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Secure Weakly Connected Domination in the Corona of Graphs

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Abstract

In this paper, we take another look at the concept of secure weakly connected domination in graphs. In particular, we determine the secure weakly connected dominating sets of the join $K_1 + G$ and the corona of two graphs and, compute their corresponding weakly connected domination numbers.

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1 Introduction

Let $G = (V(G), E(G))$ be a connected undirected graph. For any vertex $v \in V(G)$, the *open neighborhood* of v is the set $N(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. For a set $X \subseteq V(G)$, the *open neighborhood* of X is $N(X) = \bigcup_{v \in X} N(v)$ and the *closed neighborhood* of X is $N[X] = \bigcup_{v \in X} N[v]$.

The subgraph $\langle C \rangle$ of G *induced* by C is the graph having vertex-set C and whose edge set consists of those edges of G incident with two elements of C. A graph is called *connected* if every two vertices are joined by a path; otherwise, it is *disconnected*.

A S is a *dominating set* of G if for every $v \in V(G)\backslash S$, there exists $u \in S$ such that $uv \in E(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the smallest cardinality of a dominating set of G. A dominating set $C \subseteq V(G)$ is called a *weakly connected dominating set* of G if the subgraph $\langle C\rangle_w = (N_G[C], E_w)$ weakly induced by C is connected, where E_w is the set of all edges with at least one vertex in C. The *weakly connected domination number* of G, denoted by $\gamma_w(G)$, is the smallest cardinality of a weakly connected dominating set of G.

A set S is a *secure dominating set* of G if S is a dominating set of G and for every $u \in V(G)\backslash S$, there exists $v \in S$ such that $uv \in E(G)$ and $(S\backslash\{v\}) \cup \{u\}$ is a dominating set of G. The *secure domination number* of G, denoted by $\gamma_s(G)$, is the smallest cardinality of a secure dominating set of G. A set S is a *secure weakly connected dominating set* of G if S is a weakly connected dominating set of G and for every $u \in V(G)\backslash S$, there exists $v \in S$ such that $uv \in E(G)$ and $(S\backslash\{v\}) \cup \{u\}$ is a weakly connected dominating set of G. The *secure weakly connected domination number* of G, denoted by $\gamma_{sw}(G)$, is the smallest cadinality of a secure weakly connected dominating set of G.

The concept of weakly connected domination is discussed in [2] [3, and [4]. Another domination parameter is the secure domination which was discussed in [1] and [5]. A combination of these two concepts give rise to a new variant of domination called secure weakly connected domination.

2 The Join $K_1 + G$

Theorem 2.1 *Let G be a non-complete graph and let $K_1 = \langle\{v\}\rangle$. Then $S \subseteq V(K_1 + G)$ is a secure weakly connected dominating set if and only if one of the following holds:*

(i) S is a secure dominating set of G.

(ii) $v \in S$ and $S\backslash\{v\}$ is a dominating set of G.

(iii) $v \in S$ and $\langle V(G)\backslash N_G[S\backslash\{v\}]\rangle$ is complete.

Proof: Suppose S is a secure weakly connected dominating set of $K_1 + G$. If $S \subseteq V(G)$, then S is a secure weakly connected dominating set of G. Suppose $v \in S$. If $S\backslash\{v\}$ is a dominating set of G, then (ii) holds. Suppose $S\backslash\{v\}$ is not a dominating set of G. Then $V(G)\backslash N_G[S\backslash\{v\}] \neq \varnothing$. Suppose $\langle V(G)\backslash N_G[S\backslash\{v\}]\rangle$ is not complete. Then there exists $x, y \in V(G)\backslash N_G[S\backslash\{v\}]$ such that $xy \notin E(G)$. Since $vx \in E(K_1+G)$ and S is a secure weakly connected dominating set of $K_1 + G$, $(S\backslash\{v\}) \cup \{x\}$ is a weakly connected dominating set of K_1+G. This is a contradiction since S does not dominate y. Therefore, $\langle V(G)\backslash N_G[S\backslash\{v\}]\rangle$ is complete.

