Solutions of discretized affine Toda field equations for $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, $A_n^{(2)}$ and $D_{n+1}^{(2)}$

Zengo Tsuboi †

Institute of Physics, University of Tokyo
Komaba 3-8-1, Meguro-ku, Tokyo 153 Japan

Abstract. Proposed are 'affine extentions' of the transfer matrix functional equations (T-system). They may be viewed as $X_n^{(a)}$ type affine Toda field equations on discrete space time. We present, for $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, $A_n^{(2)}$ and $D_{n+1}^{(2)}$, their solutions in terms of determinants or Pfaffians.

Short title: Discretized affine Toda field equations

February 29, 2008

† E-mail: ztsuboi@po1.bell24.or.jp

1. Introduction

In [KNS1], a class of functional equations, the T-system, was proposed. They are functional equations satisfied by a set of commuting transfer matrices $T^{(a)}_m(u)$ of solvable lattice models associated with any quantum affine algebras $U_q(X^{(1)}_r)$. Here, the transfer matrices $T^{(a)}_m(u)$ are defined as traces of monodromy matrices on auxiliary spaces labeled by a and m ($m \in \mathbf{Z}, u \in \mathbf{C}, a \in \{1, 2, \ldots, r\}$). As was pointed out in [KOS], T-system are not only transfer matrix functional equations but also examples of Toda field equations [LS, MOP] on discrete space time and they have determinants or pfaffians solutions [KNH, KOS, TK].

Using Cartan matrices of simple Lie algebras X_r, T-system are written as compact forms [KNS2]. In this paper, we formally replace these Cartan matrices of simple Lie algebras with those of affine Lie algebras and present a number of functional equations (2.1-2.7h), Appendix**A**. These functional equations are different from the original T-system in that they have 0th component $T^{(0)}_m(u)$. Although their physical meanings are not always clear, they have beautiful structure as examples of discretized soliton equations (cf. [AL, DJM, H, S, Wa]). As is mentioned in section2, our functional equations may be viewed as examples of discretized affine Toda field equations. Solving them recursively, we can express $T^{(a)}_m(u)$ as a polynomial of $T^{(0)}_1(u + \text{shift}), \ldots T^{(r)}_1(u + \text{shift})$. Combining the expressions similar to those in [KNH], [KOS] and [TK], we present, for $A^{(1)}_n$, $B^{(1)}_n$, $C^{(1)}_n$, $D^{(1)}_n$, $A^{(2)}_n$ and $D^{(2)}_{n+1}$, their solutions in terms of determinants or pfaffians.

The outline of this paper is given as follows. In section2, we present our functional equations for $A^{(1)}_n$, $B^{(1)}_n$, $C^{(1)}_n$, $D^{(1)}_n$, $A^{(2)}_n$ and $D^{(2)}_{n+1}$, whose solutions are given in section4. In section3, we fix the notation. Section5 is dedicated for summary and discussion. Appendix**A** is a list of functional equations for $G^{(1)}_2$, $F^{(1)}_4$, $E^{(1)}_6$, $E^{(1)}_7$, $E^{(1)}_8$, $E^{(2)}_6$ and $D^{(3)}_4$, which have been not yet solved. Finally, it should be noted that, for the twisted quantum affine algebras $U_q(X^{(k)}_n)$ ($X^{(k)}_n = A^{(2)}_n, D^{(2)}_n, E^{(2)}_6, D^{(3)}_4$), extension of T-system different from our functional equations was proposed in [KS2].

2. Functional equations

For $T^{(a)}_m(u)$ ($m \in \mathbf{Z}, u \in \mathbf{C}, a \in \{0, 1, \ldots, r\}$), we assume the following boundary condition:

$$T^{(a)}_m(u) = 0 \quad \text{for} \quad m < 0, \quad T^{(a)}_0(u) = 1.$$

Our functional equations read explicitly as follows:
For $A^{(1)}_r (r \geq 1)$,

$$T^{(a)}_m(u-1)T^{(a)}_m(u+1) = T^{(a)}_{m+1}(u)T^{(a)}_{m-1}(u) + T^{(a-1)}_m(u)T^{(a+1)}_m(u) \quad 0 \leq a \leq r \tag{2.1}$$

with $T^{(-1)}_m(u)=T^{(r)}_m(u)$ and $T^{(r+1)}_m(u)=T^{(0)}_m(u)$.

For $B^{(1)}_r (r\geq 3)$,

$$T^{(a)}_m(u-1)T^{(a)}_m(u+1)=T^{(a)}_{m+1}(u)T^{(a)}_{m-1}(u)+T^{(2)}_m(u) \qquad a=0,1 \tag{2.2a}$$
$$T^{(2)}_m(u-1)T^{(2)}_m(u+1)=T^{(2)}_{m+1}(u)T^{(2)}_{m-1}(u)+T^{(0)}_m(u)T^{(1)}_m(u)T^{(3)}_m(u) \tag{2.2b}$$
$$T^{(a)}_m(u-1)T^{(a)}_m(u+1)=T^{(a)}_{m+1}(u)T^{(a)}_{m-1}(u)+T^{(a-1)}_m(u)T^{(a+1)}_m(u) \quad 3\leq a\leq r-2 \tag{2.2c}$$
$$T^{(r-1)}_m(u-1)T^{(r-1)}_m(u+1)=T^{(r-1)}_{m+1}(u)T^{(r-1)}_{m-1}(u)+T^{(r-2)}_m(u)T^{(r)}_{2m}(u) \tag{2.2d}$$
$$T^{(r)}_{2m}(u-1/2)T^{(r)}_{2m}(u+1/2)=T^{(r)}_{2m+1}(u)T^{(r)}_{2m-1}(u)+T^{(r-1)}_m(u-1/2)T^{(r-1)}_m(u+1/2) \tag{2.2e}$$
$$T^{(r)}_{2m+1}(u-1/2)T^{(r)}_{2m+1}(u+1/2)=T^{(r)}_{2m+2}(u)T^{(r)}_{2m}(u)+T^{(r-1)}_m(u)T^{(r-1)}_{m+1}(u). \tag{2.2f}$$

If $r=3$, (2.2b), (2.2c) and (2.2d) reduce to the following equation :

$$T^{(2)}_m(u-1)T^{(2)}_m(u+1)=T^{(2)}_{m+1}(u)T^{(2)}_{m-1}(u)+T^{(0)}_m(u)T^{(1)}_m(u)T^{(3)}_{2m}(u). \tag{2.2g}$$

For $C^{(1)}_r (r\geq 2)$,

$$T^{(0)}_m(u-1)T^{(0)}_m(u+1)=T^{(0)}_{m+1}(u)T^{(0)}_{m-1}(u)+T^{(1)}_{2m}(u) \tag{2.3a}$$
$$T^{(1)}_{2m}(u-1/2)T^{(1)}_{2m}(u+1/2)$$
$$=T^{(1)}_{2m+1}(u)T^{(1)}_{2m-1}(u)+T^{(0)}_m(u-1/2)T^{(0)}_m(u+1/2)T^{(2)}_{2m}(u) \tag{2.3b}$$
$$T^{(1)}_{2m+1}(u-1/2)T^{(1)}_{2m+1}(u+1/2)=T^{(1)}_{2m+2}(u)T^{(1)}_{2m}(u)+T^{(0)}_m(u)T^{(0)}_{m+1}(u)T^{(2)}_{2m+1}(u) \tag{2.3c}$$
$$T^{(a)}_m(u-1/2)T^{(a)}_m(u+1/2)=T^{(a)}_{m+1}(u)T^{(a)}_{m-1}(u)+T^{(a-1)}_m(u)T^{(a+1)}_m(u) \quad 2\leq a\leq r-2 \tag{2.3d}$$
$$T^{(r-1)}_{2m}(u-1/2)T^{(r-1)}_{2m}(u+1/2)$$
$$=T^{(r-1)}_{2m+1}(u)T^{(r-1)}_{2m-1}(u)+T^{(r-2)}_{2m}(u)T^{(r)}_m(u-1/2)T^{(r)}_m(u+1/2) \tag{2.3e}$$
$$T^{(r-1)}_{2m+1}(u-1/2)T^{(r-1)}_{2m+1}(u+1/2)=T^{(r-1)}_{2m+2}(u)T^{(r-1)}_{2m}(u)+T^{(r-2)}_{2m+1}(u)T^{(r)}_m(u)T^{(r)}_{m+1}(u) \tag{2.3f}$$
$$T^{(r)}_m(u-1)T^{(r)}_m(u+1)=T^{(r)}_{m+1}(u)T^{(r)}_{m-1}(u)+T^{(r-1)}_{2m}(u). \tag{2.3g}$$

