Estimating Conditional Quantiles
for Financial Time Series
by Bootstrapping and Subsampling Methods*

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Abstract

Value at Risk (VaR) has become one of the most commonly used measures of risk for financial risk management. Econometrically, a suitable conditional quantile model can provide accurate estimation for this purpose. However due to the special dependence features of financial time series, a classical econometric methodology does not lend itself for this purpose.

In this paper, the main objective is to combine bootstrap technique with nonparametric methodology to estimate the conditional quantile for financial time series. Three newly developed bootstrap based methods (nonparametric wild bootstrap, block bootstrapping and subsampling) are adopted, and local linear nonparametric estimation is then used for estimation. Moving block bootstrapping is applied to generate the confidence intervals for the conditional quantile estimates. The performance of the models is evaluated by means of Monte Carlo simulations.

*This is the first draft of this paper. Any comment is welcome.
1 Introduction

Conditional quantile regression, also simply called quantile regression has been widely used in economics and finance, see Koenker and Bassett (1978). It can be generally denoted as:

\[ Q_t = f(X_t, \beta) + u_t, \quad t = 1, 2, \ldots, n \]  

The traditional assumption for \( u_t \) is that it follows an i.i.d. (independent and identically distributed) process, something that often fails with real economic data. This is especially true when the data are exhibiting various forms of dependence and conditional heteroskedasticity at the same time, see Portnoy (1991). This problem is particularly important in financial time series, as they present special features and challenges for traditional estimators. The most obvious ones are serial correlation and conditional heteroskedasticity as stock returns are not independent and they display clustering over time.

Value at Risk (VaR) is defined as the maximum potential change in the value of a portfolio of financial instruments at a given probability over a certain horizon. Mathematically, it can be described as:

\[ P(X_t < a|X_{t-1}, X_{t-2}, \ldots) = \alpha \]  

where \( a \) is the VaR under interest. Statistically, it is the quantile of any underlying distribution. As denoted above, it is a conditional quantile given past information, where \( \alpha \) is the given probability. In the estimation of VaR, \( \alpha \) takes small values, such as 1% or 5%.

VaR corresponds to the market risk, which estimates the uncertainty of future earnings, due to changes in market conditions. Formally, a VaR calculation aims at making a statement that “the probability that we shall lose more than ”\( a \)” dollars in the next days is 1%(or 5%)”. In words, it is a number that indicates how much a financial institution can expect to lose with probability \( \alpha \) over a given time horizon. VaR reduces the (market) risk associated with any portfolio to just one number, that is the loss associated with a given probability. The usual way of estimating the Value at Risk is to estimate the cumulative density functions by either a parametric or a nonparametric method, and then use the \( \alpha \) quantile as the VaR value of interest, see Cai (2002), Li and Racine (2005). Since VaR is one application of estimating quantiles given some information, it would be interesting to employ conditional quantile techniques to see if it will be a good candidate for the same purpose. Although various methodologies have been used in estimating the VaR, conditional quantiles have rarely been applied. The main difference between conditional quantile and other VaR estimation methods
is that, for a conditional quantile method, a VaR curve rather than a single number is obtainable.

There are many different methodologies that have been developed for the purpose of estimating conditional quantiles consistently. To overcome shortcomings from parametric conditional quantile estimation methods, nonparametric techniques have been adopted recently, see Gourieroux and Monfort (1996) and Yu and Jones (1998)). Linear quantile regression, a quantile estimation method with a parametric form for the systematic part of the econometric model is typically based on the assumption of Gaussian errors. Local polynomial as in Yu and Jones (1998) improves the parametric quantile estimation of Koenker and Bassett (1978) as it performs better when the errors are not normal, see Constanot, Roncalli and Teiletche(2000). However, the main practice in the literature so far, irrespective of the quantile regression model used, is to still assume that the errors are i.i.d. In this paper, the local polynomial method, particularly local linear techniques are combined with different bootstrap methods to estimate VaR. We try to offer a more general way of estimating conditional quantiles for financial time series without making any assumptions about the error distribution or data dependence.

The essential point of the bootstrap method is to provide a framework to estimate both the statistic and its distribution without putting unrealistic or unverifiable assumptions about the DGP (data generating process). Another attraction of the bootstrap is that it is easy to apply even when the statistic is very complex to compute. Under certain circumstances, the bootstrap distribution enables us to make more accurate inferences than the asymptotic approximation.

Efron’s (1979) bootstrap (or the simple bootstrap) may not be capable to mimic the properties of financial time series data. This is because, the simple bootstrap focuses on estimated residuals, assuming that they are i.i.d. with mean zero. Secondly, heteroskedasticity presents an additional problem, especially for financial time series, even if the mean of estimated residuals is zero. If heteroskedasticity is not treated properly, the distribution of estimators and thus inferences will be incorrect. In this paper, various bootstrap based methods are combined with nonparametric techniques to estimate conditional quantiles. Note, that extreme conditional quantiles are difficult to estimate accurately, since there is less information available for this purpose as compared with the estimation of the sample mean or median.

