Control by Interconnection of Mixed Port Hamiltonian Systems
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Abstract
In this paper, the stabilization problem for mixed finite and infinite dimensional port Hamiltonian systems (m-pH systems) is discussed. A m-pH system results from the power conserving interconnection of finite and infinite dimensional systems in port Hamiltonian form. The proposed control methodology is a generalization to the infinite dimensional case of a well-established passivity-based control technique for finite dimensional port Hamiltonian systems, the control by interconnection and energy shaping, according to which the open-loop energy function is shaped so that a minimum in the desired configuration is introduced. This procedure is possible once the state variable of the controller is related to the state variable of the m-pH system to be stabilized by constraining the state of the closed-loop system on a structural invariant (defined by a set of Casimir functions). In this way, the energy function of the controller, which is freely assignable, becomes a function of the configuration of the plant and, then, it can be easily shaped in order to solve the regulation problem.

Keywords
mixed port Hamiltonian systems, Casimir function, control by interconnection and energy shaping

I. Introduction
Finite dimensional port Hamiltonian systems [6], [15] are a powerful framework for modeling and control non-linear dynamical systems, [11], [9]. Every physical system is described by means of a proper network of atomic elements, each of them characterized by a particular energetic property (e.g. energy storing, dissipation and conversion). The resulting mathematical model is able to explicitly reveal the network structure behind it and its energetic properties. In particular, a port Hamiltonian model of a system explicitly expresses its passivity properties under non restrictive conditions. This feature is of great interest both for system analysis and for the development of advanced control strategies. As a matter of fact, global stability can be easily achieved by means of passivity-based control techniques, such as damping injection [13], [15] or energy shaping, [10], [9]. The latter technique aims to develop a passive controller that shapes the total energy function of the plant in order to obtain a closed-loop energy with a minimum in the desired equilibrium configuration and stability can be proved by means of energetic considerations. Furthermore since controller, plant and, consequently, closed-loop system are passive, stability can be assured even in presence of model uncertainties.

Recently, finite dimensional port Hamiltonian systems have been extended in order to deal with distributed parameters systems, [7] and several classical distributed parameter systems have been rewritten within this novel framework, [7], [8], [4]. From control point of view, it is of great interest the generalization of the finite dimensional methodologies to the distributed parameter case: an application for the stabilization by energy shaping of a flexible beam can be found in [4], [3].

In this paper, the stabilization problem for a port Hamiltonian systems made of the interconnection of a finite dimensional system with a distributed parameter one is approached. In particular, we suppose that the finite dimensional controller can act on the finite dimensional plant through a set of transmission lines: in some sense, this paper is the generalization of [12], in which only one lossless transmission line was taken into account. Differently from [12], the problem is approached by determining necessary and sufficient conditions under which a functional defined on the mixed configuration space could be a structural invariant (i.e. Casimir functional) for the closed-loop system, that is independently from the energy functions even in presence of dissipative effects in the distributed parameter sub-system. Then, the controller state variable is related to the state variable of the plant by means of a Casimir functional, thus implementing a structural state feedback law. In this way, whatever energy function is chosen for the controller, it results in a function of the plant variables and the energy shaping procedure can be easily completed.

This paper is organized as follows. In Sec. II the control by interconnection and energy shaping is briefly discussed in the case of finite dimensional port Hamiltonian systems. In Sec. III the mixed port Hamiltonian system under study is introduced, while in Sec. IV necessary and sufficient conditions for the existence of structural invariants (Casimir function) for this class of systems are presented. The results about the existence of Casimir functions in the m-pH systems case are the starting point for the generalization of the control by energy shaping in order to deal with m-pH systems, discussed in Sec. V. Then, an example concerning the stabilization of a simple m-pH system is discussed in Sec. VI. Conclusions are presented in Sec. VII.
II. CONTROL BY INTERCONNECTION IN THE FINITE DIMENSIONAL CASE

Consider two finite dimensional port Hamiltonian systems $A$ and $B$, whose state space representations are given by:

$$
\begin{align*}
\dot{x}_a &= [J_a(x_a) - R_a(x_a)] \frac{\partial H_a}{\partial x_a} + G_a(x_a) u_a \\
y_a &= G^T_a(x_a) \frac{\partial H_a}{\partial x_a} 
\end{align*}
$$

(1a)

and

$$
\begin{align*}
\dot{x}_b &= [J_b(x_b) - R_b(x_b)] \frac{\partial H_b}{\partial x_b} + G_b(x_b) u_b \\
y_b &= G^T_b(x_b) \frac{\partial H_b}{\partial x_b} 
\end{align*}
$$

(1b)

Denote by $\mathcal{X}_a$ and $\mathcal{X}_b$ the state space of system $\text{(1a)}$ and $\text{(1b)}$ respectively, with $\dim \mathcal{X}_a = n_a$ and $\dim \mathcal{X}_b = n_b$, while $H_a : \mathcal{X}_a \to \mathbb{R}$ and $H_b : \mathcal{X}_b \to \mathbb{R}$ are the Hamiltonian functions, bounded from below. Moreover, suppose that $J_a(x_a) = -J^T_a(x_a)$ and $R_a(x_a) = R^T_a(x_a)$ for every $x_a \in \mathcal{X}_a$, that $J_b(x_b) = -J^T_b(x_b)$ and $R_b(x_b) = R^T_b(x_b)$ for every $x_b \in \mathcal{X}_b$ and that $\dim U_a = \dim U_b = m$.

If systems $A$ and $B$ are interconnected in power conserving way, that is if

$$
\begin{align*}
u_a &= -y_b \\
y_b &= u_a 
\end{align*}
$$

(2)

then, the resulting dynamics is given by the following autonomous port Hamiltonian systems, with state space $\mathcal{X}_a \times \mathcal{X}_b$ and Hamiltonian $H_a + H_b$:

$$
\begin{bmatrix}
\dot{x}_a \\
\dot{x}_b
\end{bmatrix} = \begin{bmatrix}
J_a(x_a) - R_a(x_a) & G_a(x_a) G^T_b(x_b) \\
-G^T_a(x_a) G_b(x_b) & J_b(x_b)
\end{bmatrix} \begin{bmatrix}
\partial x_a H_a \\
\partial x_b H_b
\end{bmatrix}
$$

(3)

Given a generic port Hamiltonian system, it is possible to give the following fundamental definition of structural invariant or, equivalently, of Casimir function. [5, 11, 15].

**Definition II.1 (Casimir function)** Consider the following port Hamiltonian system, with state space $\mathcal{X}$ and Hamiltonian function $H : \mathcal{X} \to \mathbb{R}$:

$$
\begin{align*}
\dot{x} &= [J(x) - R(x)] \frac{\partial H}{\partial x} + G(x) u \\
y &= G^T(x) \frac{\partial H}{\partial x}
\end{align*}
$$

A function $C : \mathcal{X} \to \mathbb{R}$ is a Casimir function for the port Hamiltonian system if and only if

$$
\frac{dC}{dt} = 0
$$

for every possible choice of Hamiltonian $H$.

