Abstract—This paper deals with the computation of time-delay margins of state feedback Model Reference Adaptive Control (MRAC) with σ-modification for single input Linear Time Invariant (LTI) plants. Unlike previous results dealing with the computation of time-delay margins, a direct MRAC (without specific modification to accommodate for the time-delay) is considered in the tracking case. The time-delay margin is derived from the Lyapunov-Razumikhin proof of Uniform Ultimate Boundedness (UUB) of the delayed system. Key assumptions of the approach are the availability of known bounds for the tracking command and its derivative and the availability of known (conservative) bounds for the matched parametric uncertainties. Under these assumptions, UUB is established locally using a quadratic Lyapunov function. The satisfaction of the conditions of the Lyapunov-Razumikhin theorem is established using Sum-Of-Squares programming.

I. INTRODUCTION

In recent years, adaptive control [1], [2] has drawn a lot of attention from the control community due to its ability to adjust the controller gains to the plant. This allows the controller to operate in a highly uncertain environment and to adjust itself in the presence of malfunctions or damages. Despite its striking capabilities, adaptive control is not yet routinely used in safety critical systems such as manned aircraft. The application to such systems is, at least partially, hampered by the absence of well accepted, practically computable measures of robustness. For example, in flight controls, gain and phase margin are traditionally used as measures of robustness. Due to the inherent nonlinearity of adaptive controllers, these LTI criteria may however not be used and hence, new measures are required [3]. One potential measure of robustness also applicable to nonlinear systems is the time-delay margin. This robustness metric is commonly seen as a valid successor of the phase margin. On the one hand, this is due to the fact that the phase margin and the time-delay margin are intimately related for LTI systems. On the other hand, robustness to time-delays also suggests robustness to unmodeled dynamics [4]. Furthermore, robustness to time-delays is desirable on behalf of its own, since time-delays are present in many practical control systems. Due to its highlighted relevance, this paper deals with the computation of the time-delay margin of a well established adaptive controller, namely Model Reference Adaptive Control (MRAC) with σ-modification [2]. While time-delays may be present at various positions within the control loop, we specifically focus on time-delays at the control input of the plant. Hence, in this paper, the time-delay margin refers to the largest admissible time-delay at the plant input which the closed-loop system may tolerate without becoming unstable.

In literature, adaptive control of time-delay systems has been tackled in two distinct ways. One approach is the explicit modification of an adaptive controller such that it accommodates for the time-delay (see e.g. [5] and references therein). The second approach, that is also taken in this paper, is to prove robustness of an existing adaptive control scheme (like MRAC) to time-delays. The largest time-delay, for which stability can be proven, may then serve as a lower bound on the true time-delay margin. In [6], the time-delay margin of MRAC with σ-modification is computed using a method originally proposed in [7] by virtue of the Lyapunov-Krasovskii stability theorem. The paper however only considers adaptive stabilization. In [8], the time-delay margin of MRAC with parameter projection is computed by analyzing the trajectories using first principles and methods presented in [9]. In [10], the time-delay margin is determined by showing robustness of MRAC with parameter projection to an unmodeled input dynamic and using Padé approximation. Finally, in [11], the time-delay margin of an adaptive controller for a second order plant is computed using the Lyapunov-Razumikhin theorem. The approach does, however, require the knowledge of an upper-bound on the regressor vector, which is not readily available when considering the state vector as regressor.

In this paper, the time-delay margin of MRAC with σ-modification is computed for a general, single-input, LTI plant with matched parametric uncertainties in the tracking case. To achieve this, a quadratic Lyapunov function is constructed that resembles the well-known Lyapunov function used in the proof of Uniform Ultimate Boundedness (UUB) in the delay-free case. Due to the time-delay, the derivative of this Lyapunov function is however not a quadratic form but a multivariate polynomial. Hence, Sum-Of-Squares (SOS) optimization [12] and an extension of the S-Procedure [13] are used to show that the Lyapunov function and its derivative satisfy the conditions of a Lyapunov-Razumikhin theorem for UUB [14].