If $r=2$, (2.3b), (2.3c), (2.3d), (2.3e) and (2.3f) reduce to the following equations :

$$T^{(1)}_{2m}(u-1/2)T^{(1)}_{2m}(u+1/2)$$
$$=T^{(1)}_{2m+1}(u)T^{(1)}_{2m-1}(u)+T^{(0)}_m(u-1/2)T^{(0)}_m(u+1/2)T^{(2)}_m(u-1/2)T^{(2)}_m(u+1/2) \tag{2.3h}$$
$$T^{(1)}_{2m+1}(u-1/2)T^{(1)}_{2m+1}(u+1/2)$$
$$=T^{(1)}_{2m+2}(u)T^{(1)}_{2m}(u)+T^{(0)}_m(u)T^{(0)}_{m+1}(u)T^{(2)}_m(u)T^{(2)}_{m+1}(u). \tag{2.3i}$$

For $D^{(1)}_r (r\geq 4)$,

$$T^{(a)}_m(u-1)T^{(a)}_m(u+1)=T^{(a)}_{m+1}(u)T^{(a)}_{m-1}(u)+T^{(2)}_m(u) \qquad a=0,1 \tag{2.4a}$$
$$T^{(2)}_m(u-1)T^{(2)}_m(u+1)=T^{(2)}_{m+1}(u)T^{(2)}_{m-1}(u)+T^{(0)}_m(u)T^{(1)}_m(u)T^{(3)}_m(u) \tag{2.4b}$$
$$T^{(a)}_m(u-1)T^{(a)}_m(u+1)=T^{(a)}_{m+1}(u)T^{(a)}_{m-1}(u)+T^{(a-1)}_m(u)T^{(a+1)}_m(u) \quad 3\leq a\leq r-3 \tag{2.4c}$$

$$T^{(r-2)}_m(u-1)T^{(r-2)}_m(u+1) = T^{(r-2)}_{m+1}(u)T^{(r-2)}_{m-1}(u) + T^{(r-3)}_m(u)T^{(r-1)}_m(u)T^{(r)}_m(u) \qquad (2.4d)$$

$$T^{(a)}_m(u-1)T^{(a)}_m(u+1) = T^{(a)}_{m+1}(u)T^{(a)}_{m-1}(u) + T^{(r-2)}_m(u) \qquad a = r-1, r. \qquad (2.4e)$$

If $r = 4$, (2.4b), (2.4c) and (2.4d) reduce to the following equation :

$$T^{(2)}_m(u-1)T^{(2)}_m(u+1) = T^{(2)}_{m+1}(u)T^{(2)}_{m-1}(u) + T^{(0)}_m(u)T^{(1)}_m(u)T^{(r-1)}_m(u)T^{(r)}_m(u). \qquad (2.4f)$$

For $A^{(2)}_{2r}(r \geq 2)$,

$$T^{(0)}_m(u-1)T^{(0)}_m(u+1) = T^{(0)}_{m+1}(u)T^{(0)}_{m-1}(u) + T^{(1)}_{2m}(u) \qquad (2.5a)$$

$$T^{(1)}_{2m}(u-1/2)T^{(1)}_{2m}(u+1/2)$$
$$= T^{(1)}_{2m+1}(u)T^{(1)}_{2m-1}(u) + T^{(0)}_m(u-1/2)T^{(0)}_m(u+1/2)T^{(2)}_{2m}(u) \qquad (2.5b)$$

$$T^{(1)}_{2m+1}(u-1/2)T^{(1)}_{2m+1}(u+1/2) = T^{(1)}_{2m+2}(u)T^{(1)}_{2m}(u) + T^{(0)}_m(u)T^{(0)}_{m+1}(u)T^{(2)}_{2m+1}(u) \qquad (2.5c)$$

$$T^{(a)}_m(u-1/2)T^{(a)}_m(u+1/2) = T^{(a)}_{m+1}(u)T^{(a)}_{m-1}(u) + T^{(a-1)}_m(u)T^{(a+1)}_m(u) \; 2 \leq a \leq r-2 \quad (2.5d)$$

$$T^{(r-1)}_m(u-1/2)T^{(r-1)}_m(u+1/2) = T^{(r-1)}_{m+1}(u)T^{(r-1)}_{m-1}(u) + T^{(r-2)}_m(u)T^{(r)}_{2m}(u) \qquad (2.5e)$$

$$T^{(r)}_{2m}(u-1/4)T^{(r)}_{2m}(u+1/4) = T^{(r)}_{2m+1}(u)T^{(r)}_{2m-1}(u) + T^{(r-1)}_m(u-1/4)T^{(r-1)}_m(u+1/4) \quad (2.5f)$$

$$T^{(r)}_{2m+1}(u-1/4)T^{(r)}_{2m+1}(u+1/4) = T^{(r)}_{2m+2}(u)T^{(r)}_{2m}(u) + T^{(r-1)}_m(u)T^{(r-1)}_{m+1}(u). \qquad (2.5g)$$

If $r = 2$, (2.5b), (2.5c), (2.5d) and (2.5e) reduce to the following equations :

$$T^{(1)}_{2m}(u-1/2)T^{(1)}_{2m}(u+1/2)$$
$$= T^{(1)}_{2m+1}(u)T^{(1)}_{2m-1}(u) + T^{(0)}_m(u-1/2)T^{(0)}_m(u+1/2)T^{(2)}_{4m}(u) \qquad (2.5h)$$

$$T^{(1)}_{2m+1}(u-1/2)T^{(1)}_{2m+1}(u+1/2) = T^{(1)}_{2m+2}(u)T^{(1)}_{2m}(u) + T^{(0)}_m(u)T^{(0)}_{m+1}(u)T^{(2)}_{4m+2}(u). \qquad (2.5i)$$

