Separation Axioms of Fuzzy Bitopological Spaces

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Abstract
In this paper, we give and study different types of separation axioms using the remoted neighbourhood of a fuzzy point and a fuzzy set in the fuzzy supratopological Spaces (X, τ₁) which is generated by the fuzzy bitopological space (X, τ₁, τ₂). Several properties on these separation axioms are researched.

Keywords:
Fuzzy supratopological, fuzzy bitopological, remoted neighbourhood, separation axioms.

1. Introduction
A.S.Mashhoue, F.H.Kehdr and M.H.Chenim [3] defined the fuzzy bitopological space (X, τ₁, τ₂). A.Kandil, A.D.Nouh and S.A. Sheikh [4] defined the fuzzy supratopological Spaces (X, τ) generated by the fuzzy bitopological space (X, τ₁, τ₂). C.K.wong [7] introduced the concepts of fuzzy point and their neighbourhood. But there are some drawbacks in this study. For overcoming the problem that traditional neighbourhood method was no longer effective in fuzzy topology, Liu and Pu [6] introduced the concept of the strong neighbourhood of the fuzzy point and he defined the fuzzy filter generated by all the open neighbourhood. Nearly at the same time, Wang [5] introduced the concept of the so-called remoted neighbourhood to study fuzzy topology. The latter concept has more extensive application than the former one. Nishimura [8] defined the strong neighbourhood of the fuzzy point and he defined the fuzzy filter generated by all the open neighbourhood and opened strong neighbourhood of the fuzzy point. In sections 3-6, we define two types of FPT-spaces and FPR-spaces and study some properties on them.

2. Notations and Preliminaries
All fuzzy sets on universe X will be denoted by I X . The class of all fuzzy points in universe X will be denoted by FP(X). Use Greek letters as μ, η, δ⋯ etc. to denote fuzzy sets on X. Also P stands for pairwise.

A fuzzy point [5] p_σ be defined as the ordered pair (p, σ) ∈ X × (I − {0}), where I = [0,1]. If σ ≤ μ(p), then (p, σ) ∈ μ and we call (p, σ) belongs to μ. Also if σ < μ(p), then (p, σ) ∈ μ and we call (p, σ) belongs strongly to μ.

Definition 2.1[3]. Let X be any set and τ₁, τ₂ be two fuzzy topologies on X. The triple (X, τ₁, τ₂) is said to be a fuzzy bitopological space.

Definition 2.2[2,8]. Let (X, τ) be an fuzzy topological space and (p, σ) ∈ FP(X). Then (i) An fuzzy set μ s.t. (p, σ) ∈ μ is said to be an open neighbourhood of (p, σ). The fuzzy filter generated by all the open neighbourhood of (p, σ) is denoted and defined as :

V_{(p,σ)} = \{μ ∈ I X : ∃ ρ ∈ τ, (p, σ) ∈ ρ ⊆ μ\}.
Each fuzzy set belonging to V_{(p,σ)} is said to be an neighbourhood of (p, σ).

(ii) An open fuzzy set μ s.t. (p, σ) ∈ μ is said to be an open *-neighbourhood of (p, σ). The fuzzy filter generated by all the open *-neighbourhood of (p, σ) is denoted and defined as :

V^*_{(p,σ)} = \{μ ∈ I X : ∃ ρ ∈ τ, (p, σ) ∈ ρ ⊆ μ\}.
Each fuzzy set belonging to V^*_{(p,σ)} is said to be an *-neighbourhood of (p, σ).
Definition 2.3[6]. The fuzzy filter generated by all the open Q-neighbourhood of \((p, \sigma)\) is denoted and defined as:
\[V^0_{(p, \sigma)} = \{ \mu \in I^X : \exists \rho \in \tau, (p, \sigma) \rho \mu \subseteq \mu \} .\]
Each fuzzy set belonging to \(V^0_{(p, \sigma)}\) is said to be an Q-neighbourhood of \((p, \sigma)\).

