Mean stability of continuous-time semi-Markov jump linear positive systems

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Abstract—In this paper we study the mean stability of continuous-time semi-Markov jump linear positive systems, which are switched linear systems such that their state variables are in positive orthants and their switching signal is a Markov renewal process. The main result of this paper shows that the mean stability is determined by the spectral radius of a matrix. In the proof we utilize a stability-preserving discretization of continuous-time semi-Markov jump linear systems. The obtained results are demonstrated through the stability analysis and the stabilization of linear time-invariant systems with controller failures.

I. INTRODUCTION

The stability analysis of switched linear positive systems, which are switched linear systems whose state variable is constrained to be in positive orthants, has received considerable attentions over the past decade [8], [12], [13], [19]. One of the motivations of the problem is their possible application in pharmacokinetics, in particular, the treatment of human immunodeficiency virus (HIV) infection, where multiple drug regimens are commonly employed to prevent the emergence of drug-resistant virus [28]. Using the switching nature of the treatment and the positivity constraint on the population of virus, the paper [14] formulates the minimization of such virus mutations as the optimal control of a positive switched linear system. The other motivation can be found in the fact that such positivity constraints arise in broad areas including communication systems [25], formation flying [17], and multi agent systems [23].

Though the stability of switched linear positive systems for an arbitrary switching signal has been extensively studied by so-called co-positive Lyapunov functions [12], [13], [19], when the switching signal is modeled as a stochastic process [20], [24], we should use other stability notions that take probability distributions into account. One of the natural notions of stability in such a case is mean stability [20], which requires that the norm of the state variable converges to zero in expectation. There are several conditions available to check the mean stability of switched linear systems that are not necessarily positive (see, e.g., [1], [2], [4], [11]).

The aim of this paper is to give a criterion for the mean stability of switched linear positive systems. We assume that the switched systems are semi-Markov jump linear systems [16] (also called stochastic hybrid systems with renewal transitions [2]), whose switching signal is a Markov renewal process [18]. We show that their mean stability is characterized by the spectral radius of a matrix. We allow the state variable to be reset [15], [21] by random linear mappings at the switching instants. The result can be considered as a generalization the stability condition [2] of the mean stability of even exponents.

One of the difficulties in analyzing semi-Markov jump linear systems is that its transition rate of discrete-modes is not time-invariant [16]. To avoid this difficulty we introduce a discretization of continuous-time semi-Markov jump linear systems that admits a certain time-invariant expression (Proposition 8). That discretization is shown to preserve stability properties and hence the system matrix of the discretization exactly determines the stability of the original system.

This paper is organized as follows. After preparing necessary mathematical notations, in Section II we introduce continuous-time semi-Markov jump linear positive systems and state the main result. Section III analyzes the mean stability of discrete-time semi-Markov jump linear positive systems. Based on the analysis Section IV gives the proof of the main result. Section V illustrates the obtained results via the stability analysis and the stabilization of linear time-invariant systems with controller failures.

A. Mathematical Preliminaries

Let $(\Omega, M, P)$ be a probability space. For an integrable random variable $X$ on $\Omega$ its expected value is denoted by $E[X]$. If $M_1 \subset M$ is a $\sigma$-algebra then $E[X | M_1]$ denotes the conditional expectation of $X$ given $M_1$. For a function $f$ on $\mathbb{R}$ its limit at $t$ from the left, if it exists, is denoted by $f(t^-)$.

A matrix $A$ is said to be nonnegative if it has only nonnegative entries and we write $A \geq 0$. Let $A$ be square. $A$ is said to be Metzler if its off-diagonal entries are nonnegative. It is well known [10] that if $A$ is Metzler then the exponential matrix $e^{tA}$ ($t \geq 0$) is nonnegative. The spectral radius of $A$ is denoted by $\rho(A)$. We say that $A$ is Schur stable if $\rho(A) < 1$.