For $A^{(2)}_{2r-1}(r \geq 3)$,

$$T^{(a)}_m(u-1/2)T^{(a)}_m(u+1/2) = T^{(a)}_{m+1}(u)T^{(a)}_{m-1}(u) + T^{(2)}_m(u) \qquad a = 0, 1 \qquad (2.6a)$$

$$T^{(2)}_m(u-1/2)T^{(2)}_m(u+1/2) = T^{(2)}_{m+1}(u)T^{(2)}_{m-1}(u) + T^{(0)}_m(u)T^{(1)}_m(u)T^{(3)}_m(u) \qquad (2.6b)$$

$$T^{(a)}_m(u-1/2)T^{(a)}_m(u+1/2) = T^{(a)}_{m+1}(u)T^{(a)}_{m-1}(u) + T^{(a-1)}_m(u)T^{(a+1)}_m(u) \; 3 \leq a \leq r-2 \quad (2.6c)$$

$$T^{(r-1)}_{2m}(u-1/2)T^{(r-1)}_{2m}(u+1/2)$$
$$= T^{(r-1)}_{2m+1}(u)T^{(r-1)}_{2m-1}(u) + T^{(r-2)}_{2m}(u)T^{(r)}_m(u-1/2)T^{(r)}_m(u+1/2) \qquad (2.6d)$$

$$T^{(r-1)}_{2m+1}(u-1/2)T^{(r-1)}_{2m+1}(u+1/2) = T^{(r-1)}_{2m+2}(u)T^{(r-1)}_{2m}(u) + T^{(r-2)}_{2m+1}(u)T^{(r)}_m(u)T^{(r)}_{m+1}(u) \quad (2.6e)$$

$$T^{(r)}_m(u-1)T^{(r)}_m(u+1) = T^{(r)}_{m+1}(u)T^{(r)}_{m-1}(u) + T^{(r-1)}_{2m}(u). \qquad (2.6f)$$

If $r = 3$, (2.6b), (2.6c), (2.6d) and (2.6e) reduce to the following equations :

$$T^{(2)}_{2m}(u-1/2)T^{(2)}_{2m}(u+1/2)$$
$$= T^{(2)}_{2m+1}(u)T^{(2)}_{2m-1}(u) + T^{(0)}_{2m}(u)T^{(1)}_{2m}(u)T^{(3)}_m(u-1/2)T^{(3)}_m(u+1/2) \qquad (2.6g)$$

$$T^{(2)}_{2m+1}(u-1/2)T^{(2)}_{2m+1}(u+1/2)$$
$$= T^{(2)}_{2m+2}(u)T^{(2)}_{2m}(u) + T^{(0)}_{2m+1}(u)T^{(1)}_{2m+1}(u)T^{(3)}_m(u)T^{(3)}_{m+1}(u). \qquad (2.6h)$$

For $D^{(2)}_{r+1} (r \geq 2)$,

$$T^{(0)}_{2m}(u-1/2)T^{(0)}_{2m}(u+1/2) = T^{(0)}_{2m+1}(u)T^{(0)}_{2m-1}(u) + T^{(1)}_{m}(u-1/2)T^{(1)}_{m}(u+1/2) \qquad (2.7a)$$

$$T^{(0)}_{2m+1}(u-1/2)T^{(0)}_{2m+1}(u+1/2) = T^{(0)}_{2m+2}(u)T^{(0)}_{2m}(u) + T^{(1)}_{m}(u)T^{(1)}_{m+1}(u) \qquad (2.7b)$$

$$T^{(1)}_{m}(u-1)T^{(1)}_{m}(u+1) = T^{(1)}_{m+1}(u)T^{(1)}_{m-1}(u) + T^{(0)}_{2m}(u)T^{(2)}_{m}(u) \qquad (2.7c)$$

$$T^{(a)}_{m}(u-1)T^{(a)}_{m}(u+1) = T^{(a)}_{m+1}(u)T^{(a)}_{m-1}(u) + T^{(a-1)}_{m}(u)T^{(a+1)}_{m}(u) \quad 2 \leq a \leq r-2 \qquad (2.7d)$$

$$T^{(r-1)}_{m}(u-1)T^{(r-1)}_{m}(u+1) = T^{(r-1)}_{m+1}(u)T^{(r-1)}_{m-1}(u) + T^{(r-2)}_{m}(u)T^{(r)}_{2m}(u) \qquad (2.7e)$$

$$T^{(r)}_{2m}(u-1/2)T^{(r)}_{2m}(u+1/2)$$
$$= T^{(r)}_{2m+1}(u)T^{(r)}_{2m-1}(u) + T^{(r-1)}_{m}(u-1/2)T^{(r-1)}_{m}(u+1/2) \qquad (2.7f)$$

$$T^{(r)}_{2m+1}(u-1/2)T^{(r)}_{2m+1}(u+1/2) = T^{(r)}_{2m+2}(u)T^{(r)}_{2m}(u) + T^{(r-1)}_{m}(u)T^{(r-1)}_{m+1}(u). \qquad (2.7g)$$

If $r = 2$, $(2.7c)$, $(2.7d)$ and $(2.7e)$ reduce to the following equation :

$$T^{(1)}_{m}(u-1)T^{(1)}_{m}(u+1) = T^{(1)}_{m+1}(u)T^{(1)}_{m-1}(u) + T^{(0)}_{2m}(u)T^{(2)}_{2m}(u). \qquad (2.7h)$$

As is mentioned in [KNH], equation (2.1) with boundary condition

$$T^{(-1)}_{m}(u) = 0, \quad T^{(0)}_{m}(u) = T^{(r+1)}_{m}(u) = 1$$

corresponds to Hirota-Miwa equation.

In [KOS], it was pointed out that T-system may be looked upon as discritized Toda field equations. By the same argument, our functional equations may be viewed as discritized affine Toda field equations. Regarding u and m as continuous space time coordinates, our functional equation reduces to two-dimensional affine Toda field equation of the form

$$(\partial_u^2 - \partial_m^2)\log\phi_a(u,m) = constant \prod_{b=0}^{r} \phi_b(u,m)^{-A_{ab}} \qquad (2.8)$$

where $\phi_a(u,m)$ denotes a scaled $T^{(a)}_{m}(u)$ and $A_{ab} = 2(\alpha_a|\alpha_b)/(\alpha_a|\alpha_a)$ the Cartan matrix of affine Lie algebra .

3. Definition

Set

$$x^{[1]}_{j}(k|u) = T_1^{(\delta_{j-1})}(u+(2j-2)/k), \quad y^{[1]}_{j}(k|u) = T_1^{(\delta_j)}(u+(2j-2)/k),$$
$$t^{[1]}_{ij}(k|u) = \mathcal{T}^{1-i+j}(u+(i+j-2)/k),$$
$$x^{[2]}_{j}(k|u) = T_1^{(r-\delta_{j-1})}(u+(2j-2)/k), \quad y^{[2]}_{j}(k|u) = T_1^{(r-\delta_j)}(u+(2j-2)/k),$$
$$t^{[2]}_{ij}(k|u) = \mathcal{T}^{r+i-j-1}(u+(i+j-2)/k), \qquad (3.1)$$
$$a^{[p]}_{ij}(k|u) = x^{[p]}_{i}(k|u)y^{[p]}_{j}(k|u) - t^{[p]}_{ij}(k|u),\ b^{[p]}_{ij}(k|u) = y^{[p]}_{i}(k|u)x^{[p]}_{j}(k|u) - t^{[p]}_{ij}(k|u),\ p = 1, 2.$$

where

$$\delta_i = \begin{cases} 0 & \text{if} \quad i \in 2\mathbf{Z} \\ 1 & \text{if} \quad i \in 2\mathbf{Z}+1. \end{cases}$$