From Def. [11.1] a scalar function $C : \mathcal{X}_a \times \mathcal{X}_b \to \mathbb{R}$ is a Casimir function for system $\text{(3)}$ if and only if the following couple of relations is satisfied:

$$
\begin{align*}
\frac{\partial C}{\partial x_a} [J_a(x_a) - R_a(x_a)] + \frac{\partial C}{\partial x_b} G_a(x_a) G^T_b(x_b) &= 0 \\
\frac{\partial C}{\partial x_b} [J_b(x_b) - R_b(x_b)] - \frac{\partial C}{\partial x_a} G^T_b(x_b) G_a(x_a) &= 0
\end{align*}
$$

(4a)

These conditions are direct consequence of the interconnection law $\text{(2)}$.

The existence of Casimir functions for the closed-loop system $\text{(3)}$ plays an important role in the control by interconnection and energy shaping methodology. Denote by $x^* \in \mathcal{X}$ a desired equilibrium configuration for system $B$. Asymptotic stability in $x^*$ can be achieved by properly choosing the Hamiltonian function of system $A$, which acts as a controller, in order to shape the closed-loop energy $H_a + H_b$ so that a (possibly) global minimum in the desired equilibrium configuration can be introduced. It is important to note that, right now, there is no relation between the state of the controller and the state of the system to be controlled. Then, it is not clear how the controller energy, which is freely assignable, has to be chosen in order to solve the regulation problem.

A possible solution can be to constraint the state of the closed-loop system $\text{(3)}$ on a certain subspace of $\mathcal{X}_a \times \mathcal{X}_b$, for example given by:

$$
\Omega_c := \{ (x_a, x_b) \in \mathcal{X}_a \times \mathcal{X}_b | x_a = S(x_b) + c \}
$$

(5)
where $c \in \mathbb{R}^{n_a}$ and $S : X_b \rightarrow X_a$ is a function still to be computed. In other words, we are looking for a set of Casimir function $C_i : X_b \times X_a \rightarrow \mathbb{R}, i = 1, \ldots, n_a$ for the closed-loop system (3) such that

$$C_i(x_a, x_b) := S_i(x_b) - x_{a,i}$$

where $[S_1(x_b), \ldots, S_{n_a}(x_b)]^T = S(x_b)$. Due to the nature of a Casimir function, it can be deduced that it is possible to introduce an intrinsic non-linear state feedback law that will be used in order to choose the energy function of the controller so that the closed-loop Hamiltonian can be properly shaped. Note that, under these hypothesis, this energy function depends on the state variables of system (11). This control methodology is called invariant function method and it is deeply discussed in [5], [1].

From (11), the set of functions (3) are Casimir functions for (3) if and only if

$$- \frac{\partial T}{\partial x_b} G_b(x_b)G_a(x_a) = J_a(x_a) - R_a(x_a)$$

Then, the following proposition can be proved, [10], [15].

**Proposition II.1:** The functions $C_i, i = 1, \ldots, n_a$, defined in (3) are Casimir functions for the system (3) if and only if the following conditions are satisfied:

$$\frac{\partial T}{\partial x_b} J_b(x_b) \frac{\partial S}{\partial x_a} = J_a(x_a)$$

$$R_b(x_b) \frac{\partial S}{\partial x_a} = 0$$

$$R_a(x_a) = 0$$

$$\frac{\partial T}{\partial x_b} J_b(x_b) = G_a(x_a)G_b(x_b)$$

Suppose that (8) are satisfied. Then, from (3) we deduce that the state variables of the controller are robustly related to the state variable of the system to be stabilized since

$$x_{a,i} = S_i(x_b) + c_i, \quad i = 1, \ldots, n_a$$

with $c_i \in \mathbb{R}$ depending on the initial conditions. Moreover, the closed-loop dynamics (3) evolves on the foliation induced by the level sets

$$\mathcal{L}^0_{C_i} = \{(x_a, x_b) \in X_a \times X_b | x_{a,i} = S_i(x_b) + c_i\}, \quad i = 1, \ldots, n_a$$

which can be expressed as a function of the $x_b$ coordinate. If conditions (8a) and (8b) are taken into account, the reduced dynamics of (3) on these level sets is given by

$$\dot{x}_b = [J_b(x_b) - R_b(x_b)] \frac{\partial H_b}{\partial x_b} - G_b(x_b)G_a(x_a) \frac{\partial H_a}{\partial x_a}$$

$$= [J_b(x_b) - R_b(x_b)] \left( \frac{\partial H_b}{\partial x_b} + \frac{\partial S}{\partial x_a} \frac{\partial H_a}{\partial x_a} \right)$$

From (9), we have that $H_a(x_a) \equiv H_a(S(x_b) + c)$: the controller energy function is finally dependent from $x_b$ through the non-linear feedback action $S(\cdot)$. If

$$H_d(x_b) := H_b(x_b) + H_a(S(x_b) + c)$$

then (11) can be written as

$$\dot{x}_b = [J_b(x_b) - R_b(x_b)] \left( \frac{\partial H_b}{\partial x_b} + \frac{\partial S}{\partial x_a} \frac{\partial H_a}{\partial x_a} \right) = [J_b(x_b) - R_b(x_b)] \frac{\partial H_d}{\partial x_b}$$

In conclusion, the following proposition has been proved.

**Proposition II.2:** Consider the closed-loop port Hamiltonian system (2) and suppose that the function $S(x_b) = [S_1(x_b), \ldots, S_{n_a}(x_b)]^T$ satisfies conditions (8). Then, the reduced dynamics on the level sets (10) is given by (13), where the closed-loop energy function $H_d$ is given by (12).

By properly choosing the controller energy function $H_d$, it is possible to shape the closed-loop energy function $H_d$ defined in (12) so that a new minimum in $x_b^*$ is introduced. Then, the desired configuration can be reached with a dynamics given by (13).
III. m-pH systems. A simple example

The m-pH under study is the result of the interconnection of systems $A$ and $B$ by means of an infinite dimensional system. Following the same idea presented in [12] for a simpler case, the interconnection law (2) can be generalized once it is supposed to interconnect systems (1a) and (1b) by means of parameters port Hamiltonian systems.

Denote by $\Omega^k(\mathcal{N})$ the space of $k$-forms on $\mathcal{N}$, where $\mathcal{N}$ is an $n$-dimensional Riemannian manifold, by $d : \Omega^k(\mathcal{N}) \rightarrow \Omega^{k+1}(\mathcal{N})$ the exterior derivative and by $* : \Omega^k(\mathcal{N}) \rightarrow \Omega^{n-k}(\mathcal{N})$ the Hodge star operator, [5]. Then, the dpH model of the $i$-th transmission line is given by (see [7]):

$$
\begin{bmatrix}
-\partial_t \alpha_{E,i} \\
-\partial_t \alpha_{M,i}
\end{bmatrix}
= \begin{bmatrix}
0 & d \\
d & 0
\end{bmatrix}
\begin{bmatrix}
G_i & 0 \\
0 & R_i
\end{bmatrix}
\begin{bmatrix}
\delta_{E,i} & \delta_{M,i}
\end{bmatrix}
\begin{bmatrix}
\mathcal{H}_{\infty,i}
\end{bmatrix}
\begin{bmatrix}
f_{b,i} \\
e_{b,i}
\end{bmatrix}
= \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
\delta_{E,i} & \delta_{M,i}
\end{bmatrix}
\begin{bmatrix}
\mathcal{H}_{\infty,i}
\end{bmatrix}
$$