We apply, three estimation strategies. The first, is a combination of classic time series analysis technique with a more generalized nonparametric wild bootstrap method. The nonparametric conditional quantile
method is then applied on the bootstrap samples. The second one aban-
dons parametric regression and applies nonparametric conditional quan-
tile estimation on the block-bootstrapped data. Finally, a subsampling
method is used and a nonparametric conditional quantile estimation is
estimated after the data are subsampled. Furthermore, moving block
bootstrapping techniques are used to generate confidence intervals for
all three estimators.

The paper is organized as follows. In section 2, the three techniques
used in the estimation of the conditional quantiles are described in detail.
In section 3 we discuss confidence interval estimation and in section 4 we
present the simulation results. Finally, section 5 concludes the paper.
In the appendix we present some additional details about confidence
interval estimation.

2 Bootstrap and subsampling approximations of esti-
mitating conditional quantiles for financial time
series data

Before proceeding with the description of the different estimation meth-
ods we need to first present some general notation that is used in the
discussion that follows.

Let \( X_t = \{X_1, X_2, ..., X_n\} \) denote a finite stretch of random variables,
observed from a weakly dependent process \( \{X_i\}_{i \in \mathbb{Z}} \) in \( \mathbb{R}^d \). Let \( Q_t \) be any
level-1 regression model, specifically here we consider the conditional
quantile of \( X_t \) given \( X_{t-1} \). The estimation of \( Q_t \) is based on last period’s
information, i.e.

\[
Q_t = q(X_{t-1}) + v_t, \quad v_t \sim d(0, \sigma^2)
\]

where \( d \) is some distribution with mean 0 and unconditional variance
\( \sigma^2 \). In theory \( q \) can take any form, either parametric or nonparametric.
We focus on the nonparametric form of \( q \). We adopt the local linear
regression model proposed by Yu and Jones (1998) to estimate the 1% and
5% conditional quantile for financial time series. The local linear
regression model can be written as:

\[
Q_t(X_t | X = x_{t-1} + h) = q + a \cdot h + v_t, \quad v_t \sim d(0, \sigma^2)
\]

We want to estimate the constant term \( q \) when \( h \to 0 \) (the band-
width). The problem now is to solve

\[
\min_{(q,a) \in \mathbb{R}^2} \sum_{t=1}^{n} w_t \rho_\tau \left( x_t - a \left( x_{t-1} - x_t \right) - q \right)
\]
where $w_t$ is the weighting function associated with local linear quantile regression. It is also possible to include a quadratic form in the regression form. Here we only use local linear analysis. The choice of bandwidth $h$ is discussed in detail in Yu and Jones (1998) and Ruppert, Sheather and Wand (1995). We use the method that was proposed in Ruppert, Sheather and Wand (1995) to calculate the optimal bandwidth.

Let $\varphi_n$ denote level–2 parameters describing the properties of $\widehat{Q}_t$, the estimate of $Q_t$, such as bias, variance or even the distribution of $\widehat{Q}_t$. In section 3 we discuss how to obtain confidence intervals of $\widehat{Q}_t$. Hence, $\varphi_n$ denotes distribution characteristics of $\widehat{Q}_t$.

### 2.1 ARIMA model combined with SCM bootstrap method

If the simple bootstrap is applied without modification to a time series, the resampled data will not preserve the properties of the original data set and thus inconsistent statistical results will be obtained.

Adesi et al (1999) have used a bootstrap procedure to compute the distribution of financial time series, and thus to obtain VaR estimates. They fit the volatility of financial time series with a particular parametric model and resample from the standardized residuals using the estimated conditional standard deviations from a GARCH (parametric) volatility model. In a subsequent study, Adesi et al. (2001) compare this method with the traditional bootstrap and analyze a specific situation with an option in the stock portfolio. Pascual and Ruiz (2003) conduct Monte Carlo simulations by using Adesi’s method and conclude that the estimated VaR are very close to the actual values. The Adesi et al. (1999) model and Pascual and Ruiz (2003) simulations are based on very specific numerical examples, specifically a real DGP of a GARCH (1,1) model, which is the same model used to fit the volatility in their simulations. However, their result that this parametric approach offers consistent estimates are based on using the same DGP as the model used in estimation. If for example serial dependence and conditional heteroskedasticity beyond what is captured by the GARCH(1,1) are ignored, the bootstrap based on the parametric GARCH(1,1) model would be inconsistent. Therefore, Pascual and Ruiz (2003)’s approach cannot be generalized.

A feasible way of implementing the nonparametric wild bootstrap analysis was proposed by Gozalo (1997) to deal with independent but not necessarily identically distributed, or heteroskedastic, data. The SCM (Smooth Conditional Moment) bootstrap allows for the presence of heteroskedasticity in cross sectional data and it is a data based nonparametric bootstrapping method. However, for time series data there
is no evidence regarding the use of the SCM method. It is also worth noting that the i.i.d. (or simple) bootstrap and the wild bootstrap are special cases of the SCM bootstrap under stronger assumptions about the data. If the data follow an i.i.d. or i.n.i.d. DGP, SCM turns out to be one of the two special bootstrap methods, either the i.i.d. bootstrap or the parametric wild bootstrap. Also because it is a nonparametric bootstrapping method, it avoids the restrictions imposed by parametric methodologies.