Define the 1-norm $\| \cdot \|$ of $x \in \mathbb{R}^n$ by $\| x \| = \sum_{i=1}^n |x_i|$. The symbol $1_\ell$ denotes the column vector of length $\ell$ whose entries are all 1. The 1-norm is linear on the positive orthant $\mathbb{R}_+^n$, because if $x \geq 0$ then $\| x \| = 1_\ell^T x$. By $e_i$ we denote the $i$-th standard unit vector in $\mathbb{R}^N$ defined by $[e_i]_j = 1$ for $j = i$ and $[e_i]_j = 0$ otherwise. It is easy to see that, for all $i$ and $x \in \mathbb{R}^n$,

$$\| x \| = \| e_i \otimes x \|$$

where $\otimes$ denotes the Kronecker product [6].
II. Continuous-time Semi-Markov Jump Linear Positive Systems

This section first introduces continuous-time semi-Markov jump linear positive systems. Then we define their exponential and stochastic mean stability. After that we state our main result, which gives the characterization of the mean stability of continuous-time semi-Markov jump linear positive systems.

Let $A_1, \ldots, A_N \in \mathbb{R}^{n \times n}$. Throughout this paper we fix a probability space $(\Omega, \mathcal{F}, P)$. Let $\{\sigma_k\}_{k=0}^{\infty}$, $\{t_k\}_{k=0}^{\infty}$, and $\{J_k\}_{k=0}^{\infty}$ be stochastic processes on $\Omega$ taking values in $\{1, \ldots, N\}$, $\mathbb{R}_+$, and $\mathbb{R}^{n \times n}$, respectively. We assume that $\{t_k\}_{k=0}^{\infty}$ is non-decreasing. We let

$$h_k = t_{k+1} - t_k, \quad k \geq 0.$$ 

Assume that $t_0 = 0$ and that $\sigma_0$ is a constant. Let $\Sigma$ be the switched linear system defined by

$$\Sigma : \begin{cases} \frac{dx}{dt} = A_{\sigma_k} x(t), & t_k \leq t < t_{k+1} \\ x(t_{k+1}) = J_k x(t_k) \end{cases}, \quad k \geq 0$$

where $x(0) = x_0 \in \mathbb{R}^n$ is a constant vector.

**Definition 1.** We say that $\Sigma$ is a continuous-time semi-Markov jump linear positive system if the following conditions hold for all $i, j \in \{1, \ldots, N\}$, $t \geq 0$, and a Borel subset $B \subset \mathbb{R}^{n \times n}$.

C1. (Markov property) It holds that

$$P(\sigma_{k+1} = j, h_k \leq t, J_k \in B \mid \sigma_k, \ldots, \sigma_0, t_0, \ldots, t_{k-1}) = P(\sigma_{k+1} = j, h_k \leq t, J_k \in B \mid \sigma_k).$$

C2. (Time homogeneity) The probability

$$P(\sigma_{k+1} = j, h_k \leq t, J_k \in B \mid \sigma_k = i)$$

is independent of $k$.

C3. (Positivity) The matrices $A_1, \ldots, A_N$ are Metzler and $J_k$ is nonnegative with probability 1.

The conditions C1 and C2 show that $\{(\sigma_k, t_k)\}_{k=0}^{\infty}$ is a time-homogeneous Markov renewal process and therefore $\{\sigma_k\}_{k=0}^{\infty}$ is a time-homogeneous Markov chain [18]. We let $[p_{ij}]_{ij}$ be the transition matrix of the Markov chain $\{\sigma_k\}_{k=0}^{\infty}$. C3 implies that $x(t) \geq 0$ for every $t \geq 0$ with probability 1 provided $x_0 \geq 0$.

We define the mean stability of $\Sigma$ defined as follows.

**Definition 2.**

- $\Sigma$ is said to be exponentially mean stable if there exist $C > 0$ and $\beta > 0$ such that $E[\|x(t)\|] \leq Ce^{-\beta t}\|x_0\|$ for all $x_0 \geq 0$ and $\sigma_0$.

- $\Sigma$ is said to be stochastically mean stable if

$$\int_0^{\infty} E[\|x(t)\|] dt < \infty$$

for all $x_0 \geq 0$ and $\sigma_0$.

We now state the next assumption.

**Assumption 3.** There exist $T > 0$ and $R > 0$ such that the following conditions hold for every $k \geq 0$.

A1. $h_k \leq T$.

A2. $\|\sigma_k\| \leq R$.

In this assumption, only A2 is essential. A1 is not restrictive because most of semi-Markov jump linear systems can be rewritten as one satisfying A1. We notice that, though the authors in [2] use constant jump matrices, this paper allows them to be random variables.