(14)

where $(\alpha_{E,i}, \alpha_{M,i}) \in \mathcal{X}_{\infty,i}$ are the state variables, with $\mathcal{X}_{\infty,i} := \Omega^1(\mathcal{D}) \times \Omega^1(\mathcal{D})$, $(f_{b,i}, e_{b,i}) \in \Omega^0(\partial \mathcal{D}) \times \Omega^0(\partial \mathcal{D})$ are the power conjugated boundary variables and $\mathcal{H}_i$ the total energy, which can be expressed, in the simplest case, by means of the following quadratic functional:

$$\mathcal{H}_{\infty,i}(\alpha_{E,i}, \alpha_{M,i}) = \frac{1}{2} \int_{\mathcal{D}} \left[ \frac{1}{G_i} \alpha_{E,i} \wedge * \alpha_{E,i} + \frac{1}{L_i} \alpha_{M,i} \wedge * \alpha_{M,i} \right]$$

Clearly, the set of $m$ transmission lines can be treated as a single dpH system, with state space

$$\mathcal{X}_\infty := \mathcal{X}_{\infty,1} \times \cdots \times \mathcal{X}_{\infty,m}$$

and total Hamiltonian

$$\mathcal{H}_\infty(\alpha_{E,1}, \ldots, \alpha_{E,m}, \alpha_{M,1}, \ldots, \alpha_{M,m}) = \sum_{i=1}^{m} \mathcal{H}_{\infty,i}(\alpha_{E,i}, \alpha_{M,i})$$

In fact, if

$$\alpha_E := \begin{bmatrix} \alpha_{E,1} & \cdots & \alpha_{E,m} \end{bmatrix}^T$$
$$\alpha_M := \begin{bmatrix} \alpha_{M,1} & \cdots & \alpha_{M,m} \end{bmatrix}^T$$
$$\delta E \mathcal{H}_\infty := \begin{bmatrix} \delta_{E,1} \mathcal{H}_{\infty,1} & \cdots & \delta_{E,m} \mathcal{H}_{\infty,m} \end{bmatrix}^T$$
$$\delta M \mathcal{H}_\infty := \begin{bmatrix} \delta_{M,1} \mathcal{H}_{\infty,1} & \cdots & \delta_{M,m} \mathcal{H}_{\infty,m} \end{bmatrix}^T$$
$$f_B := \begin{bmatrix} f_{b,1} \\
\vdots \\
e_{b,m} \end{bmatrix}^T$$

then the set of $m$ transmission lines (14) can be briefly written as

$$\begin{bmatrix}
-\partial_t \alpha_E \\
-\partial_t \alpha_M
\end{bmatrix}
= \begin{bmatrix}
0 & d \\
d & 0
\end{bmatrix}
\begin{bmatrix}
G & 0 \\
0 & R
\end{bmatrix}
\begin{bmatrix}
\delta E \mathcal{H}_\infty \\
\delta M \mathcal{H}_\infty
\end{bmatrix}
\begin{bmatrix}
f_B \\
e_B
\end{bmatrix}
= \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
\delta E \mathcal{H}_\infty |_{\partial \mathcal{D}} \\
\delta M \mathcal{H}_\infty |_{\partial \mathcal{D}}
\end{bmatrix}
$$

where $R := \text{diag}(R_1, \ldots, R_m)$ and $G := \text{diag}(G_1, \ldots, G_m)$.

Suppose to interconnect systems (1a) and (1b) by means of the $m$ transmission lines (14) in a power conserving way. An admissible interconnection law could be the following one:

$$u_a := \begin{bmatrix} e_{b,1}(0) \\
\vdots \\
e_{b,m}(0) \end{bmatrix}, \quad y_a := \begin{bmatrix} f_{b,1}(0) \\
\vdots \\
f_{b,m}(0) \end{bmatrix}, \quad u_B := \begin{bmatrix} f_{b,1}(L) \\
\vdots \\
f_{b,m}(L) \end{bmatrix}, \quad y_B := \begin{bmatrix} e_{b,1}(L) \\
\vdots \\
e_{b,m}(L) \end{bmatrix}$$

(16)

Note that, if $L = 0$, then (16) is the same as (2). The resulting system is a mixed finite and infinite dimensional port Hamiltonian system (m-pH system), with configuration space

$$\mathcal{X}_{ld} := \mathcal{X}_a \times \mathcal{X}_b \times \mathcal{X}_\infty$$

(17)
total Hamiltonian
\[ H_{cl}(x_a, x_b, \alpha_{E,1}, \ldots, \alpha_{M,m}) := H_a(x_a) + H_b(x_b) + \mathcal{H}_\infty(\alpha_{E,1}, \ldots, \alpha_{M,m}) \] (18)
and whose dynamics is described by means of the following set of ODEs and PDEs:
\[
\begin{bmatrix}
\dot{x}_a \\
\dot{x}_b \\
\dot{\alpha}_E \\
\dot{\alpha}_M
\end{bmatrix}
= \begin{bmatrix}
J_a(x_a) - R_a(x_a) & 0 & G_a(x_a) \cdot |0 & 0 \\
0 & J_b(x_b) - R_b(x_b) & 0 & -G_b(x_b) \cdot |L \\
0 & 0 & 0 & -G^* - d \\
0 & 0 & 0 & -R^*
\end{bmatrix}
\begin{bmatrix}
\dot{x}_a \\
\dot{x}_b \\
\dot{\alpha}_E \\
\dot{\alpha}_M
\end{bmatrix}
+ \begin{bmatrix}
G^T_a(x_a) \partial_{x_a} H_a \\
G^T_b(x_b) \partial_{x_b} H_b
\end{bmatrix}
\begin{bmatrix}
\delta_E H_\infty \\
\delta_M H_\infty
\end{bmatrix}
\] (19)

It is easy to verify that (19) satisfies the following power balance relation:
\[
d\mathcal{H}_{cl} \leq - \left( \frac{\partial^{T} \mathcal{H}_{cl}}{\partial x_a} R_a \frac{\partial \mathcal{H}_{cl}}{\partial x_a} + \frac{\partial^{T} \mathcal{H}_{cl}}{\partial x_b} R_b \frac{\partial \mathcal{H}_{cl}}{\partial x_b} \right) = - \left( \frac{\partial^{T} \mathcal{H}_{a}}{\partial x_a} R_a \frac{\partial \mathcal{H}_{a}}{\partial x_a} + \frac{\partial^{T} \mathcal{H}_{b}}{\partial x_b} R_b \frac{\partial \mathcal{H}_{b}}{\partial x_b} \right) \leq 0
\]

**IV. Casimir functionals for m-pH systems**

As discussed in Sec. II, the applicability of the control by interconnection and energy shaping relies on the possibility of relating the controller state variables to the state variables of the plant by means of Casimir functions. Equivalently, we can say that the controller structure is chosen in order to constrain the closed-loop trajectory to evolve on a particular generalization of Def. II.1 can be stated as follows.