Definition 2.4[6]. Let \(Y\) be a crisp subset of an fuzzy topological space \((X, \tau)\). Then \((X, \tau_y)\) is said to be a subspace of \((X, \tau)\), where \(\tau_y\) is a fuzzy topology on \(Y\) given by \(\tau_y = \{ Y \cap \mu, \mu \in \tau \}\). A subspace \((X, \tau_y)\) is open (closed) if the crisp fuzzy set \(Y\) is open (closed) in \(\tau\).

Definition 2.5[2,9]. Let \((X, \tau)\) be an fuzzy topological space and \(\mu, \rho \in I^X\). A fuzzy set \(\rho\) is said to be: (i) R-neighbourhood of a fuzzy point \((p, \sigma)\) if for some closed fuzzy set \(\lambda\) have \((p, \sigma) \notin \lambda \subseteq \rho\). (ii) R*-neighbourhood of a fuzzy point \((p, \sigma)\) if for some closed fuzzy set \(\lambda\) have \((p, \sigma) \notin \lambda \subseteq \rho\).

The collection of all the R- neighbourhoods of \((p, \sigma)\) (resp. R* - neighbourhoods of \((p, \sigma)\)) is denoted by
\[R_{(p, \sigma)}(\text{resp. } R^*_{(p, \sigma)}) \text{ i.e.}\]
\[R_{(p, \sigma)} = \{ \rho \in I^X : \exists \lambda \in \tau, (p, \sigma) \notin \lambda, \lambda \subseteq \rho \} \quad \text{and} \quad (R^*_{(p, \sigma)}) = \{ \rho \in I^X : \exists \lambda \in \tau, (p, \sigma) \notin \lambda, \lambda \subseteq \rho \}.\]

Definition 2.6[4]. Let \((X, \tau_1, \tau_2)\) be fuzzy bitopological space and \(\mu \in I^X\). Then: Associated with the fuzzy closure operators \(\tau_1 - cl\) and \(\tau_2 - cl\) define the mapping
\[C_{\tau_1} : I^X \rightarrow I^X \text{ as: } C_{\tau_1}(\mu) = \tau_1 - cl(\mu) \bigcap \tau_2 - cl(\mu).\]

Definition 2.7[4]. Let \((X, \tau_1, \tau_2)\) be fuzzy bitopological space. Then: the pair \((X, \tau_y)\) is said to be the associated fuzzy supratopological space of \((X, \tau_1, \tau_2)\), where \(\tau_y = \{ \mu \in I^X : \mu = \mu_1 \cup \mu_2, \mu_i \in \tau_i, \mu_2 \subseteq \tau_2 \} \). \(\mu \in \tau_y\) is said to be fuzzy \(\tau_y\)-open or fuzzy supropaopen in \((X, \tau_1, \tau_2)\) and its complement is said to be fuzzy supreclosed in \((X, \tau_1, \tau_2)\).

Theorem 2.1[9]. Let \((X, \tau)\) be an fuzzy topological space and \(\lambda \in I^X\). Then: (i) \(V^*_{(p, \sigma)} = V^0_{(p, \sigma)}\). (ii) \(\lambda \in V^0_{(p, \sigma)}\) iff \(\lambda' \in R^*_{(p, \sigma)}\). (iii) \(\lambda \in R^*_{(p, \sigma)}\) iff \(\lambda' \in V^*_{(p, \sigma)}\). (iv) \(\lambda \in V^0_{(p, \sigma)}\) iff \(\lambda' \in R^0_{(p, \sigma)}\).

Theorem 2.2[9]. If \((X, \tau)\) be an fuzzy topological space and \((Y, \tau_y)\) is a subspace of \((X, \tau)\), \(\mu \in I^Y\). We define:
(i) \(R^0_{(p, \sigma)} = \{ Y \cap \rho, \rho \in R^0_{(p, \sigma)} \}\).
(ii) \(R^*_{(p, \sigma)} = \{ Y \cap \rho, \rho \in R^*_{(p, \sigma)} \}\).
Then \(R^0_{(p, \sigma)}\) (resp. \(R^*_{(p, \sigma)}\)) is the collection of R-neighbourhoods (resp. R*-neighbourhoods) of \(\mu\) in the space \((Y, \tau_y)\).