The remainder of this paper is organized as follows: In Section II, some preliminaries on the stability of time-delay systems and on SOS optimization are revised. Section III then states the main result of this paper. Afterwards, the derived conditions are applied to a flight control example in Section IV. While the notation of this paper is standard, the following additional nomenclature is introduced: Let...
\( \mathbb{R} \) denote the real numbers. Then \( \mathbb{R}^+ \) denotes the positive real numbers including zero and \( \mathbb{R}_{++} \) denotes the strictly positive real numbers. Furthermore, \( \mathbb{S}^n \) denotes the set of all symmetric matrices in \( \mathbb{R}^{n \times n} \), \( \mathbb{S}^n_+ \) and \( \mathbb{S}^n_{++} \) denote the sets of all symmetric, positive semidefinite and positive definite matrices, respectively. If the dimensions of the involved variables are clear from the context, positive definiteness (or semidefiniteness) of the matrix \( Q \in \mathbb{S}^n \) is indicated by \( Q > 0 \) (or \( Q \geq 0 \), respectively).

\section*{II. Preliminaries}

\subsection*{A. Stability of Time-Delay Systems}

Let \( C = C([-r,0],\mathbb{R}^n) \) denote the set of continuous functions mapping the interval \([-r,0] \) to \( \mathbb{R}^n \). Furthermore, let \( x_\ell(\xi) \in C \) denote \( x_\ell(\xi) = x(\tau + \xi), \) \( -r \leq \xi \leq 0 \). Then, a Retarded Functional Differential Equation (RFDE) is defined as

\[ \dot{x}(t) = f(t,x_\ell), \]  

where \( x(t) \in \mathbb{R}^n \) and \( f: \mathbb{R} \times C \to \mathbb{R}^n \). In order to solve the RFDE (1), an initial condition \( \phi \in C \) has to be specified, i.e., a function \( \phi(\xi) = x(t_0 + \xi), \) \( -r \leq \xi \leq 0 \). A solution of the RFDE (1) then is denoted as \( x(t_0,\phi,t) \).

\begin{definition}[Uniform Ultimate Boundedness \cite{14}] Let \( x(t_0,\phi,t) \) be a solution of the RFDE (1). The solutions are \textit{uniformly ultimately bounded} if there is a \( b > 0 \) such that for any \( a > 0 \), there is a constant \( T(a) > 0 \) such that \( \|x(t_0,\phi,t)\| \leq b \) for \( t \geq t_0 + T(a) \) for all \( t_0 \in \mathbb{R}, \phi \in C \). \[ \|\phi\|_{C} \leq a, \] \( \|\phi\|_{C} = \sup_{-r \leq \xi \leq 0} \|\phi(\xi)\| \).

The following theorem from \cite{14} may be used to prove UUB of a RFDE:

\begin{theorem}[Razumikhin Theorem for UUB \cite{14}] Suppose \( f: \mathbb{R} \times C \to \mathbb{R}^n \) takes \( \mathbb{R} \times \) (bounded sets of \( \mathbb{R}^n \)) into bounded sets of \( \mathbb{R}^n \) and consider the RFDE (1). Suppose \( u, v, w: \mathbb{R} \to \mathbb{R}^n \) are continuous nondecreasing functions, \( u(s) \to \infty \) as \( s \to \infty \). If there is a continuous \( V: \mathbb{R} \to \mathbb{R}^n \), a continuous nondecreasing \( p: \mathbb{R}^+ \to \mathbb{R}^+ \), \( p(s) > s \) for \( s > 0 \), and a constant \( H \geq 0 \) such that

\[ u(\|x\|) \leq V(t,x) \leq v(\|x\|), \quad t \in \mathbb{R}, x \in \mathbb{R}^n \]  

and

\[ V(t,\phi) \leq -w(\|\phi(0)\|), \]  

if

\[ \|\phi(0)\| \geq H, \]  

\[ p(V(t,\phi(0))) - V(t) > 0, \quad t \in (-r,0], \]  

then the solutions of the RFDE (1) are uniformly ultimately bounded.