In the present paper, first we remove the serial correlation from the financial time series data using a certain parametric model but we leave the conditional heteroskedasticity intact. Subsequently, we use the SCM bootstrap to analyze the conditional heteroskedasticity left in residuals. The two methods combined offer a model-free estimation approach as we combine a parametric serial correlation model with a nonparametric wild bootstrap technique.

Specifically, an ARMA (1,1) model is first used to fit the financial time series. This is because serial correlation can be removed by an ARMA fit successfully for weakly dependent time series. We estimate the following model:

\[ x_t = \alpha x_{t-1} + \beta \varepsilon_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \text{i.i.d.} \]  

(6)

Here i.n.i.d. denotes errors that are independent but not necessarily identically distributed. To render the errors i.i.d., we need to remove both serial correlation and any volatility clusters that would be present. ARMA can solve the serial correlation problem but not volatility clustering. To remove volatility clustering we use the SCM bootstrap method. The procedure works by first mimicking the conditional distribution of \( X_t \) given \( X_{t-1} \) which is assumed to change in a smooth fashion. In particular, we assume that the first three moments of the conditional error distribution to be smooth functions of \( X_{t-1} \). The SCM bootstrap takes this into account, while still allowing for the possibility that each residual has a different conditional distribution.

After estimating the ARMA model, residuals are obtained as:

\[ \hat{u}_t = x_t - \hat{\alpha} x_{t-1} - \hat{\beta} \hat{u}_{t-1} \]  

(7)

Let \( \hat{\sigma}^2_{t,h_2} (x_{t-1}) \) and \( \hat{\sigma}^3_{t,h_3} (x_{t-1}) \) denote consistent nonparametric estimates of \( \sigma^2_t (x_{t-1}) \) and \( \sigma^3_t (x_{t-1}) \), respectively, based on \( \{X_{t-1}, X_t\} \). Nonparametric estimation here can rely on some smoother such as a kernel with suitable choices of bandwidths \( h_2 \) and \( h_3 \) respectively. The kernel
estimators are as follows:

\[
\hat{\sigma}^2_{t,h_2}(x_{t-1}) = n^{-1}\sum_{t=1}^{n} W_{h_2,t}(x_{t-1}) \hat{u}_t^2, \quad \hat{\sigma}^3_{t,h_2}(x_{t-1}) = n^{-1}\sum_{t=1}^{n} W_{h_2,t}(x_{t-1}) \hat{u}_t^3
\]  

(8)

where \(W_{h,t}(x_{t-1}) = K_h(x_{t-1} - X_{t-1})/\hat{f}_h(x_{t-1})\), 
\(\hat{f}_h(x_{t-1}) = n^{-1}\sum_{t=1}^{n} K_h(x_{t-1} - X_{t-1})\) is the kernel function, which possesses the properties of being symmetric, integrating to one and also being absolutely integrable. The bandwidth parameter(s) are sequences \(h\) is such that \(h \rightarrow 0\) and \(nh^d \rightarrow \infty\) as \(n \rightarrow \infty\). Picking a suitable bandwidth \(h\) is done by cross validation. The simple constant kernel estimation is better behaved at the tails of the distribution than the simple constant kernel estimator.

The next step is to apply the nonparametric wild bootstrap.

(a) For each index \(t\), randomly draw (with replacement) the bootstrap residual \(u_t^*\) from an arbitrary distribution \(\hat{F}_t^*\) such that for \(Z^* \sim \hat{F}_t^*\),

\[
E_{\hat{F}_t^*} Z = 0
\]

\[
E_{\hat{F}_t^*} Z^2 = \hat{\sigma}^2(X_{t-1})
\]

\[
E_{\hat{F}_t^*} Z^3 = \hat{\sigma}^3(X_{t-1})
\]

(9) (10) (11)

As suggested by Gozalo (1997), the distribution \(\hat{F}_t^*\) could be chosen as a discrete distribution that puts mass on two points, that is \(\hat{F}_t^* = p_t \delta_{a_t} + (1 - p_t) \delta_{b_t}\), where \(\delta_x\) denotes a probability measure that puts mass one at \(x\), \(p_t \in [0,1]\), then solving the system results in \(a_t = (\hat{\sigma}^3(X_{t-1}) - T_t) / (2\hat{\sigma}_t^2(X_{t-1}))\), \(b_t = (\hat{\sigma}_t^3(X_{t-1}) + T_t) / (2\hat{\sigma}_t^2(X_{t-1}))\), and \(p_t = (1/2) \left(1 + (\hat{\sigma}_t^3(X_{t-1}) / T_t)\right)\), where \(T_t = \sqrt{\left(\hat{\sigma}_t^3(X_{t-1})\right)^2 + 4(\hat{\sigma}_t^2(X_{t-1}))^3}\)

There is the possibility that \(E_{\hat{F}_t^*} Z \neq 0\) because the model chosen is a misspecified parametric model. Then (9) should be replaced by

\[
E_{\hat{F}_t^*} Z = \hat{\sigma}_t(X_{t-1})
\]

(12)

where \(\hat{\sigma}_t(X_{t-1})\) denotes any consistent nonparametric estimation, such as kernel or local polynomial estimation, of \(E(u_t|X_{t-1})\) (based on the residuals \(\hat{u}_t\) of the misspecified model) with bandwidth \(h_v\). The SCM bootstrap steps consisting of the moment with \(\hat{u}_t\), such as (10) and (11), should be replaced by the centered residuals \(\hat{u}_t - \hat{v}_h(x_{t-1})\). This is one of the advantages of the SCM bootstrap as discussed before. Here we
use local linear method to estimate (12) and then modify (10) and (11) accordingly.