Conversely, suppose first that S is a secure dominating set of G. Then S is a secure dominating set of $K_1 + G$. Clearly, S is a weakly connected set of $K_1 + G$. Let $z \in V(k_1 + G)\backslash S$. Then there exists $x \in S$ such that $xz \in E(K_1 + G)$ and $(S\backslash\{x\}) \cup \{z\}$ is a weakly connected set of $K_1 + G$. Hence, S is a secure weakly connected dominating set of $K_1 + G$.

Next, suppose $v \in S$ and $S\backslash\{v\}$ is a dominating set of G. Then S is a weakly connected dominating set of $K_1 + G$. Let $w \in V(K_1 + G)\backslash S$. Then $w \in V(K_1 + G)\backslash(S\backslash\{v\})$. Since $S\backslash\{v\}$ is a dominating set of G, there exists $u \in S\backslash\{v\}$ such that $uw \in E(G)$. Thus, $(S\backslash\{u\}) \cup \{w\}$ is a weakly connected dominating set of $K_1 + G$. Hence, S is a secure weakly connected dominating set of $K_1 + G$.

Finally, suppose (iii) holds. Then S is a weakly connected dominating set of K_1+G. Let $a \in V(K_1+G)\backslash S$. If $a \in N_G(S\backslash\{v\}$, then there exists $y \in (S\backslash\{v\}$ such that $ay \in E(K_1 + G)$. Thus, $(S\backslash\{y\}) \cup \{a\}$ is a weakly connected dominating set of $K_1 + G$. If $a \in V(G)\backslash N_G[S\backslash\{v\}]$, then $(S\backslash\{v\}) \cup \{a\}$ is a dominating set of $K_1 + G$ since $\langle V(G)\backslash N_G[S\backslash\{v\}]\rangle$ is complete. Moreover, $(S\backslash\{v\}) \cup \{a\}$ is a weakly connected set of $K_1 + G$. Therefore, S is a secure weakly connected dominating set of $K_1 + G$. □

Corollary 2.2 *Let G be a non-complete graph. Then*

$$\gamma_{sw}(K_1 + G) = \min\{\gamma(G) + 1, \gamma_s(G), t_G + 1\},$$

where $t_G = \min\{|D| : D \subseteq V(G)$ and $\langle V(G)\backslash N_G[D]\rangle$ is complete$\}$.

Proof: Let $r = \min\{\gamma(G) + 1, \gamma_s(G), t_G + 1\}$ and let S be a minimum secure weakly connected dominating set of $K_1 + G$. Suppose $r = \gamma(G) + 1$. Suppose further that $S\backslash\{v\}$ is not a minimum secure dominating set of G. Then there exist $S' \subset V(K_1 + G)$ such that $|S'\backslash\{v\}| < |S\backslash\{v\}|$. Thus, $|S'| < |S|$. By Theorem 3.5, S' is a secure weakly connected dominating set of $K_1 + G$. This is a contradiction since S is a minimum secure weakly connected dominating set of $K_1 + G$. Therefore, $S\backslash\{v\}$ is a minimum secure dominating set of G. Therefore,

$$\gamma_{sw}(K_1 + G) = |S| = |S\backslash\{v\}| + 1 = \gamma(G) + 1.$$

A similar argument can be used to show that the equality holds if $r = \gamma s(G)$ or $r = t_G + 1$. Accordingly, $\gamma_{sw}(K_1 + G) = \min\{\gamma(G) + 1, \gamma_s(G), t_G + 1\}$. □

3 Corona of Graphs

Let G and H be graphs of order m and n, respectively. The *corona* $G \circ H$ of G and H is the graph obtained by taking one copy of G and m copies of H, and then joining the ith vertex of G to every vertex of the ith copy of H. For every $v \in V(G)$, denote by H^v the copy of H whose vertices are attached one by one to the vertex v. Denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding to the join $\langle \{v\} \rangle + H^v$.

Corollary 3.1 *Let G be a connected graph of order $m \geq 2$ and K_n a complete graph of order n. Then $\gamma_{sw}(G \circ K_n) = m$.*

Corollary 3.2 *Let G be a connected graph of order $m \geq 2$ and H a non-complete graph. Then $S \subseteq V(G \circ H)$ is a secure weakly connected dominating set of $G \circ H$ if and only if $S \cap V(G)$ is a weakly connected dominating set of G, and the following conditions hold.*

(i) $S \cap V(H^v)$ is a secure dominating set of H^v for $v \in V(G) \backslash S$.