\begin{remark}
Let the initial conditions satisfy \( \|\phi\|_{C} \leq \bar{c} \) for some \( \bar{c} > 0 \). Then the conditions (3), (4) and (5) ensure that the Lyapunov function \( V(x) \) never leaves the set \( \mathcal{M} = \{ x | V(x) \leq \max(v(\bar{c}),v(H)) \} \). For this reason, Theorem 1 is also valid if the conditions are only satisfied locally.

\end{remark}

\subsection*{B. Polynomials}

For \( x^T = [x_1 \ldots x_n] \in \mathbb{R}^n \), a monomial is defined as the scalar function \( m_d(x) = \prod_{i=1}^{n} x_i^{d_i} \), where \( d = [d_1 \ldots d_n] \in \mathbb{N}^n \) (here, \( \mathbb{N} \) refers to the natural numbers including zero). For a polynomial of degree \( n \) variables one can determine if it is positive semidefinite or positive definite using the polynomial S-Procedure (Lemma 1) introduced in \cite{17}. The degree of a monomial is defined as \( deg m_d(x) = \sum_{j=1}^{n} d_j \). A polynomial in \( n \) variables is defined as a linear combination of \( k \) monomials \( m_d^{(j)}, j = 1,\ldots,k \), i.e., \( p(x) = \sum_{j=1}^{k} c_j \cdot m_d^{(j)} \). The degree of the polynomial is defined as the largest degree of its monomials, i.e., \( deg p(x) = \max_j (deg m_d^{(j)}) \).

Now let the degree of the polynomial \( p(x) \) be \( 2d \) with \( d \in \mathbb{N} \). Any polynomial degree \( 2d \) (or less) may be expressed in a vector representation:

\[ p(x) = c^T w(x) \]  

where \( c \in \mathbb{R}^{2d} \) is the coefficient vector and \( w(x) \in \mathbb{R}^{2d} \) is the monomial vector. The length of these vectors is given by \( l_w = (n+2d) \). Furthermore, using \( Q \in \mathbb{S}^n \) with \( l_e = (n+d) \), the polynomial may be expressed in a Gram matrix representation:

\[ p(x) = v(x)^T Q v(x) \]  

By equating the coefficients of (7) and (6), a system of linear equations may be obtained to determine \( Q \). Notice that the Gram matrix representation (7) usually is not unique.

\subsection*{C. Sum-Of-Squares Polynomials}

A polynomial \( p(x) \) of degree \( 2d \) (or less) in \( n \) variables is called Sum-of-Squares, if it can be decomposed according to:

\[ p(x) = \sum_{i} f_i^2(x), \]  

where \( deg f_i(x) \leq d \). Obviously, if a polynomial is SOS, then it is positive semidefinite, i.e., \( p(x) \geq 0 \forall x \in \mathbb{R}^n \). Furthermore, it has been shown in \cite{15} that a polynomial is SOS if and only if there exists a Gram matrix representation (7) with \( Q \in \mathbb{S}^n_+ \). Hence, the problem of verifying that a given polynomial with representation (6) is SOS may be reduced to a Linear Matrix Inequality (LMI, \cite{13}) constraint \( Q \geq 0 \) and a system of linear equations. Here, the system of linear equations relates the entries of \( Q \) to the vector representation (6). If furthermore a linear combination of the entries of the decision variable \( Q \) is to be minimized, the resulting optimization problem is called a semidefinite program \cite{16}, for which powerful solvers like SDPT3 \cite{17} are available.

Throughout this paper, the set of all SOS polynomials involving the variable \( x \) is denoted as \( \Sigma(x) \).