The next theorem is the main result of this paper.

**Theorem 4.** $\Sigma$ is exponentially mean stable if and only if the block matrix $\mathcal{A}_\Sigma \in \mathbb{R}^{n \times n}$ whose $(i, j)$-block is defined by

$$[\mathcal{A}_\Sigma]_{ij} = p_{ij} E[J_k A_{\sigma_k} h_k \mid \sigma_k = j, \sigma_{k+1} = i] \in \mathbb{R}^{n \times n}$$

is Schur stable.

This theorem can be considered as an extension of the result in [2], where the authors give a condition for the mean stability of even exponents, in particular mean square stability, without the positivity condition and under a different setting. One of the advantages of considering the mean stability given in Definition 2, not the mean square stability, is that the mean stability is weaker than the mean square stability due to the inequality $E[\|x(t)\|] \leq \sqrt{E[\|x(t)\|^2]}$ and hence gives an alternative stability when the mean square stability is hard to accomplish.

We prove Theorem 4 by first investigating the stability of its discretization. For the trajectory $x$ of $\Sigma$ we define

$$x_d(k) := x(t_k), \quad k \geq 0.$$ 

Then A1 shows

$$\mathcal{A}_\Sigma : x_d(k+1) = J_k A_{\sigma_k} h_k x_d(k), \quad k \geq 0.$$ (3)

In the next section we analyze the stability of a class of stochastic discrete-time switched systems that include the above defined $\mathcal{A}_\Sigma$. Based on the analysis Section IV gives the proof of Theorem 4.

**Remark 5.** Without the condition A2, Antunes et al. [3] study the mean square stability of impulsive renewal systems, which are special cases of semi-Markov jump linear systems. We leave the question of how to remove the assumption A2 in our setting as an open problem. They also briefly mention the connection between the stability of $\Sigma$ and that of $\mathcal{A}_\Sigma$ when $\Sigma$ is an impulsive renewal system.

III. Discrete-time Semi-Markov Jump Linear Positive Systems

The aim of this section is to introduce discrete-time semi-Markov jump linear positive systems and give the characterization of their mean stability. Let $\{\sigma_k\}_{k=0}^{\infty}$ be a time-homogeneous Markov chain taking values in $\{1, \ldots, N\}$ with the transition matrix $[p_{ij}]_{ij} \in \mathbb{R}^{N \times N}$. Also let $\{F_k\}_{k=0}^{\infty}$ be a stochastic process on $\Omega$ taking values in $\mathbb{R}^{n \times n}$. Define the discrete-time switched system $\Sigma_d$ by

$$\Sigma_d : x_d(k+1) = F_k x_d(k), \quad x_d(0) = x_0.$$
Definition 6. We say that $\Sigma_d$ is a discrete-time semi-Markov jump linear positive system if the following conditions hold for all $k \geq 0$, $i, j \in \{1, \ldots, N\}$, and a Borel subset $B$ of $\mathbb{R}^{n \times n}$.

D1. (Markov property) It holds that

$$P(\sigma_{k+1} = j, F_k \in B \mid \sigma_k, \ldots, \sigma_0, F_{k-1}, \ldots, F_0) = P(\sigma_{k+1} = j, F_k \in B \mid \sigma_k).$$

D2. (Time homogeneity) The expected probability

$$P(\sigma_{k+1} = j, F_k \in B \mid \sigma_k = i)$$

is independent of $k$.

D3. (Positivity) $F_k$ is nonnegative with probability 1.

D4. The expected value

$$E[F_k \mid \sigma_k = i, \sigma_{k+1} = j]$$

exists.

Each of the conditions D1, D2, and D3 corresponds to C1, C2, and C3 in Definition 2, respectively. Also notice that, by D2, the expectation (4) does not depend on $k$.

The stability of discrete-time semi-Markov jump linear systems is defined in a similar way as that of continuous-time ones.

Definition 7.
- $\Sigma_d$ is said to be exponentially mean stable if there exist $C > 0$ and $\beta > 0$ such that $E[\|x_d(k)\|] \leq Ce^{-\beta k}\|x_0\|$ for any $x_0 \geq 0$ and $\sigma_0$.
- $\Sigma_d$ is said to be stochastically mean stable if $\sum_{k=0}^{\infty} E[\|x_d(k)\|] < \infty$ for any $x_0 \geq 0$ and $\sigma_0$.