By the definition we have

$$x_i^{[p]}(k|u+2/k) = y_{i+1}^{[p]}(k|u), \quad y_i^{[p]}(k|u+2/k) = x_{i+1}^{[p]}(k|u), \quad t_{ij}^{[p]}(k|u+2/k) = t_{i+1\,j+1}^{[p]}(k|u),$$
$$a_{ij}^{[p]}(k|u+2/k) = b_{i+1\,j+1}^{[p]}(k|u), \qquad b_{ij}^{[p]}(k|u+2/k) = a_{i+1\,j+1}^{[p]}(k|u), \tag{3.2}$$

for $p = 1, 2$. These relations are necessary to prove Lemma4.9. Now we introduce $m \times m$ matrices $\mathcal{T}_m^a(k|u) = (\mathcal{T}_{ij}^a(k|u))_{1 \le i,j \le m}$ $(a \in \mathbf{Z})$, $(2m) \times (2m)$ matrices $\mathcal{F}_{2m}^{[p]}(k|u) = (\mathcal{F}_{ij}^{[p]}(k|u))_{1 \le i,j \le 2m}$, $(m+1) \times (m+1)$ matrices $\mathcal{H}_{m+1}^{[p]}(k|u) = (\mathcal{H}_{ij}^{[p]}(k|u))_{1 \le i,j \le m+1}$, $\mathcal{S}_{m+1}^{[p]}(k|u) = (\mathcal{S}_{ij}^{[p]}(k|u))_{1 \le i,j \le m+1}$, $\mathcal{C}_{m+1}^{[p]}(k|u) = (\mathcal{C}_{ij}^{[p]}(k|u))_{1 \le i,j \le m+1}$ and $\mathcal{R}_{m+1}^{[p]}(k|u) = (\mathcal{R}_{ij}^{[p]}(k|u))_{1 \le i,j \le m+1}$ $(p = 1, 2)$ whose (i, j) elements are given by

$$\mathcal{T}_{ij}^a(k|u) = \mathcal{T}^{a+i-j}(u + (i+j-m-1)/k) \tag{3.3a}$$

$$\mathcal{F}_{ij}^{[1]}(k|u) = \begin{cases} 0 & \text{for} \quad i = j \\ \mathcal{T}^{-i+j-1}(u + (i+j-2m-1)/k) & \text{for} \quad 1 \le i < j \le 2m \\ -\mathcal{T}^{i-j-1}(u + (i+j-2m-1)/k) & \text{for} \quad 1 \le j < i \le 2m, \end{cases} \tag{3.3b}$$

$$\mathcal{F}_{ij}^{[2]}(k|u) = \begin{cases} 0 & \text{for} \quad i = j \\ \mathcal{T}^{i-j+r+1}(u + (i+j-2m-1)/k) & \text{for} \quad 1 \le i < j \le 2m \\ -\mathcal{T}^{-i+j+r+1}(u + (i+j-2m-1)/k) & \text{for} \quad 1 \le j < i \le 2m, \end{cases} \tag{3.3c}$$

$$\mathcal{H}_{ij}^{[1]}(k|u) = \begin{cases} T_1^{(0)}(u + (2i-2)/k) & \text{for} \quad 1 \le i \le m+1 \quad \textit{and} \quad j = 1 \\ \mathcal{T}^{-i+j}(u + (i+j-5/2)/k) & \text{for} \quad 1 \le i \le m+1, \quad 2 \le j \le m+1. \end{cases} \tag{3.3d}$$

$$\mathcal{H}_{ij}^{[2]}(k|u) = \begin{cases} T_1^{(r)}(u + (2i-2)/k) & \text{for} \quad 1 \le i \le m+1 \quad \textit{and} \quad j = 1 \\ \mathcal{T}^{r+i-j}(u + (i+j-5/2)/k) & \text{for} \quad 1 \le i \le m+1, \quad 2 \le j \le m+1. \end{cases} \tag{3.3e}$$

$$\mathcal{S}_{ij}^{[1]}(k|u) = \begin{cases} 0 & \text{for} \quad i = j = 1 \\ T_1^{(\delta_j)}(u + (2j-4)/k) & \text{for} \quad i = 1 \quad \textit{and} \quad 2 \le j \le m+1 \\ -T_1^{(\delta_i)}(u + (2i-4)/k) & \text{for} \quad 2 \le i \le m+1 \quad \textit{and} \quad j = 1 \\ -\mathcal{T}^{1-i+j}(u + (i+j-4)/k) & \text{for} \quad 2 \le i, j \le m+1. \end{cases} \tag{3.3f}$$

$$\mathcal{S}_{ij}^{[2]}(k|u) = \begin{cases} 0 & \text{for} \quad i = j = 1 \\ T_1^{(r-\delta_j)}(u + (2j-4)/k) & \text{for} \quad i = 1 \quad \textit{and} \quad 2 \le j \le m+1 \\ -T_1^{(r-\delta_i)}(u + (2i-4)/k) & \text{for} \quad 2 \le i \le m+1 \quad \textit{and} \quad j = 1 \\ -\mathcal{T}^{r+i-j-1}(u + (i+j-4)/k) & \text{for} \quad 2 \le i, j \le m+1. \end{cases} \tag{3.3g}$$

$$\mathcal{C}^{[p]}_{ij}(k|u) = \begin{cases} 0 & \text{for} \quad i=j \\ x^{[p]}_{j-1}(k|u) & \text{for} \quad i=1 \quad \textit{and} \quad 2 \le j \le m+1 \\ -x^{[p]}_{i-1}(k|u) & \text{for} \quad 2 \le i \le m+1 \quad \textit{and} \quad j=1 \\ a^{[p]}_{i-1\, j-1}(k|u) & \text{for} \quad 2 \le i < j \le m+1 \\ -a^{[p]}_{j-1\, i-1}(k|u) & \text{for} \quad 2 \le j < i \le m+1, \end{cases} \tag{3.3h}$$

$$\mathcal{R}^{[p]}_{ij}(k|u) = \begin{cases} 0 & \text{for} \quad i=j \\ x^{[p]}_{j-1}(k|u) & \text{for} \quad i=1 \quad \textit{and} \quad 2 \le j \le m+1 \\ -x^{[p]}_{i-1}(k|u) & \text{for} \quad 2 \le i \le m+1 \quad \textit{and} \quad j=1 \\ b^{[p]}_{i-1\, j-1}(k|u) & \text{for} \quad 2 \le i < j \le m+1 \\ -b^{[p]}_{j-1\, i-1}(k|u) & \text{for} \quad 2 \le j < i \le m+1, \end{cases} \tag{3.3i}$$

For any matrix $\mathcal{M}(u)$, we shall let $\mathcal{M}\begin{bmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_k \end{bmatrix}(u)$ denote the minor matrix getting rid of i_l's rows and j_l's columns from $\mathcal{M}(u)$. We introduce the following pfaffians and determinants expressions, which will be used as the components of the solutions.

$$T^a_m(k|u) = \det[\mathcal{T}^a_m(k|u)] \quad \text{for } m \in \mathbf{Z}_{\ge 0}, \tag{3.4a}$$

$$F^{(0)}_m(k|u) = \mathrm{pf}[\mathcal{F}^{[1]}_{2m}(k|u)] \quad \text{for } m \in \mathbf{Z}_{\ge 0}, \tag{3.4b}$$

$$F^{(r)}_m(k|u) = \mathrm{pf}[\mathcal{F}^{[2]}_{2m}(k|u)] \quad \text{for } m \in \mathbf{Z}_{\ge 0}, \tag{3.4c}$$

$$G^{(0)}_m(k|u) = \begin{cases} \mathrm{pf}[\mathcal{C}^{[1]}_{m+1}\begin{bmatrix} 1 \\ 1 \end{bmatrix}(k|u+(-m+1)/k)] & \text{for} \quad m \in 2\mathbf{Z}_{\ge 0} \\ \mathrm{pf}[\mathcal{C}^{[1]}_{m+1}(k|u+(-m+1)/k)] & \text{for} \quad m \in 2\mathbf{Z}_{\ge 0}+1, \end{cases} \tag{3.4d}$$