\subsection*{D. Polynomial S-Procedure}

The Polynomial S-Procedure \cite{18} is an extension of the well-known S-Procedure \cite{13} and may be stated as follows:

\begin{lemma} [The Polynomial S-Procedure]
Let \( p_0(x) \) and \( p_i(x), i = 1,\ldots,q \) with \( x \in \mathbb{R}^n \) be given polynomials. The
negative semidefiniteness of \( p_i(x) \) on the set \( \mathcal{M} = \{ x \in \mathbb{R}^n | p_i(x) \leq 0 \forall i \} \) implies negative semidefiniteness of \( p_0(x) \) on \( \mathcal{M} \), if there exist multipliers \( \lambda_i(x) \in \Sigma(x) \), such that

\[
-p_0(x) + \sum_{i=1}^{q} \lambda_i(x) \cdot p_i(x) \in \Sigma(x). \tag{9}
\]

Proof: If (9) is true, then \( p_0(x) \leq \sum_{i=1}^{q} \lambda_i(x) \cdot p_i(x) \) is satisfied as well. By definition, \( \sum_{i=1}^{q} \lambda_i(x) \cdot p_i(x) \leq 0 \forall x \in \mathcal{M} \) holds and hence, \( p_0(x) \leq 0 \forall x \in \mathcal{M} \).

III. MAIN RESULT

In this section, the main result of the paper is presented. For a single-input system controlled by a Model Reference Adaptive Controller with \( \sigma \)-modification, an analytical time-delay margin is computed. For the proof of UUB, the following assumptions are required:

Assumption 1: The command signal \( r(t) \) is bounded, i.e. \( \| r(t) \|_{\mathcal{U}} \leq r_0 \), \( r_0 > 0 \). Furthermore, the first derivative of the command signal is bounded as well, i.e. \( \| \dot{r}(t) \|_{\mathcal{U}} \leq \bar{r}_0 \), \( \bar{r}_0 > 0 \).

Assumption 2: The true matched uncertainties \( \Theta^* \in \mathbb{R}^{1 \times n} \) with \( \Theta^* = [\Theta^1; \ldots; \Theta^k] \) are known within a \( n \)-dimensional hypercube \( \mathcal{U} = \{ \Theta^k | \Theta^k \leq \Theta^k_{\text{max}}, \ k = 1, \ldots, n \} \). Furthermore, the baseline controller quadratically stabilizes the plant (see e.g. [19] for a definition of quadratic stabilization) in the absence of time-delay for any \( \Theta^k \in \mathcal{U} \) (see (16)).

A. Problem Statement

Consider the time-delayed, single-input plant

\[
\dot{x}_p(t) = A_p x_p(t) + B_p u(t - \tau_c) - B_p \Theta^* x_p(t), \tag{10}
\]

where \( A_p \in \mathbb{R}^{n \times n}, B_p \in \mathbb{R}^{n \times 1} \) are known matrices, whereas the constant vector \( \Theta^* \in \mathbb{R}^{1 \times n} \) represents the matched plant uncertainties. \( \tau_c > 0 \) denotes the time-delay in the control channel. The control signal \( u(t) \) is given by:

\[
u(t) = K_r r(t) + K_s x_p(t) + \Theta(t) x_p(t), \tag{11}\]

where \( K_r \in \mathbb{R}^{1 \times n} \) and \( K_s \in \mathbb{R} \) are known baseline feedback and feedforward gains, respectively, and \( \Theta \in \mathbb{R}^{1 \times n} \) is the adaptive parameter. Using adaptation, the plant is to asymptotically follow the reference model

\[
\dot{x}_m(t) = A_m x_m(t) + B_m r(t), \tag{12}
\]

where \( A_m \triangleq A_p + B_p K_s \) and \( B_m \triangleq B_p K_r \) have to hold. In order to achieve the control objective, the tracking error \( e(t) \triangleq x_p(t) - x_m(t) \) should be driven to zero. The error dynamics \( \dot{e}(t) = \dot{x}_p(t) - \dot{x}_m(t) \) are given by

\[
\dot{e}(t) = A_p x_p(t) + B_p u(t - \tau_c) - B_p \Theta^* x_p(t) - A_m x_m(t) - B_m r(t). \tag{13}
\]