3. Separation axioms \(FP^{\rho \sigma}_{T_0}\) and \(FPT_0\)

Definition 3.1. An fuzzy bitopological space \((X, \tau_1, \tau_2)\) is said to be (i) \(\in \text{FP}^{\rho \sigma}_{T_0}\) (resp. \(\in \text{FPT}_0\)) iff \((p, \sigma)\), \((q, \delta)\) \(\in \text{FP}(X)\), \(p \neq q\) implies that there exists a fuzzy \(\tau_s\)-closed set \(\mu\) (resp. \(\mu \in \tau_1 \cap \tau_2')\) s.t. \((\mu \notin R_{(p, \sigma)}\), \(\mu \notin R_{(q, \delta)}\)) or \((\mu \notin R_{(q, \delta)}\), \(\mu \notin R_{(p, \sigma)}\)).
(ii) \(\in \text{FP}^{\rho \sigma}_{T_0}\) (resp. \(\in \text{FPT}_0\)) iff \((p, \sigma)\), \((q, \delta)\) \(\in \text{FP}(X)\), \(p \neq q\) implies that there exists a fuzzy \(\tau_s\)-closed set \(\mu\) (resp. \(\mu \in \tau_1 \cap \tau_2'\)) s.t. \((\mu \notin R_{(p, \sigma)}\), \(\mu \notin R_{(q, \delta)}\)) or \((\mu \notin R_{(q, \delta)}\), \(\mu \notin R_{(p, \sigma)}\)).

Theorem 3.1. Let \((X, \tau_1, \tau_2)\) be fuzzy bitopological space. Then: (i) \((X, \tau_1, \tau_2)\) is \(\in \text{FP}^{\rho \sigma}_{T_0}\) (resp. \(\in \text{FPT}_0\)) iff \((p, \sigma)\), \((q, \delta)\) \(\in \text{FP}(X)\), \(p \neq q\) implies that there exists a fuzzy \(\tau_s\)-open set \(\lambda\) (resp. \(\lambda \in \tau_1 \cup \tau_2\)) s.t. \((\lambda \notin V^0_{(p, \sigma)}\), \(\lambda \notin V^0_{(q, \delta)}\)) or \((\lambda \notin V^0_{(q, \delta)}\), \(\lambda \notin V^0_{(p, \sigma)}\)).
(ii) \((X, \tau_1, \tau_2)\) is \(\in \text{FP}^{T_0}\) (resp. \(\in \text{FPT}_0\)) iff \((p, \sigma)\), \((q, \delta)\) \(\in \text{FP}(X)\), \(p \neq q\) implies that there exists a fuzzy \(\tau_s\)-open set \(\lambda\) (resp. \(\lambda \in \tau_1 \cup \tau_2\)) s.t. \((\lambda \notin V^0_{(p, \sigma)}\), \(\lambda \notin V^0_{(q, \delta)}\)) or \((\lambda \notin V^0_{(q, \delta)}\), \(\lambda \notin V^0_{(p, \sigma)}\)).

Proof. (i) Follows from Theorem 2.1 (iii)(iv) and by putting \(\mu' = \lambda\) in Definition 3.1(i).
(ii) Follows from Theorem 2.1 (ii) and by putting \(\mu' = \lambda\) in Definition 3.1(ii).

Theorem 3.2. Let \((X, \tau_1, \tau_2)\) be fuzzy bitopological space. Then: \((i) \in \text{FP}^{T_0}\) \(\implies \in \text{FPT}_0\) (ii) \(\in \text{FP}^{T_0}\) \(\implies \in \text{FPT}_0\).
Proof. We prove part (i) and proof of the other part is similar. Suppose $(X, τ_1, τ_2)$ is in $FP^0 T_0$. Let $(p, σ), (q, δ) ∈ FP(X), p ≠ q$. Since $(X, τ_1, τ_2)$ is in $FP^0 T_0$, then $∃μ ∈ τ'_1 \cap τ'_2$ s.t. $μ ∈ R_{(p, q), η}$ or $μ ∈ R_{(q, δ), φ}$. But $τ'_1 \subseteq τ'_1' \cap τ'_2'$, then $∃μ ∈ τ'_1' \cap τ'_2'$ s.t. $μ ∈ R_{(p, q), η}$ or $μ ∈ R_{(q, δ), φ}$. Hence $(X, τ_1, τ_2)$ is in $FP^0 T_0$.