**Definition IV.1 (Casimir functionals)** Consider a function \( \mathcal{C} : \mathcal{X}_{cl} \rightarrow \mathbb{R} \), where \( \mathcal{X}_{cl} \) is given by (17). Then, \( \mathcal{C} \) is a Casimir functional for the system (19) if and only if
\[
d\mathcal{C} \leq 0, \quad \text{for every } \mathcal{H}_{cl} : \mathcal{X}_{cl} \rightarrow \mathbb{R}
\]
where \( \mathcal{H}_{cl} \) has the structure given in (18).

Since
\[
d\mathcal{C} = \frac{\partial \mathcal{C}}{\partial x_a} \dot{x}_a + \frac{\partial \mathcal{C}}{\partial x_b} \dot{x}_b + \frac{\partial \mathcal{C}}{\partial \alpha} \dot{\alpha}
\]
from (17), (18) and (19) we have that
\[
d\mathcal{C} = \frac{\partial \mathcal{C}}{\partial x_a} [J_a - R_a] \frac{\partial \mathcal{H}_{a}}{\partial x_a} + \frac{\partial \mathcal{C}}{\partial x_b} [J_b - R_b] \frac{\partial \mathcal{H}_{b}}{\partial x_b} + \frac{\partial \mathcal{C}}{\partial \alpha} \frac{\partial \mathcal{H}_{\infty}}{\partial \alpha}
\]
\[
\begin{bmatrix}
\delta_{E,1} H_{\infty,1} |0 \\
\vdots \\
\delta_{E,m} H_{\infty,m} |0 \\
\delta_{M,1} H_{\infty,1} |L \\
\vdots \\
\delta_{M,m} H_{\infty,m} |L
\end{bmatrix}
\]
\[
= \sum_{i=1}^{m} \int_{D} (d \delta_{M,i} H_{\infty,i} + G_i \delta_{E,i} H_{\infty,i}) \wedge \delta_{E,i} \mathcal{C}
\]
\[
+ \sum_{i=1}^{m} \int_{D} (d \delta_{E,i} H_{\infty,i} + R_i \delta_{M,i} H_{\infty,i}) \wedge \delta_{M,i} \mathcal{C}
\]

Since \( d(\alpha \wedge \beta) = d\alpha \wedge \beta + \alpha \wedge d\beta \) and \((\kappa \alpha) \wedge \beta = \alpha \wedge (\kappa \beta)\) when \( \alpha, \beta \in \Omega^{0}(D) \) and \( \kappa \in \mathbb{R} \), the integral term in (20) becomes:
\[
\sum_{i=1}^{m} \int_{D} [\delta_{E,i} H_{\infty,i} \wedge (d \delta_{M,i} \mathcal{C} - G_i \delta_{E,i} \mathcal{C}) + \delta_{M,i} H_{\infty,i} \wedge (d \delta_{E,i} \mathcal{C} - R_i \delta_{M,i} \mathcal{C})]
\]
\[
- \sum_{i=1}^{m} \int_{D} [d (\delta_{M,i} H_{\infty,i} \wedge \delta_{E,i} \mathcal{C}) + d (\delta_{E,i} H_{\infty,i} \wedge \delta_{M,i} \mathcal{C})]
\] (21)
From the Stokes’ theorem, we have that

$$\int_D d(\delta H \wedge \delta C) = \int_{\partial D} \delta H \wedge \delta C \mid_{\partial C}$$

Then, from (20) and (21), we can write that

$$\frac{dC}{dt} = \left\{ \frac{\partial^T C}{\partial x_a} [J_a - R_a] + [\delta_{E,1} C |_0 \cdots \delta_{E,m} C |_0] G_a^T \right\} \frac{\partial H_a}{\partial x_a} + \left\{ \frac{\partial^T C}{\partial x_b} [J_b - R_b] - [\delta_{M,1} C |_L \cdots \delta_{M,m} C |_L] G_b^T \right\} \frac{\partial H_b}{\partial x_b} + \sum_{i=1}^m \int_D \left[ \delta_{E,i} H_{\infty,i} \wedge \left( d\delta_{M,i} C - G_i \ast \delta_{E,i} C \right) + \delta_{M,i} H_{\infty,i} \wedge \left( d\delta_{E,i} C - R_i \ast \delta_{M,i} C \right) \right]$$

which, according to Def. IV.1, has to be equal to zero for every Hamiltonian function $H_a$, $H_b$ and $H_\infty$. It can be deduced that the following set of conditions has to hold:

\begin{align}
\frac{\partial^T C}{\partial x_a} [J_a - R_a] + [\delta_{E,1} C |_0 \cdots \delta_{E,m} C |_0] G_a^T &= 0 \quad (22a) \\
\frac{\partial^T C}{\partial x_b} [J_b - R_b] - [\delta_{M,1} C |_L \cdots \delta_{M,m} C |_L] G_b^T &= 0 \quad (22b) \\
\frac{\partial^T C}{\partial x_a} G_a + [\delta_{M,1} C |_0 \cdots \delta_{M,m} C |_0] &= 0 \quad (22c) \\
\frac{\partial^T C}{\partial x_b} G_b + [\delta_{E,1} C |_L \cdots \delta_{E,m} C |_L] &= 0 \quad (22d) \\
d\delta_{M,i} C - G_i \ast \delta_{E,i} C &= 0 \quad (22e) \\
d\delta_{E,i} C - R_i \ast \delta_{M,i} C &= 0 \quad (22f)
\end{align}

where (23) and (24) have to hold for every $i = 1, \ldots, m$. These conditions are a generalization of the classical definition of Casimir function reported in [13]. In conclusion, the following proposition has been proved.

**Proposition IV.1:** Consider the mixed finite and infinite dimensional port Hamiltonian system (19), for which $X_d$ is the configuration space, defined in (17), and $H_d$ is the Hamiltonian, defined in (18). Then, a functional $C : X_d \rightarrow \mathbb{R}$ is a Casimir functional if and only if conditions (22) are satisfied.

**Note IV.1:** Suppose that the $m$ transmission lines are lossless, that is $R_i = G_i = 0$ for every $i = 1, \ldots, m$. Then, from (22e) and (22f), we deduce that $C$ is a Casimir functional if and only if

\begin{equation}
\left\{ \begin{array}{l}
d\delta_{E,i} C = 0 \\
d\delta_{M,i} C = 0 \\
\end{array} \right. \quad i = 1, \ldots, m
\end{equation}

or, equivalently, if $\delta_{E,i} C$ and $\delta_{M,i} C$ are constant on $D$ as function of $z \in D$. Then, from (23), we have that

\begin{equation}
\delta_{E,i} C = \delta_{E,i} C |_0 = \delta_{E,i} C |_L \quad \text{and} \quad \delta_{M,i} C = \delta_{M,i} C |_0 = \delta_{M,i} C |_L \quad i = 1, \ldots, m
\end{equation}

Then, from (24) and by combining (22b) with (22d), we deduce that $C$ is a Casimir functional if satisfies relation (23) and (24). Note that (24) are the necessary and sufficient conditions for the existence of Casimir functions in the finite dimensional case, when the interconnection law is purely algebraic and given by (2); they continue to hold under the hypothesis that the interconnecting infinite dimensional system is lossless.
V. Control of m-pH systems by energy shaping

Consider the system (1b) and denote by $x^*_a$ a desired equilibrium point. As discussed in Sec. III in finite dimensions the stabilization of (1b) in $x^*_a$ by means of the controller (1a) can be solved by interconnecting both the systems according to (2) and looking for Casimir functions of the resulting closed-loop system in the form $\mathcal{C}_i(x_a, x_b) = x_{a,i} - S_i(x_b)$, $i = 1, \ldots, n_a$. Clearly, they have to satisfy conditions (13) or, equivalently, (23).