(ii) $S \cap V(H^v)$ is a dominating set of H^v or $\langle V(H^v) \backslash N_{H^v}[S \cap V(H^v)] \rangle$ is complete for $v \in S \cap V(G)$.

Proof: Suppose that $S \subseteq V(G \circ H)$ is a secure weakly connected dominating set of $G \circ H$. Then $S \cap V(G)$ is a weakly connected dominating set of G. Let $v \in V(G)$ and let us look at the subgraph $v + H^v$. Then S is a secure weakly connected dominating set of $\langle v \rangle + H^v$. Consider the following cases:

Case 1. $v \in V(G) \backslash S$.

By Theorem 2.1(i), $S \cap V(H^v)$ is a secure dominating set of H^v and (i) holds.

Case 2. $v \in S \cap V(G)$.

By Theorem 2.1, $S \cap V(H^v)$ is a dominating set of H^v or $\langle V(H^v) \backslash N_{H^v}[S \cap V(H^v)] \rangle$ is complete. Thus, (ii) and (iii) holds.

The converse is clear. □

Corollary 3.3 *Let G be a connected graph of order $m \geq 2$ and H a non-complete graph. Then $\gamma_{sw}(G \circ H) = \min\{\gamma_w(G)(1 + \gamma(G) - \gamma_s(G)) + m\gamma_s(G), \gamma_w(G)(1 + t - \gamma_w(G)) + m\gamma_s(G)\}$, where $t = \min\{|S \cap V(H^v)| : \langle V(H^v) \backslash N_{H^v}[S \cap V(H^v)] \rangle$ is complete$\}$.*

Proof: Assume that $\gamma_w(G)(1+\gamma(G)-\gamma_s(G))+m\gamma_s(G) \leq \gamma_w(G)(1+t-\gamma_s(G))+m\gamma_s(G)$. Suppose that S is a minimum secure weakly connected dominating set of $G \circ H$. By Corollary 3.2,

$$S = S \cap V(G) \cup \left(\bigcup_{v \in V(G) \backslash S} S \cap H^v \right) \cup \left(\bigcup_{v \in S \cap V(G)} S \cap H^v \right),$$

Thus,

$$\begin{aligned}
\gamma_{sw}(G \circ H) = |S| = |S \cap V(G)| + \sum_{v \in V(G) \backslash S} |S \cap H^v| + \sum_{v \in S \cap V(G)} |S \cap H^v| \\
\geq \gamma_w(G) + (m - \gamma_w(G))(\gamma_s(G)) + \gamma_w(G)\gamma(G) \\
= \gamma_w(G)(1 + \gamma(G) - \gamma_s(G)) + m\gamma_s(G).
\end{aligned}$$

Next, suppose S_1 is a minimum weakly connected dominating set of G, S_2 is a secure dominating set of H, and S_3 is a minimum dominating set of H. Set

$$S = S_1 \cup \left(\bigcup_{v \in V(G) \setminus S_1} S_2 \right) \cup \left(\bigcup_{v \in S_1 \cap V(G)} S_3 \right).$$

By Corollary 3.2, S is a secure weakly connected dominating set of $G \circ H$. Hence,

$$\begin{aligned}\gamma_{sw}(G \circ H) \leq |S| &= |S_1| + \sum_{v \in V(G) \setminus S_1} |S_2| + \sum_{v \in S_1 \cap V(G)} |S_3| \\ &= \gamma_w(G)(1 + \gamma(G) - \gamma_s(G)) + m\gamma_s(G).\end{aligned}$$

Therefore, $\gamma_{sw}(G \circ H) = \gamma_w(G)(1+\gamma(G)+\gamma_s(G))m\gamma_s(G)$. Accordingly, $\gamma_{sw}(G \circ H) = \min\{\gamma_w(G)(1+\gamma(G)-\gamma_s(G))+m\gamma_s(G), \gamma_w(G)(1+t-\gamma_w(G))+m\gamma_s(G)\}$.□

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