To analyze the stability of $\Sigma_d$ we introduce the stochastic process $\{\zeta(k)\}_{k=0}^{\infty}$ taking values in the set of standard unit vectors of $\mathbb{R}^n$ and defined by $\zeta(k) = e_{\sigma_k}$.

Proposition 8. Let $\mathcal{F} \in \mathbb{R}^{nN \times nN}$ be the block matrix whose $(i, j)$-block $\mathcal{F}_{ij}$ is defined by

$$\mathcal{F}_{ij} = p_{ji}E[F_k \mid \sigma_k = j, \sigma_{k+1} = i] \in \mathbb{R}^{n \times n}.$$

Then $\mathcal{F}$ is nonnegative. Moreover, for every $k \geq 0$,

$$E[\zeta(k) \otimes x_d(k+1)] = \mathcal{F} E[\zeta(k) \otimes x_d(k)].$$

Proof. The nonnegativity of $\mathcal{F}$ is clear from D3. To show (5) let us fix arbitrary $x_0$ and $\sigma_0$. Since $\sum_{k=1}^{N} \zeta_i(k) = 1$ we have $x_d(k+1) = \sum_{i=1}^{N} \zeta_i(k)F_kx_d(k)$. Therefore, for a fixed $j$,

$$E[\zeta_j(k+1) x_d(k+1)] = \sum_{i=1}^{N} E[\zeta_j(k+1) \zeta_i(k)F_kx_d(k)].$$

Then, from the basic properties of conditional expectations we can actually show that

$$E[\zeta_j(k+1) \zeta_i(k)F_kx_d(k)] = \mathcal{F}_{ji} E[\zeta_i(k) x_d(k)].$$

Hence, by (6),

$$E[\zeta_j(k+1) x_d(k+1)] = \mathcal{F}_{j1} \cdots \mathcal{F}_{jN} E[\zeta(k) \otimes x_d(k)].$$

Thus the definition of Kronecker products show (5).

The next theorem extends the characterization [7], [9] of the mean square stability of discrete-time Markov jump linear systems and will be used in Section IV to prove Theorem 4.

Theorem 9. The following statements are equivalent:
1) $\Sigma_d$ is exponentially mean stable.
2) $\Sigma_d$ is stochastically mean stable.
3) $\mathcal{F}$ is Schur stable.

Proof. We show the cycle $[1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1]$. The implication $[1 \Rightarrow 2]$ is trivial from the definition of mean stability. $[2 \Rightarrow 3]$: Suppose that $\Sigma_d$ is stochastically mean stable. Assume $\rho(\mathcal{F}) \geq 1$ to derive a contradiction. Since $\mathcal{F}$ is nonnegative, $\mathcal{F}$ has an eigenvector $v$ corresponding to the eigenvalue $\rho(\mathcal{F})$ (see, e.g., [26]). Since the set $\{e_i \otimes x_0\}_{1 \leq i \leq N, x_0 \in \mathbb{R}^n}$ in $\mathbb{R}^{nN}$ spans $\mathbb{R}^{nN}$ over $\mathbb{R}$, there exist $c_i \in \mathbb{R}$, $\sigma_0(i) \in \{1, \ldots, N\}$, and $x_0(i) \in \mathbb{R}^n$ such that $v = \sum_{i=1}^{N} c_i e_i \otimes x_0(i)$. Let $x_0(i)$ be the solution of $\Sigma_d$ with the initial state $x_0(i)$ and mode $\sigma_0(i)$. Then (5) shows $\rho(\mathcal{F})^k v = \mathcal{F}^k v = \sum_{i=1}^{N} E[\zeta(i) \otimes x_0(i)]$. Therefore, by (1),

$$\rho(\mathcal{F})^k \|v\| \leq \sum_{i=1}^{N} |c_i| E[\|\zeta(i) \otimes x_0(i)\|]$$

because $\zeta(i)$ takes its value in the set of standard unit vectors. Thus, by taking the summation over $k$ in (7), the stochastic mean stability of $\Sigma_d$ shows $\sum_{k=0}^{\infty} \rho(\mathcal{F})^k \|v\| < \infty$. This gives a contradiction because we assumed $\rho(\mathcal{F}) \geq 1$ and $v \neq 0$.