(b) The new set of bootstrapped data is generated as \( \{ X_{t-1}, X_t^* = \hat{X}_t + u_t^* \} \). At this point, a local linear conditional quantile regression method, as discussed above from Yu and Jones (1998), is applied to estimate the 1% and 5% conditional quantile by using \( \{ X_{t-1}, X_t^* \} \). The bandwidth \( h \), is picked following their recommendation.

(c) Repeat process (a) to (b) B times. Then for each \( X_{t-1} \), take the median of B estimated conditional quantiles to be the SCM bootstrapping counterpart. The idea behind it is that, due to the possibility of misspecification, the mean value sometimes is distorted by the extreme value of estimated residuals and thus influences the accuracy of the estimates. Hence, we use the median instead of the mean of the residuals moments conditional on \( X_{t-1} \).

2.2 Block bootstrap method to estimate conditional quantile for financial time series

An alternative way of investigating the properties of conditional quantiles of financial time series is the block bootstrap. Historically in the application of the original bootstrap methodology, Efron (1979) derived the bootstrap within a context of an i.i.d. framework by focusing on residuals of some general regression model. In that case, residuals are resampled, but not the original observations. It has been shown that the simple (or i.i.d.) bootstrap behaves well in many situations, which include linear regression, see Freedman (1981), Freedman (1984), Wu (1986), Liu (1988) for example, autoregressive time series, see Efron and Tibhirani (1986), Bose (1988), and nonparametric regression and nonparametric kernel spectral estimation, see Franke and Hardle (1992). However, the success of this approach is based on the reliability of the regression model.

A more general approach is to apply resampling on the original data sequence by considering blocks of data rather than single data points as in the i.i.d. setup. If we have a m-dependent sequence and still use the i.i.d. bootstrap method to estimate the parameters of interest, that will yield inconsistent estimates. The motivation behind the block bootstrap is that within each block the dependence structure of the underlying model is preserved and if the block size is allowed to tend to infinity as the sample size increases, asymptotically one will obtain correct inferences. In such a way, the data are divided into blocks, and these blocks are then resampled. Carlstein (1986) proposed a non-overlapping block bootstrap. As compared with resampling a single observation at
a time, the dependent structure of the original observations is preserved within each block. Kunsch (1989) and Liu and Singh (1992) introduced the Moving Block Bootstrap (MBB) which employs overlapping blocks. The moving block bootstrap method divides the data into overlapping blocks of fixed length and resampling is done with replacement from these blocks. Any block bootstrap technique can be applied to dependent data without any parametric model assumptions.

In both Carlstein (1986) and Kunsch (1989) bootstrap blocks of fixed length are resampled. However, the newly generated pseudo-time series might not be stationary as it may have been initially. To fix this problem Politis and Romano (1994a) proposed the stationary block bootstrap (SBB). The SBB method is to resample blocks of data with random lengths, applied to stationary time series. The whole process could be described as:

1. Wrap the data \( \{X_1, \ldots, X_N\} \) around a circle.
2. Let \( i_1, i_2, \ldots \) be i.i.d. draws from a uniform distribution on the set \( \{1, 2, \ldots, N\} \); these are the starting points of the new blocks.
3. Let \( b_1, b_2, \ldots \) be i.i.d. draws from some distribution \( F_b(\cdot) \) that depends on a parameter \( b \); these are the block sizes.
4. Construct a bootstrap pseudo-series \( X^*_1, \ldots, X^*_N \). The starting observation for the first block is \( X_{i_1} \) and the block size is \( b_1 \). Thus the first block contains observations \( X_{i_1}, \ldots, X_{i_1+b_1-1} \). Then we move on to the next \( b_2 \) observations in the pseudo-time series, which includes \( X_{i_2}, \ldots, X_{i_2+b_2-1} \). This process is stopped once \( N \) observations in the pseudo-time series have been generated.
5. Finally, we focus on the first \( N \) observations of the bootstrapped series and construct the bootstrap sample statistics. Local linear techniques are employed here to estimate the conditional quantiles.
6. Repeat the process \( B \) times, take the median level of \( B \) estimated conditional quantiles for each point as the SBB conditional quantile.

In general the approach here is designed to use MBB or SBB to preserve the dependence structure of the time series, while the nonparametric methodology is applied to solve the heteroskedasticity problem afterwards. It is worth noting that as the main problem with nonparametric estimation is to pick a suitable bandwidth \( h \), the key point for block bootstrapping technique is to find the appropriate block size.

