(0) = 2\mathbf{i}$

64.  $\mathbf{r}'(t) = 3t^2\mathbf{j} + 6\sqrt{t}\mathbf{k}, \quad \mathbf{r}(0) = \mathbf{i} + 2\mathbf{j}$

65.  $\mathbf{r}''(t) = -32\mathbf{j}, \quad \mathbf{r}'(0) = 600\sqrt{3}\mathbf{i} + 600\mathbf{j}, \quad \mathbf{r}(0) = \mathbf{0}$

66.  $\mathbf{r}''(t) = -4 \cos t\mathbf{j} - 3 \sin t\mathbf{k}, \quad \mathbf{r}'(0) = 3\mathbf{k}, \quad \mathbf{r}(0) = 4\mathbf{j}$

67.  $\mathbf{r}'(t) = te^{-t}\mathbf{i} - e^{-t}\mathbf{j} + \mathbf{k}, \quad \mathbf{r}(0) = \frac{1}{2}\mathbf{i} - \mathbf{j} + \mathbf{k}$

68.  $\mathbf{r}'(t) = \frac{1}{1 + t^2}\mathbf{i} + \frac{1}{t^2}\mathbf{j} + \frac{1}{t}\mathbf{k}, \quad \mathbf{r}(1) = 2\mathbf{i}$

### Writing About Concepts

69. State the definition of the derivative of a vector-valued function. Describe how to find the derivative of a vector-valued function and give its geometric interpretation.
70. How do you find the integral of a vector-valued function?
71. The three components of the derivative of the vector-valued function  $\mathbf{u}$  are positive at  $t = t_0$ . Describe the behavior of  $\mathbf{u}$  at  $t = t_0$ .
72. The  $z$ -component of the derivative of the vector-valued function  $\mathbf{u}$  is 0 for  $t$  in the domain of the function. What does this information imply about the graph of  $\mathbf{u}$ ?

In Exercises 73–80, prove the property. In each case, assume  $\mathbf{r}, \mathbf{u},$  and  $\mathbf{v}$  are differentiable vector-valued functions of  $t$ ,  $f$  is a differentiable real-valued function of  $t$ , and  $c$  is a scalar.

73.  $D_t[c\mathbf{r}(t)] = c\mathbf{r}'(t)$

74.  $D_t[\mathbf{r}(t) \pm \mathbf{u}(t)] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$

75.  $D_t[f(t)\mathbf{r}(t)] = f(t)\mathbf{r}'(t) + f'(t)\mathbf{r}(t)$

76.  $D_t[\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$


77.  $D_t[\mathbf{r}(f(t))] = \mathbf{r}'(f(t))f'(t)$

78.  $D_t[\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}(t) \times \mathbf{r}''(t)$

79.  $D_t\{\mathbf{r}(t) \cdot [\mathbf{u}(t) \times \mathbf{v}(t)]\} = \mathbf{r}'(t) \cdot [\mathbf{u}(t) \times \mathbf{v}(t)] + \mathbf{r}(t) \cdot [\mathbf{u}'(t) \times \mathbf{v}(t)] + \mathbf{r}(t) \cdot [\mathbf{u}(t) \times \mathbf{v}'(t)]$

80. If  $\mathbf{r}(t) \cdot \mathbf{r}(t)$  is a constant, then  $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$ .

81. **Particle Motion** A particle moves in the  $xy$ -plane along the curve represented by the vector-valued function  $\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}$ .

 (a) Use a graphing utility to graph  $\mathbf{r}$ . Describe the curve.

(b) Find the minimum and maximum values of  $\|\mathbf{r}'\|$  and  $\|\mathbf{r}''\|$ .

82. **Particle Motion** A particle moves in the  $yz$ -plane along the curve represented by the vector-valued function  $\mathbf{r}(t) = (2 \cos t)\mathbf{j} + (3 \sin t)\mathbf{k}$ .

(a) Describe the curve.

(b) Find the minimum and maximum values of  $\|\mathbf{r}'\|$  and  $\|\mathbf{r}''\|$ .

**True or False?** In Exercises 83–86, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

83. If a particle moves along a sphere centered at the origin, then its derivative vector is always tangent to the sphere.

84. The definite integral of a vector-valued function is a real number.

85.  $\frac{d}{dt}[\|\mathbf{r}(t)\|] = \|\mathbf{r}'(t)\|$

86. If  $\mathbf{r}$  and  $\mathbf{u}$  are differentiable vector-valued functions of  $t$ , then  $D_t[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}'(t) \cdot \mathbf{u}'(t)$ .

87. Consider the vector-valued function

$$\mathbf{r}(t) = (e^t \sin t)\mathbf{i} + (e^t \cos t)\mathbf{j}.$$

Show that  $\mathbf{r}(t)$  and  $\mathbf{r}''(t)$  are always perpendicular to each other.