$[3 \Rightarrow 1]$: Let $x_0 \geq 0$ and $\sigma_0$ be arbitrary. Then clearly $x_d(k)$ is nonnegative by D3. Now suppose $\rho(\mathcal{F}) < 1$. Then Proposition 8 shows that $E[\zeta(k) \otimes x_d(k)]$ exponentially converges to 0 as $k \to \infty$. This proves the exponential mean stability of $\Sigma_d$ because the positivity of $\Sigma_d$ shows $E[\|x_d(k)\|] = E[\|\zeta(k) \otimes x_d(k)\|] < \infty$ for every $k \geq 0$.

IV. PROOF OF THE MAIN RESULT

This section gives the proof of Theorem 4. We separately prove sufficiency and necessity. Let $\Sigma$ be a continuous-time semi-Markov jump linear positive system and let $\mathcal{F}$ be its discretization defined by (3).

A. Sufficiency

Let us first observe the next corollary of Theorem 9.

Corollary 10. The following statements are equivalent:
1) $\mathcal{F}$ is exponentially mean stable.
2) $\mathcal{F}$ is stochastically mean stable.
3) $\mathcal{F}$ is Schur stable.

Proof. Using basic manipulations of conditional probabilities we can show that $\mathcal{F}$ is a discrete-time semi-Markov jump linear positive system. Then the conclusion follows from the definition (2) of the matrix $\mathcal{F}$ and Theorem 9.

□
The next lemma helps us to relate the stability of $\Sigma$ and $\mathcal{S}\Sigma$ and will be used repeatedly.

**Lemma 11.** There exist positive constants $C_1$ and $C_2$ such that, for every sample path $x$ of $\Sigma$ and $k \geq 0$,

$$C_1 \|x(t_k)\| \leq \|x(t)\|, \quad t_k \leq t < t_{k+1}$$

$$\|x(t)\| \leq C_2 \|x(t_k)\|, \quad t \leq t_k \leq t_{k+1}.$$  \hfill (8)

**Proof.** First take an arbitrary $t \in [t_k, t_{k+1})$. Then, by A2, there exist $h \in [0, T]$ and $i \in \{1, \ldots, N\}$ such that $x(t) = e^{A_h} x(t_k)$ and therefore $x(t_k) = e^{-A_h} x(t)$. Hence it holds that $\|x(t_k)\| \leq e^{\|A\|} \|x(t)\| \leq e^{\max_{i \in [0, N]} \|A\|} \|x(t)\|$. Thus the constant $C_1 := e^{-\max_{i \in [0, N]} \|A\| T}$ satisfies the inequality (8).
We can similarly prove the existence of $C_2 > 0$ such that the other inequality (9) holds. $\Box$

Let us prove sufficiency for Theorem 4.

**Proof of sufficiency for Theorem 4.** For $t \geq 0$ define the random variable $k_t$ by

$$k_t(\omega) = \max \{k \in \mathbb{N} : t_k(\omega) \leq t\}.$$  

Assume that $\mathcal{S}\Sigma$ is Schur stable. Then, by Corollary 10, $\mathcal{S}\Sigma$ is exponentially mean stable. We shall show that $\Sigma$ is exponentially mean stable. Let $x_0 \geq 0, \sigma_0$, and $t \geq 0$ be arbitrary. Since $t_k \leq t < t_{k+1}$, the inequality (9) gives $\|x(t)\| \leq C_2 \|x(t_k)\| = C_2 \|x_d(k_t)\|$. Now notice that A2 shows $t < t_k \leq T(k_t + 1)$ and therefore $k_t > T^{-1} t - 1$. Hence the exponential mean stability of $\mathcal{S}\Sigma$ shows

\[
E[\|x(t)\|] \leq C_2 \int_{t}^{t_{T^{-1} t - 1}} \|x_d(k_t)\| \, dP
\]

\[
= C_2 \sum_{t > T^{-1} t - 1} \int_{t_{k_t} = t}^{t_{k_t+1}} \|x_d(k_t)\| \, dP
\]

\[
= C_2 \sum_{t > T^{-1} t - 1} E[\|x_d(k_t)\|]
\]

\[
\leq \frac{CC_2 e^\beta}{1 - e^{-\beta T^{-1}}} \|x_0\|.
\]

Thus $\Sigma$ is exponentially mean stable.