Theorem 3.3. A subspace of $a ∈ FP^0 T_0$ (resp. $*FP^0 T_0$) is $FP^0 T_0$ (resp. $*FP^0 T_0$).

Proof. We prove the case $FP^0 T_0$, and the proof of the other case is similar. Suppose $(Y, τ'_0, τ''_0)$ is a subspace of a $FP^0 T_0$. Let $(p, σ), (q, δ) ∈ FP(Y), p ≠ q$. Then $(p, σ), (q, δ) ∈ FP(X)$. Since $(X, τ_1, τ_2)$ is in $FP^0 T_0$, then $∃μ ∈ τ'_1 \cap τ'_2$ s.t. $μ ∈ R_{(p, q), η}$ or $μ ∈ R_{(q, δ), φ}$. So $∃μ' = μ ∩ Y ∈ τ'_1' \cap τ'_2'$ s.t. $μ' ∈ R^Y_{(p, q), η}$ or $μ' ∈ R^Y_{(q, δ), φ}$, where $R^Y_{(p, q), η} = Y ∩ η \cap φ ∈ R_{(p, q), η}$. Hence $(Y, τ'_1, τ''_0)$ is in $FP^0 T_0$.

Theorem 3.4. A subspace of $a ∈ FP^0 T_0$ (resp. $*FP^0 T_0$) is $FP^0 T_0$ (resp. $*FP^0 T_0$).

Proof. It is similar to that of Theorem 3.3.

4. Separation axioms $FP^0 T_i$ and $FP T_i$

Definition 4.1. An fuzzy bitopological space $(X, τ_1, τ_2)$ is said to be (i) $FP^0 T_i$ (resp. $FP T_i$) iff $(p, σ), (q, δ) ∈ FP(X), p ≠ q$ implies that there exists a fuzzy $τ_s$-open set $μ, λ$ (resp. $μ, λ ∈ τ'_1 \cap τ'_2$) s.t. $μ ∈ R_{(p, q), η}$ and $λ ∈ R_{(q, δ), φ}$. (ii) $FP^0 T_i$ (resp. $FP T_i$) iff $(p, σ), (q, δ) ∈ FP(X), p ≠ q$ implies that there exists a fuzzy $τ_s$-closed set $μ, λ$ (resp. $μ, λ ∈ τ'_1 \cap τ'_2$) s.t. $μ ∈ R^c_{(p, q), η}$ and $λ ∈ R^c_{(q, δ), φ}$.

Theorem 4.1. Let $(X, τ_1, τ_2)$ be fuzzy bitopological space. Then : (i) $(X, τ_1, τ_2)$ is in $FP^0 T_i$ (resp. $FP T_i$) iff $(p, σ), (q, δ) ∈ FP(X), p ≠ q$ implies that there exists a fuzzy $τ_s$-open set $η, ρ$ (resp. $η, ρ ∈ τ'_1 \cup τ'_2$) s.t. $η ∈ V^0_{(p, q), η}$ and $ρ ∈ V^0_{(q, δ), φ}$.
(q, δ) ∈ FP(X), p ≠ q implies that there exists a fuzzy τ, -closed set μ, λ (resp. μ, λ ∈ τ, ∩ τ, ) where μ ∈ R(p,σ), λ ∈ R(q,δ) s.t. λ ∪ μ = 1X . (i) *FPPT 2 (resp. *FPPT 2 ) iff (p, σ), (q, δ) ∈ FP(X), p ≠ q implies that there exists a fuzzy τ, -closed set μ, λ (resp. μ, λ ∈ τ, ∩ τ, ) where μ ∈ R(p,σ), λ ∈ R(q,δ) s.t. λ ∪ μ = 1X .