In this way, since $\mathcal{C}_i = 0$, we have that $x_a = S(x_b) + \kappa$ for every energy function $H_a$ and $H_b$. This relation defines, then, a structural state feedback law. Furthermore, $H_a$, which is freely assignable, can be expressed as a function of $x_a$: the problem of shaping the closed-loop energy in order to introduce a minimum in $x^*_a$ can be solved by properly choosing $H_a$. Finally, if dissipation is added, then this new minimum is reached.

The stabilization of the m-pH system (19) can be stated as follows. Denote by $(\chi^*, x^*_b)$ a desired equilibrium configuration for the m-pH system, where $\chi^*$ is a configuration of the infinite dimensional system that is compatible with the desired equilibrium point $x^*_b$ of the finite dimensional sub-system. In order to stabilize the configuration $(\chi^*, x^*_b)$, it is necessary to chose the finite dimensional controller (1a) so that the open-loop energy function could be shaped by acting on $H_a$. As in the finite dimensional case, there is no a priori relation between the state of the controller and the state of the system to be controlled and it is not clear how the controller energy, which is freely assignable, has to be chosen in order to solve the regulation problem. By generalization of the finite dimensional approach, a possible solution can be to robustly relate the controller state variable with the plant state variable by means of a set of Casimir functions. Consider the m-pH system (19): if

$$\mathcal{C}_i(x_{a,i}, x_b, \alpha_{E,1}, \ldots, \alpha_{M,m}) = x_{a,i} - S_i(x_b) - S_i(\alpha_{E,1}, \ldots, \alpha_{M,m}), \quad i = 1, \ldots, n_a$$

are a set of Casimir functionals, then, independently from the energy functions $H_a$, $H_b$ and $H_{\infty}$, we have that:

$$x_{a,i} = S_i(x_b) + S_i(\alpha_{E,1}, \ldots, \alpha_{E,m}, \alpha_{M,1}, \ldots, \alpha_{M,m}) + \kappa_i, \quad i = 1, \ldots, n_a$$

The constants $\kappa_i$ depends only on the initial conditions and can be set to zero if the initial state is known. In this way, the controller state variable is expressed as function of the state variable of the system (1a) and of the configuration of the $m$ transmission lines (14). Consequently, the closed-loop energy function (18) becomes:

$$H_{cl}(x_a, x_b, \alpha_{E,1}, \ldots, \alpha_{E,m}, \alpha_{M,1}, \ldots, \alpha_{M,m}) = H_b(x_b) + \sum_{i=1}^{m} H_{\infty,i}(\alpha_{E,i}, \alpha_{M,i}) + H_a(S_1(x_b) + S_1(\alpha_{E,1}, \ldots, \alpha_{M,m}), \ldots, S_{n_a}(x_b) + S_{n_a}(\alpha_{E,1}, \ldots, \alpha_{M,m}))$$

where $H_a$ can be freely chosen in order to introduce a minimum in $(\chi^*, x^*_b)$.

The $n_a$ functionals (25) are Casimir functionals for (19) if and only if conditions (22) are satisfied. In particular, the couple of relations (22a) and (22b) can be equivalently written as

$$\begin{cases} \frac{d\delta_{E,i}S_i}{dt} - G_j* \delta_{E,j}S_i = 0 & i = 1, \ldots, n_a; \quad j = 1, \ldots, m \\ \frac{d\delta_{E,j}S_i}{dt} - R_j* \delta_{M,j}S_i = 0 & \end{cases}$$

which is a system of partial differential equations that has to be solved for every $S_i, i = 1, \ldots, n_a$.

As in the finite dimensional case (see Sec. II), dissipation introduces strong constraints on the applicability of passivity-based control techniques or, equivalently, on the admissible Casimir functions for the closed-loop system. Clearly, these limitations are present also when dealing with m-pH systems. In order to initially simplify the problem, it assumed that the infinite dimensional subsystem is lossless, as already discussed in Note IV.3. In this case, condition (25) becomes

$$\begin{cases} \frac{d\delta_{E,i}S_j}{dt} = 0 & i = 1, \ldots, m; \quad j = 1, \ldots, n_a \\ \frac{d\delta_{M,j}S_i}{dt} = 0 & \end{cases}$$

which expresses the fact that $\delta_{E,i}S_i$ and $\delta_{M,j}S_i$ are constant along $D$. Then, we have that, for every $i = 1, \ldots, n_a$ and $j = 1, \ldots, m$

$$\delta_{E,j}S_i = \delta_{E,j}S_i \big|_0 = \delta_{E,j}S_i \big|_L \quad \text{and} \quad \delta_{M,j}S_i = \delta_{M,j}S_i \big|_0 = \delta_{M,j}S_i \big|_L,$$
From (25) and (27), conditions (22a–d) can be written as:

\[
\begin{bmatrix}
\delta_{E,1}S_1 & \cdots & \delta_{E,m}S_1 \\
\vdots & \ddots & \vdots \\
\delta_{E,1}S_{na} & \cdots & \delta_{E,m}S_{na}
\end{bmatrix}
G_a^T = 264 \pm E; 1 S_1 \cdots \pm E; 1 S_{na}. \quad (28a)
\]

\[
\begin{bmatrix}
\delta_{M,1}S_1 & \cdots & \delta_{M,m}S_1 \\
\vdots & \ddots & \vdots \\
\delta_{M,1}S_{na} & \cdots & \delta_{M,m}S_{na}
\end{bmatrix}
G_b^T = 264 \pm M; 1 S_1 \cdots \pm M; 1 S_{na}. \quad (28b)
\]

\[
G_a = \begin{bmatrix}
\delta_{M,1}S_1 & \cdots & \delta_{M,m}S_1 \\
\vdots & \ddots & \vdots \\
\delta_{M,1}S_{na} & \cdots & \delta_{M,m}S_{na}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\delta_{E,1}S_1 & \cdots & \delta_{E,m}S_1 \\
\vdots & \ddots & \vdots \\
\delta_{E,1}S_{na} & \cdots & \delta_{E,m}S_{na}
\end{bmatrix}
G_b = -264 \pm M; 1 S_1 \cdots \pm M; 1 S_{na}. \quad (28c)
\]

\[
\begin{bmatrix}
\delta_{E,1}S_1 & \cdots & \delta_{E,m}S_1 \\
\vdots & \ddots & \vdots \\
\delta_{E,1}S_{na} & \cdots & \delta_{E,m}S_{na}
\end{bmatrix}
G_b = -264 \pm M; 1 S_1 \cdots \pm M; 1 S_{na}. \quad (28d)
\]

By substitution of (28c) in (28a), and of (28d) in (28b), after a post-multiplication by \( \frac{\partial S}{\partial x_b} \), we deduce that

\[
J_a + R_a = \frac{\partial^T S}{\partial x_b} [J_b - R_b] \frac{\partial S}{\partial x_b} \quad (29)
\]

Since \( J_a \) and \( J_b \) are skew-symmetric and \( R_a \) and \( R_b \) are symmetric and positive definite, we deduce that, necessarily,

\[
J_a = \frac{\partial^T S}{\partial x_b} J_b \frac{\partial S}{\partial x_b} \quad (30a)
\]

\[
R_a = 0 \quad (30b)
\]

Furthermore, from (29) and (30a), we deduce that

\[
R_b \frac{\partial S}{\partial x_b} = 0 \quad (30c)
\]

and from (28a), (28b) and (30c) that

\[
\frac{\partial^T S}{\partial x_b} J_b = G_a G_b^T \quad (30d)
\]

In conclusion, the following proposition has been proved.