**B. Necessity**

Then let us prove necessity for Theorem 4. For the proof we introduce a family of continuous-time semi-Markov jump linear positive systems. Let $\tau > 0$ and define $J_k^{(\tau)} := \mathcal{X}[h_t \geq \tau] J_k$, where $\mathcal{X}$ denotes a characteristic function. Then define $\Sigma^{(\tau)}$ by

\[
\Sigma^{(\tau)} : \begin{cases}
\frac{dx}{dt} = A_{\chi}\tau x(t), & t_k \leq t < t_{k+1} \\
x(t_{k+1}) = J_{k}^{(\tau)} x(t_{k-1}), & k \geq 0
\end{cases}
\]

The reset matrix $J_{k}^{(\tau)}$ makes the state of $\Sigma^{(\tau)}$ jump to the origin whenever it observes a dwell time $h_k$ less than $\tau$ (see Fig. 1). The next proposition shows that this system $\mathcal{S}\Sigma^{(\tau)}$ inherits the stability of $\Sigma$.

**Proposition 12.** If $\Sigma$ is exponentially mean stable then $\mathcal{S}\Sigma^{(\tau)}$ is stochastically mean stable.

\[
\text{Fig. 1. Sample paths of } \Sigma \text{ (dashed) and } \Sigma^{(\tau)} \text{ (solid).}
\]

**Proof.** Assume that $\Sigma$ is exponentially mean stable. Let $x$ be a sample path of $\Sigma$. First assume that there exists $k \geq 0$ such that $h_k < \tau$. Let $k_0$ be the minimum of such $k$. Then $x_d(k) = 0$ for every $k > k_0$ by the definition of $J_{k}^{(\tau)}$. Therefore, by (9),

\[
\sum_{k=0}^{\infty} \|x_d(k)\| = \|x(t_0)\| + \sum_{k=0}^{k_0-1} \|x(t_k)\|
\]

\[
\leq C_2 \|x(t_0)\| + \sum_{k=0}^{k_0-1} \|x(t_k)\|
\]

\[
\leq 2C_2 \sum_{k=0}^{k_0-1} \|x(t_k)\|.
\]

Since $h_k \geq \tau$ and therefore $1 \leq \tau^{-1} \int_{h_k}^{h_{k+1}} dt$ for every $k = 0, \ldots, k_0 - 1$, using (8) we can show that

\[
\int_{k=0}^{k_0-1} \|x(t_k)\| \leq \tau^{-1} \sum_{k=0}^{k_0-1} \int_{h_k}^{h_{k+1}} \|x(t)\| \, dt
\]

\[
\leq \tau^{-1} C_1 \sum_{k=0}^{k_0-1} \int_{h_k}^{h_{k+1}} \|x(t)\| \, dt
\]

\[
\leq \tau^{-1} C_1 \int_{0}^{\infty} \|x(t)\| \, dt.
\]

Therefore $\Sigma^{(\tau)}$ is exponentially mean stable, taking expectations in (10) and using Fubini’s theorem show the stochastic mean stability of $\mathcal{S}\Sigma^{(\tau)}$. $\Box$

The next proposition shows the continuity of the matrix $\mathcal{S}\Sigma^{(\tau)}$ at $\tau = 0$. The proof naturally follows from the conditional monotone convergence theorem (see, e.g., [5]) and hence is omitted.

**Proposition 13.** It holds that $\lim_{\tau \to 0} \mathcal{S}\Sigma^{(\tau)} = \mathcal{S}\Sigma$.

We also need another family of semi-Markov jump linear positive systems given by

\[
\Sigma_{\alpha} : \begin{cases}
\frac{dx}{dt} = (A_{\chi} + \alpha I) x(t), & t_k \leq t < t_{k+1} \\
x(t_{k+1}) = J_{k}^{(\tau)} x(t_{k-1}), & k \geq 0
\end{cases}
\]

for $\alpha \in \mathbb{R}$. Notice that $\Sigma_0$ equals $\Sigma$. About this system $\Sigma_{\alpha}$ we can prove the next proposition using the monotonicity of the spectral radius of positive matrices [27]. The proof is omitted due to limitations of space.
Proposition 14. $\rho(\mathcal{A}_\Sigma) < \rho(\mathcal{A}_{\Sigma\alpha})$ for every $\alpha > 0$.