By adding and subtracting \( B_p u(t) \) and using the definition of \( \Theta(t) \triangleq \Theta(t) - \Theta^* \), the error dynamics may be rewritten as

\[
\dot{e}(t) = A_m e(t) + B_p \tilde{\Theta}(t) x_p(t) + B_p (u(t - \tau_c) - u(t)). \tag{14}
\]

It follows from (14), that even if the parameter estimation error \( \Theta(t) \) is driven to zero, a disturbance \( u(t - \tau_c) - u(t) \) is still acting on the error dynamics. For this reason, the gradient-based update law for \( \Theta(t) \) is enhanced by a \( \sigma \)-modification [2] to avoid parameter drift. The update law is given by

\[
\dot{\Theta}^T(t) = -\Gamma \left( x_p(t) e^T(t) P B_p + \sigma \Theta^T(t) \right), \tag{15}
\]

where \( \Gamma \in \mathbb{R}^{q_+} \) is the learning rate and \( \sigma \in \mathbb{R}^{q_+} \) is the modification gain. In contrast to the proof of UUB in the delay-free case, \( P \in \mathbb{R}^{q_+} \) is not a solution of a Lyapunov equality but the solution of the Lyapunov inequality

\[
(A_m - B_p \Theta^*)^T P + P(A_m - B_p \Theta^*) \leq -Q \forall \Theta^* \in \mathcal{U}. \tag{16}
\]

Eq. (16) implies that the baseline controller quadratically stabilizes the uncertain plant. In (16), \( Q \in \mathbb{R}^{q_+} \) is a design parameter. Note that (16) imposes infinitely many LMI constraints, but may be reduced to a finite number of LMIs using the methods proposed in [20].

B. Model Transformation

In order to let (14) explicitly depend on the time-delay, a model-transformation is applied [9]. For this, notice that

\[
\dot{u}(t - \tau_c) - u(t) = - \int_{t - \tau_c}^{t} \dot{u}(\xi) d\xi \quad \text{with} \quad \dot{u}(\xi) = \frac{du(t)}{dt} = \frac{d}{dt} |_{t = \xi} \tag{17}
\]

holds. Upon substitution of \( \xi = \xi + t \), (17) becomes:

\[
\dot{u}(t - \tau_c) - u(t) = - \int_{-\tau_c}^{0} \dot{u}(t + \xi) d\xi. \tag{18}
\]

The tracking error dynamics (14) may hence be written as:

\[
\dot{e}(t) = A_m e(t) + B_p \tilde{\Theta} x_p(t) - B_p \int_{-\tau_c}^{0} \dot{u}(t + \xi) d\xi. \tag{19}
\]

Using \( A \triangleq A_m - B_p \Theta^* \), the derivative \( \dot{u}(t) \) is given by:

\[
\frac{d}{dt} = \Theta(t) x_p(t) + (\Theta(t) + K_c) \dot{x}_p(t) + K_r \dot{r}(t)
\]

\[
= -B_p^T P e^T x_p(t) + (\Theta(t) + K_c) \dot{x}_p(t) + K_r \dot{r}(t)
\]

\[
= A_p x_p(t) + B_p u(t - \tau_c) - B_p \Theta^* x_p(t) + B_m r(t - \tau_c) + B_p \tilde{\Theta}(t) x_p(t - \tau_c).
\]

(20)

\[
= A x_p(t) + B_p K_s (x_p(t - \tau_c) - x_p(t)) + B_m r(t - \tau_c) + B_p \tilde{\Theta}(t) x_p(t - \tau_c).
\]

(21)