Theorem 5.1. Let (X, τ, ) be fuzzy bitopological space. Then: (i) (X, τ, ) is ∈ FPPT 2 (resp. ∈ FPPT 2 ) iff (p, σ), (q, δ) ∈ FP(X), p ≠ q implies that there exists a fuzzy τ, -open set η, ρ (resp. η, ρ ∈ τ, ∪ τ, ) where η ∈ V (p,σ), ρ ∈ V (q,δ) and η ∩ ρ = 0X . If(p, σ), (q, δ) ∈ FP(X), p ≠ q implies that there exists a fuzzy τ, -open set η, ρ (resp. η, ρ ∈ τ, ∪ τ, ) where η ∈ V (p,σ), ρ ∈ V (q,δ) and δ ∩ ρ = 0X .

Theorem 5.2. Let (X, τ, ) be fuzzy bitopological space. Then: (i) ∈ FPPT 2 ⇒ ∈ FPPT 2 (ii) *FPPT 2 ⇒ *FPPT 2 .

Proof. It follows from the face that τ, ≤ τ, ∩ τ, .

Theorem 5.3. Let (X, τ, , τ, ) be fuzzy bitopological space. Then: (i) ∈ FPPT 2 (resp. ∈ FPPT 2 ) ⇒ ∈ FPPT 1 (resp. ∈ FPPT 1 ). (ii) *FPPT 2 (resp. *FPPT 2 ) ⇒ *FPPT 1 (resp. *FPPT 1 ).

Proof. We prove the case (i) and the proof of the other case is similar. Suppose (X, τ, , τ, ) is ∈ FPPT 2 . Let (p, σ), (q, δ) ∈ FP(X), p ≠ q . then: ∀ μ, λ ∈ τ, ∩ τ, where μ ∈ R(p,σ), λ ∈ R(q,δ) s.t. λ ∪ μ = 1X . Since μ ∈ R(p,σ), then (p, σ) ∈ μ and since λ ∪ μ = 1X , then (p, σ) ∈ λ . So λ ∈ R(p,σ). Similar we have μ ∈ R(q,δ).

Hence (X, τ, , τ, ) is ∈ FPPT 1 .

Theorem 5.4. A subspace of a ∈ FPPT 2 (resp. *FPPT 2 ) is ∈ FPPT 2 (resp. *FPPT 2 ).

Proof. We prove the case ∈ FPPT 2 , and the proof of the other case is similar. Suppose (Y, τ, 0, τ, 0) is a subspace of (X, τ, , τ, ). Let (p, σ), (q, δ) ∈ FP(Y), p ≠ q . Then (p, σ), (q, δ) ∈ FP(X). Since (X, τ, , τ, ) is ∈ FPPT 2 , then: ∀ μ, λ ∈ τ, ∩ τ, where μ ∈ R(p,σ), λ ∈ R(q,δ) s.t. λ ∪ μ = 1Y . Hence (Y, τ, 0, τ, 0) is ∈ FPPT 2 .
an ∈ FPR₀. Let μ ∈ R_{(p, l, σ)}⁺, then μ ∈ R_{(p, l, σ)}. Since 
(X, τ₁, τ₂) ∈ FPR₀, then C_{τ₂}((p, σ)) = C_{τ₁}((p, σ)) ∩ Y and 
R_{(p, l, σ)}⁺ = \{μ ∈ R_{(p, l, σ)}⁺ \}. Hence (Y, τ₁₀, τ₂₀) is ∈ FPR₂₀.

Theorem 6.4. A subspace of a ∈ FP₀^ɔ R₀ (resp. *FP₀^ɔ R₀) is ∈ FP₀^ɔ R₀ (resp. *FP₀^ɔ R₀).

Proof. It is similar to that of Theorem 6.3.

Similar with 4 and 5, we can further study Separation axioms FP₀^ɔ R₀ and FPR_0 (resp. *FP₀^ɔ R₀ and FPR_0). Omitted here.

References