Now we are at the position to prove necessity for Theorem 4. Notice that the two mappings $\Sigma \mapsto \Sigma^{(\tau)}$ and $\Sigma \mapsto \Sigma_\alpha$ obviously commute so that we can without confusion denote $(\Sigma^{(\tau)})_\alpha = (\Sigma_\alpha)^{(\tau)}$ by $\Sigma^{(\tau)}_\alpha$.

Proof of necessity for Theorem 4. Assume that $\Sigma$ is exponentially mean stable. Then we can see that, for some small $\alpha > 0$, the system $\Sigma_\alpha$ is also exponentially mean stable. By Proposition 14, it suffices to prove $\rho(\mathcal{A}_{\Sigma_{\alpha}}) \leq 1$. Notice that, by Proposition 12, $\mathcal{A}_{\Sigma_{\alpha}}^{(\tau)}$ is stochastically mean stable for every $\tau > 0$. Then Corollary 10 shows $\rho(\mathcal{A}_{\Sigma_{\alpha}}^{(\tau)}) < 1$. Therefore Proposition 13 does yield $\rho(\mathcal{A}_{\Sigma_{\alpha}}^{(\tau)}) = \lim_{\tau \to 0} \rho(\mathcal{A}_{\Sigma_{\alpha}}^{(\tau)}) \leq 1$ because the spectral radius is a continuous function of a matrix. \qed

V. EXAMPLES

In this section, following the formulation in [4], we illustrate Theorem 4 by discussing the stability analysis and the stabilization of a failure-prone linear time-invariant system. Let us consider the linear time-invariant system $\Sigma: dx/dt = Ax + Bu$ and the fault-prone controller

$$u(t) = \begin{cases} 0, & \text{a fault occurs,} \\ Kx(t), & \text{otherwise.} \end{cases}$$

The controlled system can be modeled as a semi-Markov jump linear system, denoted by $\Sigma_K$, if we let $N = 2$, $A_1 = A + BK$, $A_2 = A$, $p_{11} = p_{22} = 0$, and $p_{12} = p_{21} = 1$. The discrete modes 1 and 2 correspond to the state of no-fault and fault, respectively. We assume $\sigma_0 = 1$, i.e., there is no fault initially.

The probability $P(\sigma_{k+1} = 2, h_k \leq t \mid \sigma_k = 1)$ models the length of the time the controller does not experience a failure and $P(\sigma_{k+1} = 1, h_k \leq t \mid \sigma_k = 2)$ models the length of the time needed to repair the controller. We assume that

$$P(\sigma_{k+1} = 2, h_k \leq t \mid \sigma_k = 1) = \begin{cases} F(t; k, \lambda), & t \leq t_p \\ 1, & t \geq t_p \end{cases}$$

where $F(\cdot; k, \lambda)$ denotes the probability distribution function of the Weibull distribution with the shape parameter $k > 0$ and the scale parameter $\lambda > 0$, i.e., $F(t; k, \lambda) = 1 - e^{-(t/\lambda)^k}$ for $t \geq 0$ and $F(t; k, \lambda) = 0$ for $t < 0$, and $t_p > 0$ is the unique number satisfying $F(t_p; k, \lambda) = 1 - p$ ($p > 0$). In this section we set $\lambda = 3$, $k = 10$, and $p = 0.1$. Also we assume that the time required to repair the controller is uniformly distributed on $[\tau/2, 3\tau/2]$ for some $\tau > 0$. Then $\Sigma_K$ clearly satisfies Assumption 3 with $T = \max(3\tau/2, t_p)$ and $R = 1$.

The following examples illustrate the analysis and synthesis of the controlled system $\Sigma_K$ using Theorem 4. The computation of the matrix $\mathcal{A}_{\Sigma_K}$ is done by the function on MATLAB.