$$G^{(1)}_m(k|u) = \begin{cases} \mathrm{pf}[\mathcal{C}^{[1]}_{m+2}\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}(k|u+(-m-1)/k)] & \text{for} \quad m \in 2\mathbf{Z}_{\ge 0} \\ \mathrm{pf}[\mathcal{C}^{[1]}_{m+2}\begin{bmatrix} 2 \\ 2 \end{bmatrix}(k|u+(-m-1)/k)] & \text{for} \quad m \in 2\mathbf{Z}_{\ge 0}+1, \end{cases} \tag{3.4e}$$

$$G^{(r)}_m(k|u) = \begin{cases} \mathrm{pf}[\mathcal{C}^{[2]}_{m+1}\begin{bmatrix} 1 \\ 1 \end{bmatrix}(k|u+(-m+1)/k)] & \text{for} \quad m \in 2\mathbf{Z}_{\ge 0} \\ \mathrm{pf}[\mathcal{C}^{[2]}_{m+1}(k|u+(-m+1)/k)] & \text{for} \quad m \in 2\mathbf{Z}_{\ge 0}+1, \end{cases} \tag{3.4f}$$

$$G^{(r-1)}_m(k|u) = \begin{cases} \mathrm{pf}[\mathcal{C}^{[2]}_{m+2}\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}(k|u+(-m-1)/k)] & \text{for} \quad m \in 2\mathbf{Z}_{\ge 0} \\ \mathrm{pf}[\mathcal{C}^{[2]}_{m+2}\begin{bmatrix} 2 \\ 2 \end{bmatrix}(k|u+(-m-1)/k)] & \text{for} \quad m \in 2\mathbf{Z}_{\ge 0}+1, \end{cases} \tag{3.4g}$$

$$H^{(0)}_{m+1}(k|u) = \det[\mathcal{H}^{[1]}_{m+1}(k|u-m/k)] \quad \text{for} \quad m \in \mathbf{Z}_{\ge 0}, \tag{3.4h}$$

$$H^{(r)}_{m+1}(k|u) = \det[\mathcal{H}^{[2]}_{m+1}(k|u-m/k)] \quad \text{for} \quad m \in \mathbf{Z}_{\ge 0}. \tag{3.4i}$$

For $a \in \mathbf{Z}$, $k \in \mathbf{Z} - \{\mathbf{0}\}$ and $u \in \mathbf{C}$, we introduce the following functional relations, combinations of which will be used in solving the equations.

$$\mathcal{T}^a(u) + \mathcal{T}^{1-a}(u) = T_1^{(0)}(u + (a-1/2)/k) T_1^{(0)}(u + (-a+1/2)/k) \qquad (3.5a)$$

$$\mathcal{T}^a(u) + \mathcal{T}^{2r-1-a}(u) = T_1^{(r)}(u + (-r+a+1/2)/k) T_1^{(r)}(u + (r-a-1/2)/k) \qquad (3.5b)$$

$$\mathcal{T}^a(u) + \mathcal{T}^{-2-a}(u) = 0 \qquad (3.5c)$$

$$\mathcal{T}^a(u) + \mathcal{T}^{2r-a+2}(u) = 0 \qquad (3.5d)$$

$$\begin{aligned} \mathcal{T}^a(u) + \mathcal{T}^{2-a}(u) &= T_1^{(0)}(u + (a-1)/k) T_1^{(\delta_a)}(u + (-a+1)/k) \\ &+ T_1^{(1)}(u + (a-1)/k) T_1^{(\delta_{a-1})}(u + (-a+1)/k), \end{aligned} \qquad (3.5e)$$

$$\begin{aligned} \mathcal{T}^a(u) + \mathcal{T}^{2r-a-2}(u) &= T_1^{(r)}(u + (r-a-1)/k) T_1^{(r-\delta_{r-a})}(u + (-r+a+1)/k) \\ &+ T_1^{(r-1)}(u + (r-a-1)/k) T_1^{(r-\delta_{r-a-1})}(u + (-r+a+1)/k), \end{aligned} \qquad (3.5f)$$

Remark1 : The functional relation (3.5b) for $k = 1$ with the boundary condition

$$\mathcal{T}^a(u) = \begin{cases} 0 & a < 0 \\ 1 & a = 0 \\ T_1^{(a)}(u) & 1 \le a \le r-1 \end{cases} \qquad (3.6a)$$

was firstly introduced in [KOS]. The functional relation (3.5f) for $k = 1$ with the boundary condition

$$\mathcal{T}^a(u) = \begin{cases} 0 & a < 0 \\ 1 & a = 0 \\ T_1^{(a)}(u) & 1 \le a \le r-2 \end{cases} \qquad (3.6b)$$

was firstly introduced in [TK].

Remark2 : Under the functional relation (3.5e) (resp., (3.5f)) , the following relations for $p = 1$ (resp., $p = 2$) hold.

$$\mathcal{C}_{m+1}^{[p]}(k|u) = \mathcal{S}_{m+1}^{[p]}(k|u) \prod_{j=2}^{m+1} P(1, j; -y_{j-1}^{[p]}(k|u)) \qquad (3.7a)$$

$$\mathcal{R}_{m+1}^{[p]}(k|u) = \left(\prod_{i=2}^{m+1} P(i, 1; y_{i-1}^{[p]}(k|u)) \right) \mathcal{S}_{m+1}^{[p]}(k|u) \qquad (3.7b)$$

Here

$$P(i, j; c) = I + c I_{ij} \qquad (3.8)$$

is the $m+1$ by $m+1$ matrix with I the identity and I_{ij} the matrix unit. We further have

$$\mathcal{F}^{[1]}_{2m}(k|u) = {}^t\mathcal{T}^{-1}_{2m}(k|u) = -\mathcal{T}^{-1}_{2m}(k|u) \qquad (3.9a)$$

under the functional relations (3.5c),

$$\mathcal{F}^{[2]}_{2m}(k|u) = \mathcal{T}^{r+1}_{2m}(k|u) \qquad (3.9b)$$

under the functional relations (3.5d), where the index ^{t} denotes transposition of a matrix.
Remark3 : In deriving the solutions in section4, the parameter k in this section is determined by the value of $t_a = 2/(\alpha_a|\alpha_a)$, where $t_a = 1$ if α_a is the longest root.

4. Solutions

The solutions of our functional equations (2.1)-(2.7h) are given as follows.

Theorem 4.1. (*The* $A^{(1)}_r$ *case,* $r \geq 1$) For $m \in Z_{\geq 0}$,

$$T^{(a)}_m(u) = T^a_m(1|u) \qquad (4.1a)$$

solves the functional equations (2.1) under the condition

$$\mathcal{T}^{b+g}(u) = \mathcal{T}^b(u), \quad g = r+1, \quad b \in \mathbf{Z}. \qquad (4.1b)$$

Theorem 4.2. (*The* $B^{(1)}_r$ *case,* $r \geq 3$) For $m \in Z_{\geq 0}$,

$$\begin{aligned} &T^{(0)}_m(u) = G^{(0)}_m(1|u), \quad T^{(1)}_m(u) = G^{(1)}_m(1|u) \\ &T^{(a)}_m(u) = T^a_m(1|u), \quad 2 \leq a \leq r-1 \\ &T^{(r)}_{2m}(u) = T^r_m(1|u), \quad T^{(r)}_{2m+1}(u) = H^{(r)}_{m+1}(1|u) \end{aligned} \qquad (4.2)$$

solves the functional equations (2.2a-2.2g) under the relations (3.5b) and (3.5e) for $k = 1$.

Theorem 4.3. (*The* $C^{(1)}_r$ *case,* $r \geq 2$) For $m \in Z_{\geq 0}$,

$$\begin{aligned} &T^{(0)}_m(u) = F^{(0)}_m(2|u), \\ &T^{(a)}_m(u) = T^a_m(2|u), 1 \leq a \leq r-1 \\ &T^{(r)}_m(u) = F^{(r)}_m(2|u) \end{aligned} \qquad (4.3)$$

solves the functional equations (2.3a-2.3i) under the relations (3.5c) and (3.5d) for $k = 2$.