In order to simplify the subsequent derivations, the compact notation according to Table I is used. Hence, \( \dot{u}(t + \xi) \) in compact notation is given by:

\[
\dot{u}(t + \xi) = -B_p^T P e^T x_p(t + \xi) + (\Theta_\xi + K_s) (A x_p(t + \xi) + B_p K_s (x_p(t - \tau_c - \xi) + (\Theta_\xi + K_s) (B_m r(t) + \Theta(t) x_p(t) - \tau_c).
\]

(22)
C. Proof of Ultimate Boundedness

In this section, UUB of the time-delayed adaptive control system is proven. Similar to the delay-free case, a quadratic Lyapunov function candidate is chosen:

\[ V(z) = \frac{1}{2} e^T P e + \frac{1}{2} \bar{\Theta} T_{\bar{P}} \Theta + \frac{1}{2} \Theta \Gamma^{-1} \Theta^T, \]  

(23)

where \( P_m \in \mathbb{S}^{n_{++}} \) solves the Lyapunov equation

\[ A_{m}^T P_m + P_m A_m = -Q_m. \]  

(24)

In (24), \( Q_m \in \mathbb{S}^{n_{++}} \) is a design parameter.

For the subsequent derivations, let \( \|z\|_W \) denote the weighted 2-norm of the vector \( z = [x^T \ z^T \ 0 \ 0 \ 0 \ \Theta^T], \) satisfying \( \|z\|_W = z^T W z, \) \( W \in \mathbb{S}^{n_{++}} \). With

\[ W = \frac{1}{2} \begin{bmatrix} P & 0 & 0 & 0 & 0 \\ 0 & P_m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \]  

(25)

the Lyapunov function may be written as \( V(z) = \|z\|_W^2 \). Hence, using \( u(s) = s^2 \) and \( v(s) = s^2 \), the conditions of Theorem 1 concerning the functions \( u(s) \) and \( v(s) \) are met and \( u(\|z\|_W) = V(z) = v(\|z\|_W) \) holds. Differentiating the Lyapunov function (23) along the solutions of the RFDE (19), (15) and (12) yields:

\[ \dot{V}(z) = e^T P \left( A_m e + B_p \bar{\Theta} x_p - B_p \int_{-\tau}^{0} \dot{u}(t + \xi) d\xi \right) + T_{\bar{P}} P_m \left( A_{m_{xm}} + B_m r - \Theta (x^T \overline{P} \Theta + \alpha \Theta^T) \right). \]  

(26)

Using \( \bar{\Theta}(t) = \Theta - \Theta^* \) and (16), (26) may be upper-bounded by:

\[ \dot{V}(z) \leq -\frac{1}{2} e^T Q e - e^T P \bar{\Theta} x_m \]

\[ + \frac{1}{2} \bar{\Theta} T_{\bar{P}} \Theta x_m + e^T F \]  

(27)

In order to comply with condition (3) of Theorem 1, \( V(z) \leq -w(\|z\|_W) \) with \( w(s) = es^2 \), \( e > 0 \), if (4), (5) hold will be proven in Theorem 2. To simplify the subsequent derivations, \( w(\|z\|_W) = eV(z) \) is added and subtracted in (27):

\[ \dot{V}(z) \leq -eV(z) - \frac{1}{2} e^T Q e - e^T P \bar{\Theta} x_m \]

\[ + \frac{1}{2} \bar{\Theta} T_{\bar{P}} \Theta x_m + e^T F \]  

(28)

Since the integration in (28) is performed over \( \xi \), the upper bound on \( V(z) \) may be rewritten as:

\[ \dot{V}(z) \leq -eV(z) + \int_{-\tau}^{0} v(x) d\xi \]  