If the distribution \( F_b \) is a unit mass on the positive integer \( b \); the procedure described above becomes the CBB (circular block bootstrap) of Politis and Romano (1992). It is an asymptotically equivalent variation of the MBB (moving block bootstrap) of Kunsch (1989) and Liu and Singh (1992). The only difference is that in CBB, the original data are packed circularly but not in MBB. By using this \( F_b \), the block size is
fixed. The problem of using a fixed block bootstrap, as mentioned above, is that it may produce a nonstationary time series after bootstrapping even if the original series is stationary. The SBB (stationary bootstrap) of Politis and Romano (1994a) uses a Geometric distribution with mean equal to the real number $b$. This distribution is also used in our paper. The requirement for SBB block size selection is less restricted as compared with MBB. It only requires that when $b \to \infty$ when $n \to \infty$, while $n/b \to \infty$ as $n \to \infty$. We also use an experimental method to pick $b$. When $n$ is small, a smaller value of $b$ is used. A larger value of $b$ is associated with a larger number of observations. It seems that more dependent data should be associated with a larger $b$ to have the strong dependence preserved. In that case, a pre-test procedure could be employed before $b$ is decided, whereas stronger dependence would lead to a bigger value of $b$.

2.3 Subsampling

Another way of estimating the conditional quantile is to adopt the subsampling method proposed by Politis and Romano (1994b). The subsampling process can be described as:

(a) define $Y_i$ to be the subsample $(X_i, X_{i+1}, \ldots, X_{i+b_n+1})$, for $i = 1, \ldots, q$ and $q = n - b_n + 1$; $Y_i$ consists of $b_n$ consecutive observations from the $X_1, \ldots, X_n$ sequence, and the order of the observations is preserved.

(b) Within each subsample, use the local linear method to estimate the conditional quantile.

(c) For each $X_i$, for $i = 1, \ldots, n$, take the median of all the corresponding estimated conditional quantiles from various subsamples.

The difference between the bootstrap and subsampling is that subsampling takes samples without replacement of size $b$ from the original sample of size $n$, with $b$ much smaller than $n$. Subsamples are themselves samples of size $b$ from the true unknown distribution $F$ of the original sample. Efron’s bootstrap on the other hand takes samples with replacement of size $n$ from the original sample of size $n$ from the empirical distribution $\hat{F}_n$ associated with the original sample.

The advantage of subsampling as compared with the bootstrap can be summarized as:

Firstly, as the underlying sample distribution is unknown, each subsample as a part of the original series is naturally thought to mimic this distribution. It then seems reasonable to expect that one can gain information about the sampling distribution of a statistic by evaluating it on all subsamples, or "subsamples".

Secondly, there are fewer assumptions needed in order to apply this procedure, see Politis and Romano (1997) who investigated the issue of
heteroskedasticity that may arise in time series by applying subsampling methods.

Bertail et al (2004) gives the theoretical proof of consistency. They pointed out that for the asymptotic approximation to be useful in a finite-sample situation, \( b \) not only has to be fixed, but it has to be small as compared to \( n \), so that \( b/n \) is small. They also remark that for a constant \( b \), the consistency is true under few assumptions. Both Politis and Romano (1994b) and Bertail et al (2004) produce simulation results to examine the robustness of subsampling when applied to financial time series. Simulations in Politis and Romano (1994b) focus on univariate mean estimation and its variance for heteroskedastic time series. Bertail et al (2004) explore subsampling estimation in estimating the VaR for some stock portfolios. In our paper, we combine subsampling method with conditional quantile techniques to assess its performance.

3 Block bootstrap estimation for confidence intervals

Since the bootstrap is a widely used technique to estimate the properties of an unknown distribution, it is reasonable to use it in order to construct confidence intervals of the conditional quantiles. We use the MBB method to do that. Fitenberger (1997) used MBB for the purpose of inference in parametric quantile regressions and he proved that MBB provides a valid asymptotic procedure for linear quantile regressions. In our paper, MBB is combined with local linear quantile regression technique to generate the point-wise confidence intervals for all three estimation methods presented above.

The MBB literature points out that the behavior of block bootstrap estimation critically depends on the block size. In a recent paper Lahiri, Furukawa and Lee (2003) propose a plug-in method for selecting the optimal block length of MBB estimation of confidence interval. The optimal block size \( l_n^0 \) can be showed to be:

\[
\hat{l}_n^0 = \left[ \frac{2\hat{B}^2_n}{r\hat{v}_n} \right]^{\frac{1}{r+2}} n^{\frac{1}{r+2}}
\]

where \( r = 2 \) for this purpose according to discussion in Lahiri, Furukawa and Lee (2003). In the appendix we analyze the above expression in more detail.
4 Monte-Carlo simulation result

4.1 DGP’s and simulation settings

Below we present the framework of our simulation analysis. We generate a Generalized Autoregressive Conditional Heteroskedastic (GARCH) model that combines both an autoregressive and heteroskedastic structure. The GARCH model is generated according to the following equations:

\[ y_t = \varepsilon_t \sigma_t \]  \hspace{1cm} (14)

\[ \sigma_t^2 = 0.1275 + 0.16264 \cdot y_{t-1}^2 + (0.76 - 0.16264)\sigma_{t-1}^2 \]  \hspace{1cm} (15)

We also consider the non-stationary case, where we generate

\[ y_t = \varepsilon_t \sigma_t \]  \hspace{1cm} (16)

\[ \sigma_t^2 = 0.53102592 + 0.16264 \cdot y_{t-1}^2 + (1 - 0.16264)\sigma_{t-1}^2 \]  \hspace{1cm} (17)

This is the Integrated GARCH (IGARCH) model which has been used extensively in empirical financial economics. We can see that the sum of two coefficients in front of the lagged return and the lagged volatility term equals to one, but it is less than one in the simple GARCH model.