Theorem 4.4. (*The* $D^{(1)}_r$ *case,* $r \geq 4$) For $m \in Z_{\geq 0}$,

$$\begin{aligned} &T^{(0)}_m(u) = G^{(0)}_m(1|u), \quad T^{(1)}_m(u) = G^{(1)}_m(1|u) \\ &T^{(a)}_m(u) = T^a_m(1|u), 2 \leq a \leq r-2 \\ &T^{(r-1)}_m(u) = G^{(r-1)}_m(1|u), \quad T^{(r)}_m(u) = G^{(r)}_m(1|u) \end{aligned} \qquad (4.4)$$

solves the functional equations (2.4a-2.4f) under the relations (3.5e) and (3.5f) for $k = 1$.

Theorem 4.5. (*The $A^{(2)}_{2r}$ case, $r \geq 2$*) For $m \in Z_{\geq 0}$,

$$
\begin{aligned}
&T^{(0)}_m(u) = F^{(0)}_m(2|u)\\
&T^{(a)}_m(u) = T^a_m(2|u), 1 \leq a \leq r-1\\
&T^{(r)}_{2m}(u) = T^r_m(2|u), \quad T^{(r)}_{2m+1}(u) = H^{(r)}_{m+1}(2|u)
\end{aligned} \tag{4.5}
$$

solves the functional equations (2.5a-2.5i) under the relations (3.5b) and (3.5c) for $k = 2$.

Theorem 4.6. (*The $A^{(2)}_{2r-1}$ case, $r \geq 3$*) For $m \in Z_{\geq 0}$,

$$
\begin{aligned}
&T^{(0)}_m(u) = G^{(0)}_m(2|u), \quad T^{(1)}_m(u) = G^{(1)}_m(2|u)\\
&T^{(a)}_m(u) = T^a_m(2|u), 2 \leq a \leq r-1\\
&T^{(r)}_m(u) = F^{(r)}_m(2|u)
\end{aligned} \tag{4.6}
$$

solves the functional equations (2.6a-2.6h) under the relations (3.5d) and (3.5e) for $k = 2$.

Theorem 4.7. (*The $D^{(2)}_{r+1}$ case, $r \geq 2$*) For $m \in Z_{\geq 0}$,

$$
\begin{aligned}
&T^{(0)}_{2m}(u) = T^0_m(1|u), \quad T^{(0)}_{2m+1}(u) = H^{(0)}_{m+1}(1|u),\\
&T^{(a)}_m(u) = T^a_m(1|u), 1 \leq a \leq r-1\\
&T^{(r)}_{2m}(u) = T^r_m(1|u), \quad T^{(r)}_{2m+1}(u) = H^{(r)}_{m+1}(1|u)
\end{aligned} \tag{4.7}
$$

solves the functional equations (2.7a-2.7h) under the relations (3.5a) and (3.5b) for $k = 1$.

Remark1 : In (4.1a)-(4.7), the following type of boundary conditions are valid.

$$\text{For} \quad \alpha \leq a \leq \beta, \quad \mathcal{T}^a(u) = T^{(a)}_1(u)$$

where $(\alpha, \beta) = (0, r)$ for (4.1a); $(\alpha, \beta) = (2, r-1)$ for (4.2); $(\alpha, \beta) = (0, r)$ for (4.3); $(\alpha, \beta) = (2, r-2)$ for (4.4); $(\alpha, \beta) = (0, r-1)$ for (4.5); $(\alpha, \beta) = (2, r)$ for (4.6); $(\alpha, \beta) = (1, r-1)$ for (4.7).
Remark2 : In (4.1a), if $T^{(a)}_1(u)$ does not depend on $u \in \mathbf{C}$ for all a, then the following truncation holds:

$$T^{(b)}_g(u) = 0 \qquad \text{for} \quad \text{any} \quad b.$$

Now we enumerate lemmas that are necessary for the proofs of the theorems.

Lemma 4.8. For $m \in \mathbf{Z}_{\geq 0}$ and $u \in \mathbf{C}$, $F^{(0)}_m(k|u)$ (3.4b) (resp., $F^{(r)}_m(k|u)$ (3.4c)) satisfy the following relations for $\alpha = 0$ (resp., $\alpha = r$) under the relation (3.5c) (resp., (3.5d)).

$$F^{(\alpha)}_m(k|u - 1/k)F^{(\alpha)}_m(k|u + 1/k) = T^\alpha_{2m}(k|u) \tag{4.8a}$$

$$F^{(\alpha)}_m(k|u)F^{(\alpha)}_{m+1}(k|u) = T^\alpha_{2m+1}(k|u) \tag{4.8b}$$

Lemma 4.9. For $m \in \mathbf{Z}_{\geq 1}$, $G^{(1)}_{m}(k|u)$ (3.4e) and $G^{(0)}_{m}(k|u)$ (3.4d) (resp., $G^{(r-1)}_{m}(k|u)$ (3.4g) and $G^{(r)}_{m}(k|u)$ (3.4f)) satisfy the following relations for $(p,\alpha,\beta)=(1,1,0)$ (resp., $(p,\alpha,\beta)=(2,r-1,r)$) under the relation (3.5e) (resp., (3.5f)).

$$G^{(\alpha)}_{m}(k|u+1/k)G^{(\beta)}_{m-1}(k|u) = \begin{cases} \det[\mathcal{S}^{[p]}_{m+1}\begin{bmatrix} m+1 \\ 1 \end{bmatrix}(k|u+(-m+2)/k)] & \text{for} \quad m\in 2\mathbf{Z}_{\geq 1} \\ \det[\mathcal{S}^{[p]}_{m+2}\begin{bmatrix} 2 & m+2 \\ 1 & 2 \end{bmatrix}(k|u-m/k)] & \text{for} \quad m\in 2\mathbf{Z}_{\geq 0}+1, \end{cases} \tag{4.9a}$$

$$G^{(\alpha)}_{m}(k|u)G^{(\beta)}_{m}(k|u) = (-1)^{m}\det[\mathcal{S}^{[p]}_{m+1}\begin{bmatrix} 1 \\ 1 \end{bmatrix}(k|u+(-m+1)/k)], \tag{4.9b}$$

$$G^{(\alpha)}_{m}(k|u+1/k)G^{(\beta)}_{m+1}(k|u) = (-1)^{m+1}\det[\mathcal{S}^{[p]}_{m+2}\begin{bmatrix} 1 \\ 2 \end{bmatrix}(k|u-m/k)], \tag{4.9c}$$

$$G^{(\alpha)}_{m-1}(k|u)G^{(\alpha)}_{m}(k|u+1/k) = (-1)^{m}\det[\mathcal{S}^{[p]}_{m+2}\begin{bmatrix} 1 & 2 \\ 2 & m+2 \end{bmatrix}(k|u-m/k)], \tag{4.9d}$$

$$G^{(\beta)}_{m-1}(k|u-1/k)G^{(\beta)}_{m}(k|u) = (-1)^{m}\det[\mathcal{S}^{[p]}_{m+1}\begin{bmatrix} 1 \\ m+1 \end{bmatrix}(k|u+(-m+1)/k)], \tag{4.9e}$$

$$G^{(\alpha)}_{m}(k|u)G^{(\beta)}_{m-1}(k|u+1/k) = (-1)^{m}\det[\mathcal{S}^{[p]}_{m+2}\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}(k|u+(-m-1)/k)]. \tag{4.9f}$$

Remark3 : Lemma 4.8 for $\alpha=r, k=1$ with different boundary condition for $\mathcal{T}^{a}(u)$ was firstly proved in [KNH].
Remark4 : Lemma 4.9 for $(p,\alpha,\beta,k) = (2,r-1,r,1)$ with different boundary condition(3.6b) for $\mathcal{T}^{a}(u)$ was firstly proved in [TK].
The proofs of the theorems and lemmas in this section are quite similar to those in [KNH], [KOS] and [TK]. That is, most of them reduce to the Jacobi identity:

$$\det\mathcal{M}\begin{bmatrix} b \\ b \end{bmatrix}\det\mathcal{M}\begin{bmatrix} c \\ c \end{bmatrix} - \det\mathcal{M}\begin{bmatrix} b \\ c \end{bmatrix}\det\mathcal{M}\begin{bmatrix} c \\ b \end{bmatrix} = \det\mathcal{M}\begin{bmatrix} b & c \\ b & c \end{bmatrix}\det\mathcal{M}, (b\neq c).$$

So we shall not go into the detailed proofs here.