(29)

where \( x^T = [x^T \ z^T \ z^T \ 0 \ r \ \bar{r} \ \Theta^T] \). In (29), the vector \( x \) is chosen such that it contains all variables of a SOS optimization problem, which is going to be presented in Theorem 2. Using \( \kappa_e = 1/\tau_e \), \( v(x) \) becomes

\[ v(x) = \kappa_e \left( eV(z) - \frac{1}{2} e^T Q e - \frac{1}{2} \bar{\Theta} T_{\bar{P}} \Theta x_m \right) \]

\[ + \kappa_e \left( -e^T P \bar{\Theta} x_m + x^T \bar{P} B_m r - \Theta \sigma \Theta^T \right) \]  

(30)

By inspecting (29), a general idea about the sets involved in the proof of UUB may be obtained. In order to ensure \( V(z) \leq -eV(z) \), \( v(x) \leq 0 \) is shown. Due to the reference signal \( r(t) \), \( v(x) \) is indefinite inside a level set \( V(z) \leq \xi \) with \( \xi > 0 \). Hence, \( V(z) \) is indefinite in this level set, too. Similarly, \( v(x) \) is indefinite outside a level set \( V(z) \geq \bar{\xi} \) with \( \bar{\xi} > 0 \), since \( v(x) \) is a polynomial of degree 4 and the largest negative definite terms in \( v(x) \) are of degree 2. This is illustrated in Fig. 1. One arrives at the following theorem:

**Theorem 2:** Consider the transformed adaptive control system (19), (15), (12) and let \( p_0 > 1 \), \( e > 0 \), \( \xi > 0 \), \( \bar{\xi} > 0 \) be given constants. Consider the polynomial \( f(x) \) with

\[ f(x) = v(x) + a_1 \left( p_0 V(z) - V(z_\xi) \right) \]

\[ + a_2 \left( p_0 V(z) - V(z_{\xi}) \right) \]  

(31)

where \( v(x) \) is defined in (30). Let the initial condition satisfy \( \|\Phi\|_C^2 = \sup_{0 \leq \xi \leq 0} \|\Phi(\xi)\|_W \leq \bar{\xi} \). If there are multipliers \( \lambda_1, \lambda_2 \in \Sigma(x), \lambda_3, \lambda_4, \lambda_5, \ldots, \lambda_{n_{+}} \in \mathbb{R}_{++} \) and constants \( \alpha_1, \alpha_2 \in \mathbb{R}_{++} \) such that the SOS optimization problem

\[ \min_{\lambda, \alpha_1, \alpha_2} \]  

s.t. \[ -f(x) + \lambda_1 (x - V(z)) \]

\[ + \lambda_2 (x^T \overline{P} \overline{P} \Theta^T - \Theta \sigma \Theta^T) \]

\[ + \lambda_3 (\tau^2 - \tau_0^2) + \lambda_4 (\bar{r}^2 - \bar{r}_0^2) + \lambda_5 (\bar{r}^2 - \bar{r}_0^2) \]  

\[ + \alpha_1 (\sum_{k=1}^{n} \lambda_{k+4}(\Theta^2_{k, \max} - \Theta^2_{k,\max}) \in \Sigma(x) \]  

(32)
is feasible, then the adaptive control system (19), (15), (12) is UUB with a guaranteed time-delay margin of $\tau^* = 1/\kappa^*$. Furthermore, the ultimate bound is given by $H^2 = \zeta$.

Proof: Let $\kappa^*$ be the minimizer of the optimization problem. Then, $\tau^* = 1/\kappa^*$ is largest time-delay for which (33) is satisfied and for which the Polynomial S-Procedures prove the implication:

$$V(z) \geq \zeta \land V(z) \leq \hat{\zeta} \land V(z_e) \leq \hat{\zeta}$$

$$\land r^2 \leq \hat{r}_0^2 \land r^2 \leq \hat{r}_0^2 \land \hat{r}_0 \leq \hat{r}_0$$

$$\land \Theta_k^2 \leq \Theta_{k,\max}^2 \land f \leq 0 \land \Theta_k \leq \Theta_{k,\max}$$