An explosive time series model is rarely considered in the literature. An IGARCH model can be transformed into an explosive GARCH model with a little modification.

\[ y_t = \varepsilon_t \sigma_t \]  \hspace{1cm} (18)

\[ \sigma_t^2 = 0.53102592 + 0.16264 \cdot y_{t-1}^2 + (1.05 - 0.16264)\sigma_{t-1}^2 \]  \hspace{1cm} (19)

where the parameters are such that the roots of the characteristic polynomial all lie in the region of the complex plane. Note that unlike the stationary case, the error variables \( \varepsilon_t \)'s in (18) are not required to have zero mean.

Lahiri (2003) examined the performance of the bootstrap in estimating the coefficients of an explosive autoregressive model, under a Model-Based bootstrap framework. It is shown that consistency of the estimator depends greatly on the initial values of the estimation model.

The parameters \( p \) and \( q \) for all 3 \( GARCH(p, q) \) related models are picked to be \( p = q = 1 \). Sample with observations of 50, 100 and 250 are investigated separately for each DGP.\(^1\)

\(^1\)At this stage, we have 500 replications done for when \( n = 50 \). Due to the time requirements of the simulation when \( n \) goes up, 200 replications have been done for \( n = 100 \) and \( n = 500 \).
4.2 Simulation Results

The Mean Squared Error (MSE) for each estimator is listed in Table 1-3 for the 1% and 5% conditional quantiles for different DGP’s and sample sizes. The MSE results for the three methods are as follows. When the time series is stationary, both the SCM nonparametric wild bootstrap (SCM-NWB) and SBB attain a smaller MSE. When the 1% conditional quantile is under investigation, SCM-NWB always performs better. MSE attained by SBB is less than that from SCM-NWB if we consider the 5% conditional quantile. When we look at the nonstationary case, SBB always outperforms the other two methods, for both the 1% and 5% conditional quantile estimates. For the nonstationary time series (IGARCH) the MSE’s from SCM-NWB are larger as compared with SBB and subsampling. When the time series is explosive, neither SCM-NWB nor SBB attains small MSE’s. In that case the smallest MSE is achieved by subsampling for both the 1% and 5% conditional quantile estimates. When the sample size is small and the time series is explosive, it is not easy to judge if any of these three methodologies offers a better choice for conditional quantile estimation. Even for the subsampling method, we do not observe the convex shape of the MSE function when subsample size increases. When the sample size is larger, the MSE keeps decreasing. For small samples and explosive time series, conditional quantile estimation is hard to achieve reliable results by any of the three techniques discussed. When the sample size increases, we do observe certain subsampling size which minimizes MSE. In other words, the convex shape of the MSE is obtained when sample size is large enough, as when n=250 where the optimal subsample size for 1% and 5% conditional quantile estimation is around 20 and 15 respectively.\footnote{We do not discuss how to pick the optimal subsample size in this paper. We use an empirical way of finding the subsample size which results in the smallest MSE.}

Two types of coverage rates are calculated by using the critical values estimated from the MBB process. If we use two sided critical values (at a 95% confidence interval), the coverage rate is not good enough. Generally speaking, the coverage rate is higher for the 5% conditional quantile than for the 1% conditional quantile. The coverage rate is also higher for the stationary time series process than for the nonstationary and explosive cases. The distribution of the conditional quantile has an unclear shape, especially for the extreme lower quantile level. The distribution of the conditional quantile seems to be skewed severely to one side and thus leads to inaccurate inferences using a two-sided confidence interval. In addition to the two-sided confidence interval, we establish an one-sided 95% confidence interval to see if the coverage rate will im-
prove. We notice that the undercoverage of the two-sided confidence intervals is largely improved when the one-sided confidence interval is applied, especially for the stationary time series process\(^3\). The undercoverage is also diminished for nonstationary process and the explosive time series, but to a lesser extent than the stationary case. Generally speaking, when the sample size is bigger, the coverage rate is close to the confidence interval level.

To check the stability of the three conditional quantile estimators, we generate another set of measurements called "rejection rates" by using the MBB confidence intervals. As discussed above, the one sided MBB confidence interval outperforms the two-sided one in calculating the coverage rates. We also calculate rejection rates by using an one-sided confidence interval. The results show that SBB method provides the most stable estimators among all three methodologies. The SCM-NWB behaves well when the time series is stationary but it is very volatile for the nonstationary and explosive cases. The rejection rates for subsampling method decrease when the subsample size increases. It is natural to expect that the bigger the subsample size, the more information is utilized. However, the higher rejection rates for the subsampling estimator indicates that it overestimates the real conditional quantiles, especially for the lowest quantiles such as 1%.