5. Summary and discussion

In this paper, we have given 'affine extentions' of the T-system (2.1)-(2.7h), AppendixA , which may be looked upon as discretized affine Toda field equations. Solving them recursively, we have given, for $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, $A_n^{(2)}$ and $D_{n+1}^{(2)}$, their solutions in terms of determinants or pfaffians.

In [KOS, TK], relations between solutions of T-system and analytic Bethe ansatz [KS1, R] were discussed (cf. [KLWZ,Wi,Z]). Especially for B_r case [KOS], $T_m^{(a)}(u)$ is represented as summations over Yangian analogues of the semi-standard Young tableaux. Whether we can establish such relations between our solutions and analytic Bethe ansatz is not always clear. There remain problems to solve the functional equations presented in Appendix A.

Acknowledgments

The author thanks Prof. A Kuniba for useful discussions and comment on the manuscript, Prof. R Hirota and Dr. S Tsujimoto for kind interests.

Appendix A. Other functional equations

For $G_2^{(1)}$,

$$T_m^{(0)}(u-1)T_m^{(0)}(u+1) = T_{m+1}^{(0)}(u)T_{m-1}^{(0)}(u) + T_m^{(1)}(u) \tag{1.1a}$$

$$T_m^{(1)}(u-1)T_m^{(1)}(u+1) = T_{m+1}^{(1)}(u)T_{m-1}^{(1)}(u) + T_m^{(0)}(u)T_{3m}^{(2)}(u) \tag{1.1b}$$

$$\begin{aligned} &T_{3m}^{(2)}(u-1/3)T_{3m}^{(2)}(u+1/3) \\ &\qquad = T_{3m+1}^{(2)}(u)T_{3m-1}^{(2)}(u) + T_m^{(1)}(u-2/3)T_m^{(1)}(u)T_m^{(1)}(u+2/3) \end{aligned} \tag{1.1c}$$

$$\begin{aligned} &T_{3m+1}^{(2)}(u-1/3)T_{3m+1}^{(2)}(u+1/3) \\ &\qquad = T_{3m+2}^{(2)}(u)T_{3m}^{(2)}(u) + T_m^{(1)}(u-1/3)T_m^{(1)}(u+1/3)T_{m+1}^{(1)}(u) \end{aligned} \tag{1.1d}$$

$$\begin{aligned} &T_{3m+2}^{(2)}(u-1/3)T_{3m+2}^{(2)}(u+1/3) \\ &\qquad = T_{3m+3}^{(2)}(u)T_{3m+1}^{(2)}(u) + T_m^{(1)}(u)T_{m+1}^{(1)}(u-1/3)T_{m+1}^{(1)}(u+1/3) \end{aligned} \tag{1.1e}$$

For $F_4^{(1)}$,

$$T_m^{(0)}(u-1)T_m^{(0)}(u+1) = T_{m+1}^{(0)}(u)T_{m-1}^{(0)}(u) + T_m^{(1)}(u) \tag{1.2a}$$

$$T_m^{(1)}(u-1)T_m^{(1)}(u+1) = T_{m+1}^{(1)}(u)T_{m-1}^{(1)}(u) + T_m^{(0)}(u)T_m^{(2)}(u) \tag{1.2b}$$

$$T_m^{(2)}(u-1)T_m^{(2)}(u+1) = T_{m+1}^{(2)}(u)T_{m-1}^{(2)}(u) + T_m^{(1)}(u)T_{2m}^{(3)}(u) \tag{1.2c}$$

$$\begin{aligned} &T_{2m}^{(3)}(u-1/2)T_{2m}^{(3)}(u+1/2) \\ &\qquad = T_{2m+1}^{(3)}(u)T_{2m-1}^{(3)}(u) + T_m^{(2)}(u-1/2)T_m^{(2)}(u+1/2)T_{2m}^{(4)}(u) \end{aligned} \tag{1.2d}$$

$$\begin{aligned} &T_{2m+1}^{(3)}(u-1/2)T_{2m+1}^{(3)}(u+1/2) \\ &\qquad = T_{2m+2}^{(3)}(u)T_{2m}^{(3)}(u) + T_m^{(2)}(u)T_{m+1}^{(2)}(u)T_{2m+1}^{(4)}(u) \end{aligned} \tag{1.2e}$$

$$T_m^{(4)}(u-1/2)T_m^{(4)}(u+1/2) = T_{m+1}^{(4)}(u)T_{m-1}^{(4)}(u) + T_m^{(3)}(u) \tag{1.2f}$$

For $E_6^{(1)}$,

$$T_m^{(0)}(u-1)T_m^{(0)}(u+1) = T_{m+1}^{(0)}(u)T_{m-1}^{(0)}(u) + T_m^{(6)}(u) \tag{1.3a}$$
$$T_m^{(a)}(u-1)T_m^{(a)}(u+1) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) + T_m^{(a-1)}(u)T_m^{(a+1)}(u), \quad a=2,4 \tag{1.3b}$$
$$T_m^{(1)}(u-1)T_m^{(1)}(u+1) = T_{m+1}^{(1)}(u)T_{m-1}^{(1)}(u) + T_m^{(2)}(u) \tag{1.3c}$$
$$T_m^{(3)}(u-1)T_m^{(3)}(u+1) = T_{m+1}^{(3)}(u)T_{m-1}^{(3)}(u) + T_m^{(2)}(u)T_m^{(4)}(u)T_m^{(6)}(u) \tag{1.3d}$$
$$T_m^{(5)}(u-1)T_m^{(5)}(u+1) = T_{m+1}^{(5)}(u)T_{m-1}^{(5)}(u) + T_m^{(4)}(u) \tag{1.3e}$$
$$T_m^{(6)}(u-1)T_m^{(6)}(u+1) = T_{m+1}^{(6)}(u)T_{m-1}^{(6)}(u) + T_m^{(0)}(u)T_m^{(3)}(u) \tag{1.3f}$$

For $E_7^{(1)}$,

$$T_m^{(0)}(u-1)T_m^{(0)}(u+1) = T_{m+1}^{(0)}(u)T_{m-1}^{(0)}(u) + T_m^{(1)}(u) \tag{1.4a}$$
$$T_m^{(a)}(u-1)T_m^{(a)}(u+1) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) + T_m^{(a-1)}(u)T_m^{(a+1)}(u), \quad a=1,2,4,5 \tag{1.4b}$$
$$T_m^{(3)}(u-1)T_m^{(3)}(u+1) = T_{m+1}^{(3)}(u)T_{m-1}^{(3)}(u) + T_m^{(2)}(u)T_m^{(4)}(u)T_m^{(7)}(u) \tag{1.4c}$$
$$T_m^{(6)}(u-1)T_m^{(6)}(u+1) = T_{m+1}^{(6)}(u)T_{m-1}^{(6)}(u) + T_m^{(5)}(u) \tag{1.4d}$$
$$T_m^{(7)}(u-1)T_m^{(7)}(u+1) = T_{m+1}^{(7)}(u)T_{m-1}^{(7)}(u) + T_m^{(3)}(u) \tag{1.4e}$$