**Proposition V.1:** Consider the \( m \)-pH system (19) and suppose that the \( m \) transmission lines are lossless, that is \( R_i = G_i = 0, i = 1, \ldots, m \). Then, the \( n_a \) functionals (25) are Casimir functionals for this system if the conditions (27), (28a), (28b) and (30a) are satisfied.

Note that conditions (30), involving the finite dimensional subsystem of (19), are the same of (8), which are required in the finite dimensional energy Casimir method. Furthermore, Prop. V.1 generalizes the already cited results presented in [12], which can be easily treated as a particular case.

Problems arise when the infinite dimensional system is not lossless. In order to not complicate calculations too much, in (14) suppose that \( G_i = 0 \) and \( R_i \neq 0, i = 1, \ldots, m \). Then, from (25), conditions (22e) and (22f) can be written as the following set of ODEs in the spatial variable \( z \)

\[
\begin{cases}
\delta_{M,j}S_i = 0 \\
\delta_{E,j}S_i = R_j \delta_{M,j}S_i
\end{cases}
\]

\[
\delta_{M,j}S_i = M_{ij} \quad i = 1, \ldots, n_a; \quad j = 1, \ldots, m
\]

\[
\delta_{E,j}S_i = R_j M_{ij} z + E_{ij} \quad i = 1, \ldots, n_a; \quad j = 1, \ldots, m
\]

where \( E_{ij}, M_{ij} \in \mathbb{R} \) are constants still to be specified and \( z \in \mathcal{D} \). Consequently, in (25), we have that for every \( i = 1, \ldots, n_a \)

\[
S_i(\alpha_{E,1}, \ldots, \alpha_{M,m}) = \int_{\mathcal{D}} \sum_{j=1}^{m} [M_{ij}(\alpha_{M,j} + (R_j M_{ij} z + E_{ij}) \alpha_{E,j})]
\]
If
\[ E := [E_{ij}] \quad M := [M_{ij}] \]
then conditions (22a–d) can be written as
\[ [J_a - R_a] = E G_a^T \]
\[ \partial^T S [J_b - R_b] = M G_b^T \]
\[ G_a = M \]
\[ \partial^T S G_b = -[M RL + E] \]

From (33b) and (33c), we have that
\[ \partial^T S [J_b - R_b] \frac{\partial S}{\partial x_b} = G_a G_b^T \frac{\partial S}{\partial x_b} \]
(34)
Then, combining (33b), (33c) and (34), it can be obtained that
\[ \partial^T S [J_b - R_b] \frac{\partial S}{\partial x_b} = J_a + R_a - G_a R G_a^T L \]
(35)
Since \( J_a \) and \( J_b \) are skew-symmetric and \( R_a \) and \( R_b \) are symmetric and positive definite, the controller interconnection and damping matrices can be chosen as
\[ J_a = \frac{\partial^T S}{\partial x_b} J_b \frac{\partial S}{\partial x_b} \]
(36a)
\[ R_a = G_a R G_a^T L \]
(36b)
Consequently, in (25), the set of functions \( S_i(\cdot), i = 1, \ldots, n_a \) taking into account the finite dimensional part of the m-pH system (19) are the solution of the following PDE:
\[ R_b \frac{\partial S}{\partial x_b} = 0 \]
(36c)
This is the same equation that has to be solved in the finite dimensional case and in the m-pH system case when the transmission lines are lossless. Finally the input matrix \( G_a \) of the controller is the solution of the following equation, resulting from (33b), (33c) and (36c):
\[ \frac{\partial^T S}{\partial x_b} J_b = G_a G_b^T \]
(36d)
The same result holding in finite dimensions and when the dpH part of the system is lossless has been obtained again. Then, the following proposition has been proved.

Proposition V.2: Consider the m-pH system (19) and suppose that \( R_i \neq 0 \) and \( G_i = 0, i = 1, \ldots, m \). Then, the \( n_a \) functionals (25) are Casimir functionals for this system if each \( S_i \) is given as in (32), \( i = 1, \ldots, n_a \), and conditions (36) are satisfied.

The presence of dissipative phenomena in the dpH subsystem (14) obviously modifies the expression of the Casimir functions (25) of the closed-loop system. It is important to note, however, that, if \( x_a,i - S_i(\cdot), i = 1, \ldots, n_a \), are Casimir functions in finite dimensions, then the same functions \( S_i \) appear in the expression of the Casimir functions (25) for the m-pH system if the damping matrix \( R_a \) of the controller is chosen according to (36b). What happens is that the contribution to the structural invariants (25) of the finite dimensional part is still the solution of the PDE (36c), which is the same as (8b)); furthermore, in order to compensate the effect of dissipation in the m-pH system (14) which interconnects plant (system \( B \)) and controller (system \( A \)), it is necessary to introduce some dissipation by means of (36b) also in the controller. The damping matrix \( R_a \) of the controller takes into account the overall dissipation inside the set of transmission lines.

An analogous result holds if, in (14), it is supposed that \( G_i \neq 0 \) and \( R_i = 0, i = 1, \ldots, n_a \). In this case, conditions (22b) and (22d) can be written as the following set of ODEs in the spatial variable \( z \):
\[
\begin{align*}
\frac{d\delta M_{j,i} S_i}{dz} &= G_i * \delta E_{j,i} S_i \\
\frac{d\delta E_{j,i} S_i}{dz} &= 0
\end{align*}
\]
i = 1, \ldots, \( n_a \); \( j = 1, \ldots, m \)
whose solution is given by
\[
\begin{align*}
\delta_{M,j} S_i &= G_j E_{ij} z + M_{ij} \
\delta_{E,j} S_i &= E_{ij} 
\end{align*}
\tag{37}
\]
with \(E_{ij}, M_{ij} \in \mathbb{R}\). In (25), we have that for every \(i = 1, \ldots, n_a\)
\[
S_i(\alpha_{E,1}, \ldots, \alpha_{M,m}) = \int_{\mathcal{D}} \sum_{j=1}^{m} [E_{ij} \alpha_{E,j} + (G_j E_{ij} z + M_{ij}) \alpha_{M,j}]
\tag{38}
\]
and conditions (22a–d) can be written as
\[
\begin{align*}
[J_a - R_a] &= E G_a^T \
\frac{\partial^T S}{\partial x_b} [J_b - R_b] &= [EGL + M] G_b^T \
G_a &= M \
\frac{\partial^T S}{\partial x_b} G_b &= -E
\end{align*}
\tag{39a-39d}
\]
It is possible to verify that
\[
\frac{\partial^T S}{\partial x_b} [J_b - R_b] \frac{\partial S}{\partial x_b} = J_a + R_a - \frac{\partial^T S}{\partial x_b} G_b G_b^T \frac{\partial S}{\partial x_b} L
\tag{40}
\]
Consequently, if the interconnection and damping matrices of the controller are given by
\[
\begin{align*}
J_a &= \frac{\partial^T S}{\partial x_b} J_b \frac{\partial S}{\partial x_b} \
R_a &= \frac{\partial^T S}{\partial x_b} G_b G_b^T \frac{\partial S}{\partial x_b} L
\end{align*}
\tag{41a-41b}
\]
then the functions \(S_i(\cdot), i = 1, \ldots, n_a\) are again the solutions of the set of PDEs (36c), which is the same as in finite dimensions. In this case, the input matrix \(G_a\) is solution of the following equation:
\[
\frac{\partial^T S}{\partial x_b} [J_b + G_b G_b^T L] = G_a G_a^T
\tag{41c}
\]
Also in this case, it is necessary to introduce dissipation in the controller by means of the matrix \(R_a\) in order to compensate an analogous phenomenon in the transmission lines.