Example 15. This example illustrates how to determine the maximum value of $\tau$ such that the controlled system is stable for given $\Sigma$ and $K$. Let

$$A = \begin{bmatrix} -0.6 & 0.6 \\ 0.7 & 0.4 \end{bmatrix}, \quad B = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix},$$

and $K = [-0.7 \ 1]$. This feedback gain $K$ is chosen in such a way that the closed loop system is stable provided no failure occurs (i.e., the eigenvalues of $A + BK$ have negative real parts). Notice that $\Sigma_K$ is positive both $A$ and

$$A + BK = \begin{bmatrix} -0.88 & 0.2 \\ 0.28 & -0.2 \end{bmatrix}$$

are Metzler. We assume $J_k = I$. Fig. 2 shows the graph of $\rho(\mathcal{A}_{\Sigma_K})$ as the constant $\tau$ moves over $[0, 1]$. We can see that $\Sigma_K$ is exponentially mean stable if $\tau < 0.5$ by Theorem 4.

In Example 15, the reset matrix $J_k$ was set to the identity matrix. The next example illustrates the effect of the presence of resets on stability.

Example 16. Define $A$, $B$, and $K$ as in Example 15. In this example we assume that, when a controller failure occurs at time $t$, the second state variable $x_2(t)$ is reset to zero with probability $q$ in order to improve stability. In other words we let $P(\sigma_{k+1} = 2, J_k = \text{diag}(1, 0) \mid \sigma_k = 1) = q$ and $P(\sigma_{k+1} = 2, J_k = I \mid \sigma_k = 1) = 1 - q$, while it still holds $P(\sigma_{k+1} = 1, J_k = I \mid \sigma_k = 2) = 1$. Fig. 3 shows the supremum of $\tau$ such that $\Sigma_K$ is exponentially mean stable as $q$ varies over $[0, 1]$. For a fixed $q$, such $\tau$ can be obtained as in Example 15. Recall that $\tau$ equals the expected length of the time it takes to repair the controller. We can see that, the more frequent the reset by $J_k = \text{diag}(1, 0)$ is, the longer repair-time is allowed stability-wise. Fig. 4 shows a sample path of $\Sigma_K$ when $q = 1/2$ and $\tau = 0.8$.

Based on Theorem 4 we can also design an optimal feedback gain. The next proposition can be proved in a similar way as [22, Proposition 6.5] and hence its proof is omitted.
Proposition 17. Assume $B \geq 0$. Define $\bar{\mathbf{K}} = \mathbb{R}^{1 \times n}$ by $\bar{K}_j = \max_{1 \leq i \leq n, i \neq j} \left( -a_{ij}/b_i \right)$ for every $1 \leq j \leq n$. Then $K = \bar{K}$ minimizes $\rho(\sigma_{\Sigma K})$ over $K \in \mathbb{R}^{1 \times n}$ such that $\Sigma K$ is positive.

This proposition shows that, when $B$ is nonnegative, we can explicitly find an optimal feedback gain $K$ in the sense that $\rho(\sigma_{\Sigma K})$, which determines the growth rate of $E[\|x(t)\|]$, is minimized over the gains $K$ that keeps $\Sigma K$ positive. Let us see an example.

Example 18. With the parameters (11), since $B \geq 0$, we can apply Proposition 17 and obtain $\bar{K} = \left[ \begin{array}{c} -7/6 \\ 3/2 \end{array} \right]$. Then $\Sigma K$ is positive because $A + BK = \text{diag}(-16/15, -3/2)$ is Metzler. For $\tau = 1.6$ the matrix $\sigma(\Sigma K)$ has the spectral radius 0.829 and therefore $\Sigma K$ is exponentially mean stable by Theorem 4.

Notice that this value of $\tau$ is larger than the maximum value of $\tau$ observed in Example 15. Fig. 5 shows a sample path of the system.

VI. CONCLUSION
This paper studied the mean stability of semi-Markov jump linear positive systems. We showed that the stability is determined by the spectral radius of an associated matrix that is easy to compute. For deriving the condition we used a discretization of a semi-Markov jump linear positive system that preserves stability. The result is illustrated through the stability analysis and stabilization of controlled linear time-invariant systems with controller failures.

REFERENCES