For $E_8^{(1)}$,

$$T_m^{(0)}(u-1)T_m^{(0)}(u+1) = T_{m+1}^{(0)}(u)T_{m-1}^{(0)}(u) + T_m^{(7)}(u) \tag{1.5a}$$
$$T_m^{(a)}(u-1)T_m^{(a)}(u+1) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) + T_m^{(a-1)}(u)T_m^{(a+1)}(u), \quad a=2,4,5,6 \tag{1.5b}$$
$$T_m^{(1)}(u-1)T_m^{(1)}(u+1) = T_{m+1}^{(1)}(u)T_{m-1}^{(1)}(u) + T_m^{(2)}(u) \tag{1.5c}$$
$$T_m^{(3)}(u-1)T_m^{(3)}(u+1) = T_{m+1}^{(3)}(u)T_{m-1}^{(3)}(u) + T_m^{(2)}(u)T_m^{(4)}(u)T_m^{(8)}(u) \tag{1.5d}$$
$$T_m^{(7)}(u-1)T_m^{(7)}(u+1) = T_{m+1}^{(7)}(u)T_{m-1}^{(7)}(u) + T_m^{(0)}(u)T_m^{(6)}(u) \tag{1.5e}$$
$$T_m^{(8)}(u-1)T_m^{(8)}(u+1) = T_{m+1}^{(8)}(u)T_{m-1}^{(8)}(u) + T_m^{(3)}(u) \tag{1.5f}$$

For $E_6^{(2)}$,

$$T_m^{(0)}(u-1/2)T_m^{(0)}(u+1/2) = T_{m+1}^{(0)}(u)T_{m-1}^{(0)}(u) + T_m^{(1)}(u) \tag{1.6a}$$
$$T_m^{(1)}(u-1/2)T_m^{(1)}(u+1/2) = T_{m+1}^{(1)}(u)T_{m-1}^{(1)}(u) + T_m^{(0)}(u)T_m^{(2)}(u) \tag{1.6b}$$
$$T_{2m}^{(2)}(u-1/2)T_{2m}^{(2)}(u+1/2) = T_{2m+1}^{(2)}(u)T_{2m-1}^{(2)}(u) + T_{2m}^{(1)}(u)T_m^{(3)}(u-1/2)T_m^{(3)}(u+1/2) \tag{1.6c}$$
$$T_{2m+1}^{(2)}(u-1/2)T_{2m+1}^{(2)}(u+1/2) = T_{2m+2}^{(2)}(u)T_{2m}^{(2)}(u) + T_{2m+1}^{(1)}(u)T_m^{(3)}(u)T_{m+1}^{(3)}(u) \tag{1.6d}$$
$$T_m^{(3)}(u-1)T_m^{(3)}(u+1) = T_{m+1}^{(3)}(u)T_{m-1}^{(3)}(u) + T_{2m}^{(2)}(u)T_m^{(4)}(u) \tag{1.6e}$$
$$T_m^{(4)}(u-1)T_m^{(4)}(u+1) = T_{m+1}^{(4)}(u)T_{m-1}^{(4)}(u) + T_m^{(3)}(u) \tag{1.6f}$$

For $D_4^{(3)}$,

$$T_m^{(0)}(u-1/3)T_m^{(0)}(u+1/3) = T_{m+1}^{(0)}(u)T_{m-1}^{(0)}(u) + T_m^{(1)}(u) \tag{1.7a}$$

$$T_{3m}^{(1)}(u-1/3)T_{3m}^{(1)}(u+1/3) = T_{3m+1}^{(1)}(u)T_{3m-1}^{(1)}(u) + T_{3m}^{(0)}(u)T_m^{(2)}(u-2/3)T_m^{(2)}(u)T_m^{(2)}(u+2/3) \tag{1.7b}$$

$$T_{3m+1}^{(1)}(u-1/3)T_{3m+1}^{(1)}(u+1/3) = T_{3m+2}^{(1)}(u)T_{3m}^{(1)}(u) + T_{3m+1}^{(0)}(u)T_m^{(2)}(u-1/3)T_m^{(2)}(u+1/3)T_{m+1}^{(2)}(u) \tag{1.7c}$$

$$T_{3m+2}^{(1)}(u-1/3)T_{3m+2}^{(1)}(u+1/3) = T_{3m+3}^{(1)}(u)T_{3m+1}^{(1)}(u) + T_{3m+2}^{(0)}(u)T_m^{(2)}(u)T_{m+1}^{(2)}(u-1/3)T_{m+1}^{(2)}(u+1/3) \tag{1.7d}$$

$$T_m^{(2)}(u-1)T_m^{(2)}(u+1) = T_{m+1}^{(2)}(u)T_{m-1}^{(2)}(u) + T_{3m}^{(1)}(u) \tag{1.7e}$$

References

[AL] Ablowitz M J and Ladik F J 1976 *Stud. Appl. Math.* **55** 213; 1977 *Stud. Appl. Math.* **57** 1

[DJM] Date E, Jimbo M and Miwa T 1982 *J. Phys. Soc. Japan* **51** 4116; 4125; 1983 *J. Phys. Soc. Japan* **52** 388; 761; 766

[H] Hirota R 1977 *J. Phys. Soc. Japan* **43** 1424; 1978 *J. Phys. Soc. Japan* **45** 321; 1981 *J. Phys. Soc. Japan* **50** 3785; 1987 *J. Phys. Soc. Japan* **56** 4285

[KLWZ] Krichever I, Lipan O, Wiegmann P and Zabrodin A preprint 1996 *ESI* 330

[KNH] Kuniba A, Nakamura S and Hirota R 1996 *J. Phys. A: Math. Gen.* **29** 1759

[KNS1] Kuniba A, Nakanishi T and Suzuki J 1994 *Int. J. Mod. Phys.* **A9** 5215

[KNS2] Kuniba A, Nakanishi T and Suzuki J 1994 *Int. J. Mod. Phys.* **A9** 5267

[KOS] Kuniba A, Ohta Y and Suzuki J 1995 *J. Phys. A: Math. Gen.* **28** 6211

[KS1] Kuniba A and Suzuki J 1995 *Commun. Math. Phys.* **173** 225

[KS2] Kuniba A and Suzuki J 1995 *J. Phys. A: Math. Gen.* **28** 711

[LS] Leznov A N and Saveliev M V 1979 *Lett. Math. Phys.* **3** 489

[MOP] Mikhailov A V, Olshanetsky M A and Perelomov A M 1981 *Commun. Math. Phys.* **79** 473

[R] Reshetikhin N Yu 1983 *Sov. Phys. JETP* **57** 691

[S] Suris Yu B 1990 *Phys. Lett.* **145A** 113; 1991 *Phys. Lett.* **156A** 467

[TK] Tsuboi Z and Kuniba A preprent *hep-th* 9608002 , to appear in *J. Phys. A: Math. Gen.*

[Wa] Ward R S 1995 *Phys. Lett.* **199A** 45

[Wi] Wiegmann P preprent *cond-mat* 9610132

[Z] Zabrodin A preprent *hep-th* 9607162; *hep-th* 9610039