**Proposition V.3:** Consider the m-pH system (19) and suppose that \(R_i = 0\) and \(G_i \neq 0, i = 1, \ldots, m\). Then, the \(n_a\) functional (25) are Casimir functionals for this system if each \(S_i\) is given as in (38), \(i = 1, \ldots, n_a\), and conditions (41) are satisfied.

Once it is possible to choose the controller interconnection, damping and input matrices in order to render the set of functions (25) structural invariants for the closed-loop system (19), the energy function of the controller (1a), that is its Hamiltonian function \(H_a\) can be chosen in order to introduce a minimum in the desired equilibrium configuration \((\chi^*, x_b^*) \in \mathcal{X}_\infty \times \mathcal{X}_b\) in the closed-loop energy function (18). At this point, (asymptotic) stability can be verified following the method proposed in [14] and based on Arnold’s first and second stability theorem for linear and nonlinear distributed parameter systems or by applying the generalization of La Salle’s Invariance Principle to infinite dimensions proposed in [2].

In the next section, the stabilization of a simple m-pH system for which the stability proof relies on the application of the method proposed [14] is presented. This example is a generalization of the one proposed in [12].

**VI. Stabilization of a Simple m-pH System by Energy Shaping**

Consider the series RLC circuit whose port Hamiltonian model is given by
\[
\begin{align*}
\begin{bmatrix}
\dot{x}_{b,1} \\
\dot{x}_{b,2}
\end{bmatrix} &= 
\begin{bmatrix}
0 & 1 \\
-1 & -R
\end{bmatrix}
\begin{bmatrix}
\partial x_{b,1} H_b \\
\partial x_{b,2} H_b
\end{bmatrix} + 
\begin{bmatrix}
0 \\
1
\end{bmatrix} u_b \\
y_b &= \frac{\partial H_b}{\partial x_{b,2}}
\end{align*}
\tag{42}
\]
in which $x_b = [x_{b,1}^T x_{b,2}]^T$ is the state variable, with $x_{b,1}$ the charge stored in the capacitor and $x_{b,2}$ the flux in the inductance, and

$$H_b(x_{b,1}, x_{b,2}) = \frac{1}{2} x_{b,1}^2 \frac{1}{C_b} + \frac{1}{2} x_{b,2}^2 \frac{1}{L_b}$$

the total energy. As reported in Fig. 1, suppose to interconnect system (24) to a transmission line in $z = L$, where $L$ denotes the length of the line. Furthermore, suppose that $G \neq 0$ and that $R = 0$ in (14). Finally, suppose to interconnect the controller (14) to the transmission line in $z = 0$. If $(f_0, e_0)$ and $(f_L, e_L)$ are the power conjugated port variables at $z = 0$ and $z = L$ respectively, then, after a port dualization, at both sides of the line in order to give physical consistency to efforts and flows, the interconnection law (16) can be written as

$$\begin{cases}
y_a = -e_0 \\
u_a = f_0
\end{cases} \quad \begin{cases}
y_b = f_L \\
u_b = e_L
\end{cases}$$

Furthermore, denote by $\chi$ the state of the resulting m-pH system, by $C_\infty$ the distributed capacitance and by $L_\infty$ the distributed inductance of the line. Consequently, the total closed-loop energy function becomes

$$\mathcal{H}_{cl}(\chi) = \frac{1}{2} x_{b,1}^2 \frac{1}{C_b} + \frac{1}{2} x_{b,2}^2 \frac{1}{L_b} + H_a(x_a) + \frac{1}{2} \int_0^L \left( \alpha_E^2 \frac{1}{C_\infty} + \alpha_M^2 \frac{1}{L_\infty} \right) dz \quad (43)$$

In order to apply the control by interconnection methodology, it is necessary to find the Casimir functions of the form (25) for the closed-loop dynamics. Since it is desired to conserve the finite dimensional part of the Casimir function even in presence of an interconnection involving a dpH systems, condition (8b) of Prop. 11 has to be satisfied. As in the finite dimensional case, we obtain that

$$\frac{\partial S}{\partial x_{b,1}} = 1 \quad \text{and} \quad \frac{\partial S}{\partial x_{b,2}} = 0$$

which implies that $S(x_b) = x_{b,1}$. Moreover, as regard the functional $S(\alpha_E, \alpha_M)$, we obtain that

$$\delta_M S(\alpha_E, \alpha_M) = E \quad \text{and} \quad \delta_E S(\alpha_E, \alpha_M) = G E z + M \quad (44)$$

where $E, M \in \mathbb{R}$ are constants that will be specified later on. It is easy to verify that conditions (43) can be satisfied by choosing $J_a = 0, R_a = G L$ and $G_a = -1$ and, consequently, $E = 1$ and $M = -G L$ in (14), as presented in Prop. 13. Then, a Casimir function for the closed-loop system is given by

$$\mathcal{C}(x_a, x_b, \alpha_E, \alpha_M) = x_a - x_{b,1} - \int_0^L [\alpha_E + G(z - L) \alpha_M] dz \quad (45)$$

and the closed-loop energy function (43) can be written as

$$\mathcal{H}_{cl}(\chi) = \frac{1}{2} x_{b,1}^2 \frac{1}{C_b} + \frac{1}{2} x_{b,2}^2 \frac{1}{L_b} + H_a \left( x_{b,1} + \int_0^L [\alpha_E + G(z - L) \alpha_M] dz + \kappa \right) + \frac{1}{2} \int_0^L \left( \alpha_E^2 \frac{1}{C_\infty} + \alpha_M^2 \frac{1}{L_\infty} \right) dz$$

since, from (45), we have that

$$x_a = x_{b,1} + \int_0^L [\alpha_E + G(z - L) \alpha_M] dz + \kappa \quad (46)$$