5 Conclusion

In this paper we apply three different types of bootstrap based methodologies to the estimation of the extreme lower conditional quantiles of financial time series data, which exhibit dependence and heteroskedasticity simultaneously. Estimation of such quantiles is valuable for risk management in financial economics. The conditional quantile is interpreted as the VaR in risk management. The SCM bootstrap is originally proposed to be applied to cross sectional data to tackle problems of heteroskedasticity. In this paper the SCM is combined with time series analysis techniques to estimate point-wise conditional quantiles. SBB has the advantage of preserving the dependence properties of financial time series. Subsampling is a newly developed technique to estimate the sample distribution properties of statistics of interest, even though it has not yet been applied to estimate point-wise conditional quantiles. We analyze the performance of the three estimators by means of Monte Carlo simulations. The results show that both SCM-NWB and SBB performs well when the time series is stationary, with SCM-NWB per-

\(^3\)As the bootstrap combined with nonparametric method is very time consuming for large sample size, here for n=250, the result is only for 100 replications. More work is being currently undertaken to obtain more accurate coverage rates.
forming better for the 1% and SBB for the 5% conditional quantile. For the nonstationary or explosive cases, SCM-NWB does not work well at all, while SBB still does reasonably well for the nonstationary case but not so for the explosive case. Results from the subsampling method depend on the sample size and the choice of the subsample size. Subsampling performs better when the sample size is large enough and it also outperforms the other methods for the explosive case. Finally, MBB estimation provides better critical values for one-side than two-side confidence intervals.
6 Appendix

The most important point turns out to be how to choose the constant term in front of the term (13). Consistent estimation of $\hat{B}$ is equivalent to consistent estimation of the bias of the confidence interval estimate, and this is what is called a level 3 parameter. A consistent estimation of $\hat{v}$ is actually an estimate of the variance of the confidence interval, also a level 3 parameter. As shown in Lahiri, Furukawa and Lee (2003), for estimating $\hat{B}$ consistently, we need to combine two moving block bootstrap estimators suitably. Estimation of the level 3 parameter associated with the variance part employ the Jackknife-After-Bootstrap (JAB) method of Efron (1992) and Lahiri (2002).

For constructing the bias estimator, it is suggested that a consistent estimator of $\text{Bias}(\hat{\varphi}_n(l_1))$ may be constructed as

$$\hat{\text{BIAS}}_n = \hat{\text{BIAS}}_n(l_1) = 2 (\hat{\varphi}_n(l_1) - \hat{\varphi}_n(2l_1)) \quad (20)$$

with $1 < l_1 < n^{\frac{1}{5}}$ as $n \to \infty$

where $l_1$ is equivalent to the optimal block size $\hat{l}_n^0$ for bias estimation. $\hat{\varphi}_n(l_1)$ and $\hat{\varphi}_n(2l_1)$ denote MBB estimation of the statistic $\varphi_n$ of interest with block size $l_1$ and $2l_1$ respectively.

A specific choice of $\{l_{1n}\}_{n \geq 1}$ will be suggested later for the plug-in estimator $\hat{\varphi}_n^0$. Note that, as pointed out earlier, the estimator $\hat{\text{BIAS}}_n$ is based on only two block bootstrap estimators of $\varphi_n$ and may be computed using only one level of resampling.

The JAB estimator is applied to estimate the variance of the conditional quantile. The JAB method was proposed by Efron (1992) for assessing the accuracy of bootstrap estimators based on the i.i.d. bootstrap for independent data. A modified version of the method for block bootstrap estimators in the case of dependent data was proposed by Lahiri (2002). The JAB method for dependent data applies a version of the block jackknife method to a block bootstrap estimators.

Let $\hat{\varphi}_n(l)$ be the MBB estimator of a level 2 parameter $\varphi_n$ based on (overlapping) blocks of size $l$ from $\chi_n = \{X_1, ..., X_n\}$. Let $B_i = \{X_i, ..., X_{i+l-1}\}$, $i = 1, ..., N$ (with $N = n - l + 1$) denote the collection of all overlapping blocks contained in $\chi_n$ that are used for defining the MBB estimator $\hat{\varphi}_n(l)$. Also, let $m$ be an integer such that $m^{-1} + n^{-1}m = o(1)$ as $n \to \infty$. Note that the MBB estimator $\hat{\varphi}_n(l)$ is defined in terms of the "basic building blocks" $B_i$’s. Hence, instead of deleting blocks of original observations $\{X_i, ..., X_{i+m-1}\}$, as done in the traditional jackknife method, the JAB method of Lahiri(2002) defines the jackknife point-values by deleting $m$ blocks of $B_i$’s.
Since there are $N$ observed blocks of length $l$, we can define $M = N - m + 1$ many JAB point-values corresponding to the bootstrap estimator $\hat{\varphi}_n$, by deleting the overlapping "blocks of blocks" $\{B_i, ..., B_{i+m-1}\}$ of size $m$ for $i = 1, ..., M$. Let $I_i^0 = \{1, ..., N\} \setminus \{i, ..., i + m - 1\}$, $i = 1, ..., M$. To define the $ith$ JAB point-value $\hat{\varphi}_n^{(i)} \equiv \hat{\varphi}_n^{(i)}(l)$, we need to resample $b = \lceil n/l \rceil$ blocks randomly and with replacement from the reduced collection $\{B_j : j \in I_i^0\}$ and construct the MBB estimator of $\varphi$ using these resampled blocks.