1 The port dualization changes the role of efforts and flows variables in the transmission line. Differently from Sec. 13 and Sec. 17 in which the finite dimensional systems are supposed to have an effort as input, in this case the input signal is a flow (i.e. a current). The immediate consequence is that, even if it is supposed that $G \neq 0$ and $R = 0$, the result concerning the existence of Casimir functions in the form (25) that has to be used is the one presented in Prop. 13.
where $\kappa \in \mathbb{R}$. If the initial configuration is known, it is possible to assume $\kappa = 0$. Denote by $\chi^*$ a desired equilibrium configuration, that is

$$\chi^* = [x^*_b, 1, x^*_b, 2, \alpha^*_E, \alpha^*_M]^T = \left[ x^*_b, 1, 0, C\infty x^*_b, 1, L\infty x^*_b, 1, G(L - z) \right]^T$$

and by $x^*_a$ the value of $x_a$ given by (45) and evaluated in $\chi^*$. Then, in the remaining part of this section, it will be shown that, by selecting

$$H_a(x_a) = \frac{1}{2} \frac{1}{C} x^2_a + k x_a$$

with $C_a > 0$, $k \in \mathbb{R}$ to be specified, it is possible to shape the closed-loop energy in such a way that it has a minimum at the equilibrium point $\chi^*$. Furthermore, it will be verified that this configuration is stable in the sense of the following definition. [13].

**Definition VI.1** (Lyapunov stability for distributed param. systems) Denote by $\chi^* \in \mathcal{X}_\infty$ an equilibrium configuration for a distributed parameter system. Then, $\chi^*$ is said to be stable in the sense of Lyapunov with respect to the norm $\|\cdot\|$ if, for every $\epsilon > 0$ there exists $\delta_0 > 0$ such that

$$\|\chi(0) - \chi^*\| < \delta_0 \implies \|\chi(t) - \chi^*\| < \epsilon$$

for all $t > 0$, where $\chi(0) \in \mathcal{X}_\infty$ is the initial configuration of the system.

**Remark VI.1:** In finite dimensions, the positive definiteness of the second differential of the closed-loop Hamiltonian function calculated at the equilibrium configuration is sufficient to show that the steady state solution corresponds to a strict extremum of the Hamiltonian, thus implying (asymptotic) stability of the configuration itself. On the other hand, as pointed out in [14], in infinite dimensions, the same condition on the second variation of the Hamiltonian evaluated at the equilibrium is not, in general, sufficient to guarantee asymptotic stability. This is due to the fact that, when dealing with distributed parameter systems, it is necessary to specify the norm associated with the stability argument, because stability with respect to a one norm does not necessarily imply stability with respect to another norm. This is a consequence of the fact that, unlike finite dimensional vector spaces, all norms are not equivalent in infinite dimensions. In particular, in infinite dimensions, not every convergent sequence on the unit ball converges to a point on the unit ball, that is infinite dimensional vector spaces are not compact.

It is possible to prove that the configuration $\chi^*$ is stable in the sense of the previous definition. This property can be verified by following the procedure illustrated in [14] and already applied in [12]. [4]. [3] which relies on the Arnold’s first stability theorem. First of all, it is necessary to verify that $\nabla H_{cl}(\chi^*) = 0$, relation that holds if in (48) we chose $k = - \frac{x^*_1}{C}$. In fact

$$\nabla H_{cl}(\chi^*) = \begin{bmatrix} \frac{x^*_1}{C} + k & 0 \\ 0 & \frac{\alpha^*_E}{C\infty} + k \\ \frac{\alpha^*_M}{L\infty} + k \end{bmatrix} = 0$$

Then, it is necessary to compute the nonlinear functional $\mathcal{N}(\Delta \chi) := H_{cl}(\chi^* + \Delta \chi) - H_{cl}(\chi^*)$, which is proportional to the second variation of $H_{cl}$ evaluated in $\chi^*$. It can be obtained that

$$\mathcal{N}(\Delta \chi) = \frac{1}{2} \frac{\Delta x^2_{b, 1}}{C} + \frac{1}{2} \frac{\Delta x^2_{b, 2}}{L} + \int_0^L \left( \frac{\Delta \alpha^2_E}{C\infty} + \frac{\Delta \alpha^2_M}{L\infty} \right) dz + \frac{1}{2} C_a \left( \Delta x_{b, 1} + \int_0^L \Delta \alpha_E + G(z - L) \Delta \alpha_M \right) dz$$

The stability proof is completed if it is possible to find $\alpha, \gamma_1, \gamma_2 > 0$ such that

$$\gamma_1 \|\Delta \chi\|^2 \leq \mathcal{N}(\Delta \chi) \leq \gamma_2 \|\Delta \chi\|^\alpha$$

(49)

where $\|\cdot\|$ is a suitable norm on $\mathcal{X}_\infty$. If the following norm is assumed:

$$\|\chi\| := \left( \Delta x^2_1 + \Delta x^2_2 + \int_0^L \Delta \alpha^2_E dz + \int_0^L \Delta \alpha^2_M dz \right)^{\frac{1}{2}}$$

the constant $\gamma_1$ can be easily estimated in

$$\gamma_1 = \frac{1}{2} \min \left\{ \frac{1}{C}, \frac{1}{L}, \frac{1}{C\infty}, \frac{1}{L\infty} \right\}$$
Moreover, note that
\[
\Delta x_{1,1} \leq \frac{1}{2} \int_0^L \left( \Delta a_E + G(z - L) \Delta a_M \right) dz \leq \frac{1}{2} \Delta E_{b,1} + \frac{1}{2} \left( \int_0^L \left( \Delta a_E + G(z - L) \Delta a_M \right) dz \right)^2
\]

Consequently, it is possible to chose
\[
\gamma_2 = \frac{1}{2} \max \left\{ \frac{1}{C_b} + \frac{2}{1} \frac{1}{C_a}, \frac{1}{L_b}, \frac{1}{C_{\infty}}, \frac{2}{1} \frac{L}{L_b}, \frac{1}{L_{\infty}}, \frac{2}{1} \frac{G^2 L^3}{C_a} \right\}
\]

and \( \alpha = 2 \) in order to complete stability proof of the desired configuration \( \chi^* \). In other words, the following proposition has been proved.

**Proposition VI.1:** Consider the m-pH system of Fig. [1] where the transmission line is not lossless since \( R = 0 \) and \( G \neq 0 \). If the controller interconnection, damping and input matrices are equal to 0, \( GL \) and \(-1\) respectively, then, if the controller energy function is given by \( (I) \), the configuration \( (I) \) is stable in the sense of Def. [VI.1].

VII. Conclusions

In this paper, the well known control by interconnection and energy shaping methodology is generalized in order to deal with dynamical systems resulting from the power conserving interconnection of finite and infinite dimensional port Hamiltonian systems, that we call mixed port Hamiltonian systems (m-pH systems). More in details, it has been shown that, for a particular class of m-pH systems, it is possible to generalize the definition of Casimir functions to infinite dimensions and necessary and sufficient conditions for their existence are presented. The presence of a set of structural invariants in the closed-loop systems is of great importance when the control problem is approached, since, in this way, it is possible to robustly relate the configuration variables of the plant to the state variables of the controller, independently from the energy function of the controller itself. Then, the energy function of the controller, which results to depend on the configuration variables of the plant, can be properly chosen in order to shape the total energy so that a minimum can be introduced in the desired equilibrium configuration. Finally, a simple applicative example is presented, together with the stability proof.

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References