Let $\chi_n^{s(i)}$ denote the resampled data obtained, and the JAB point-value $\hat{\varphi}_n^{(i)}$ is given by applying the functional to the conditional distribution as

$$\hat{\varphi}_n^{(i)} = \varphi \left( \hat{G}_{n,j} \right)$$

(21)

The JAB variance estimator of $\hat{\varphi}_n$ is calculated as

$$\widehat{VAR}_{JAB}(\hat{\varphi}_n) = \frac{m}{N - m} \cdot \frac{1}{M} \sum_{i=1}^{M} \left( \hat{\varphi}_n^{(i)} - \hat{\varphi}_n \right)^2$$

(22)

where $\hat{\varphi}_n^{(i)} = m^{-1} \left( N\hat{\varphi}_n - (N - M)\hat{\varphi}_n^{(i)} \right)$ denotes the $ith$ JAB pseudo-value corresponding to $\hat{\varphi}_n$.

At last, the nonparametric plug-in procedure depends on the choice of the smoothing parameter $l_1$, and on the JAB "blocks of blocks" deletion parameter $m$.

It turns out that a reasonable choice of $l_1$ is of the form (Lahiri 2003)

$$l_1 = C_3 n^{r \frac{1}{3}}$$

(23)

where $r$ is as in (13), and $C_3$ is a population parameter. As for the other smoothing parameter, Lahiri (2002a) suggests that a reasonable choice of the JAB parameter $m$ is given by

$$m = C_4 n^{1/3} l_1^{2/3}$$

(24)

Results of Lahiri, Furukawa and Lee (2003) show that the choice $C_3 = 1$, and $C_4 = 0$ yields good results for the distribution estimation.

---

4 Any consistent estimation of conditional distribution can be used. Here the Kernel estimation of conditional quantile is applied to accelerate the computational speed.
Table 1: MSE Results (n=50)

<table>
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<tr>
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Table 2: MSE Results (n=100)

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Table 3: MSE Results (n=250)

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Table 4: Coverage Rate by MBB Two-Side Confidence Intervals

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Table 5: Coverage Rate by MBB One-Side Confidence Interval

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### Table 6: Rejection Rate by MBB Two-Side Confidence Intervals (n=50)

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<td>SCM NWB</td>
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### Table 7: Rejection Rate by MBB Two-Side Confidence Intervals (n=100)

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### Table 8: Rejection Rate by MBB Two-Side Confidence Intervals (n=250)

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<td>SCM NWB</td>
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Table 9: Rejection Rate by MBB One-Side Confidence Intervals (n=50)

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<td>1%</td>
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Table 10: Rejection Rate by MBB One-Side Confidence Intervals (n=100)

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<tr>
<td>SCM</td>
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Table 11: Rejection Rate by MBB One-Side Confidence Intervals (n=250)

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<td>0.3680</td>
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<td>0.3014</td>
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<tr>
<td>b=45</td>
<td>0.1140</td>
<td>0.0547</td>
<td>0.1303</td>
</tr>
</tbody>
</table>
Figure 1

1% Conditional Quatile Estimation for GARCH Model (n=50)

Figure 2

1% Conditional Quatile Estimation for GARCH Model (n=100)
Figure 3

1% Conditional Quartile Estimation for GARCH Model (n=250)

![Graph showing 1% Conditional Quartile Estimates for GARCH Model with True Value, SCM, SBB, SUB, and C.I.](image-url)
Figure 4

5% Conditional Quatile Estimation for GARCH Model (n=50)

Figure 5

5% Conditional Quatile Estimation for GARCH Model (n=100)
Figure 6

5% Conditional Quantile Estimation for GARCH Model (n=250)
Figure 7
1% Conditional Quatile Estimation for IGARCH Modle (n=50)

Figure 8
1% Conditional Quatile Estimation for IGARCH Modle (n=100)
Figure 9

1% Conditional Quatile Estimation for IGARCH Model (n=250)

- True Value
- SCM
- SBB
- SUB
- C.I

X(i-1)

-3 -2 -1 0 1 2 3

-12 -10 -8 -6 -4 -2
Figure 10

5% Conditional Quatile Estimation for IGARCH Model (n=50)

Figure 11

5% Conditional Quatile Estimation for IGARCH Model (n=100)
Figure 12

5% Conditional Quatile Estimation for IGARCH Model (n=250)

![Graph showing 5% Conditional Quatile Estimation for IGARCH Model (n=250)]
Figure 13

1% Conditional Quatile Estimation for EX-GARCH Model (n=50)

Figure 14

1% Conditional Quatile Estimation for EX-GARCH Model (n=100)
Figure 15

1% Conditional Quantile Estimation for Ex-GARCH Model (n=250)

True Value
SCM
SBB
SUB
CI
Figure 16

5% Conditional Quatile Estimation for Ex-GARCH Modle (n=50)

Figure 17

5% Conditional Quatile Estimation for Ex-GARCH Modle (n=100)
Figure 18

5% Conditional Quartile Estimation for ExGARCH Model (n=250)
References


[23] Li, Q. and Racine, J. (2005), "Nonparametric Estimation of Conditional CDF and Quantile Functions with Mixed Categorical and Continuous Data", CESG 2005.
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[34] Yu, K. and M.C. Jones (1998), "Local Linear Quantile Regression", *Journal of the American Statistical Association*, 93, 228-237