Thus, we have $f(x) \leq 0 \forall x \in \mathcal{M}$ with

$$\mathcal{M} = \{ x | \zeta \leq V(z) \leq \hat{\zeta}, V(z_e) \leq \hat{\zeta}, V(z) \leq \hat{\zeta}, r^2 \leq \hat{r}_0^2, r^2 \leq \hat{r}_0^2, \hat{r}_0 \leq \hat{r}_0 \}$$

Now let $w(x) = e^2 \xi^2$. According to Theorem 1, $\hat{V}(z) \leq -w(\|z\|^2_W) = -e^2V(z)$ has to hold, whenever

$$\|z\|^2_W \geq H^2, H^2 = \zeta$$

$$poV(z) - V(e(t + \xi).x_m(t + \zeta).\Theta(t + \xi)) > 0$$

are satisfied for $\xi \in [-2\tau, 0]$. Eq. (37) is however true if and only if

$$poV(z) - V(z) > 0$$

$$poV(z) - V(z_e) > 0$$

hold for $\xi = [-\tau, 0]$. Thus, $f(x) \leq 0 \forall x \in \mathcal{M}$ implies $v(x) \leq 0 \forall x \in \mathcal{M}$, if (37) is satisfied. Furthermore, note that (36) is satisfied as well, since (34) ensures $V(z) \geq \zeta$. Because of

1) Assumptions 1 and 2, the bounds on the reference signal $r$, its delayed version $r_\xi$, its delayed derivative $r_\xi^2$ and the bounds on the true matched uncertainties $\Theta_k$ are always satisfied.

2) Remark 1 and the boundedness of the initial conditions (i.e. $\|\Theta_k\| \leq \zeta$, the Lyapunov function will never exceed the level set $V(z) = \|z\|^2_W \leq \hat{\zeta}$. Hence, the delayed states satisfy $\|z_\xi\|^2_W \leq \hat{\zeta}, \|z_\tau\|^2_W \leq \hat{\zeta}$.

Due to the preceding two remarks, $v(x) \leq 0 \forall x \in \mathcal{M}$ implies without loss of generality $V(z) \leq e^2V(z), \forall x \in \{ z | \zeta \geq \|z\|^2_W \geq \zeta \}$, if (37) is true. Hence, all conditions of Theorem 1 are satisfied and the adaptive control system (19), (15), (12) is UUB with a time-delay margin $\tau^*$ and an ultimate bound $H^2 = \zeta$

Remark 2: In Theorem 2, the multipliers $\lambda_i(x)$ with $i = 1, \ldots, 5 + n$ have been chosen such that the computational burden is kept small. Other choices of the multipliers, like separate polynomial multipliers for each inequality on the left hand side of (34), are also admissible and may lead to larger estimates of the time-delay margin at the price of an increased computational complexity.

Remark 3: For the application of Theorem 2, the question whether a time-delay may always be computed is of great importance. In order to obtain an insight into this question, consider (28) and denote the upper bound of $V(z)$ as $V_u(z)$.

For $\tau \to 0$, it follows that $V_u(z) \to -\frac{1}{2}e^TQe - e^TPB_p\Theta^Tx_m - \frac{1}{2}x_m^TQ_mx_m + x_m^TP_mB_m\tau - \Theta_0^T\Theta^T$. This however implies that for sufficiently large $\zeta$, the set containment condition of Theorem 2 is always solvable if $\tau \to 0$.

IV. EXAMPLE

To illustrate the effectiveness of Theorem 2, consider the short-period dynamics of a X15 aircraft taken from [6]:

$$\dot{x}_p(t) = A_p(x_p(t) + B_p(t - \tau_c) - B_p^T \Theta x_p(t))$$

Where $A_p = \begin{bmatrix} -0.2950 & 1.0000 \\ -13.0798 & -0.2084 \end{bmatrix}$, $B_p = \begin{bmatrix} 0.0000 \\ -9.4725 \end{bmatrix}$.

The short-period dynamics are controlled by the controller (11). The baseline controller feedback and feedforward gains are given by $K_c = [0.0577, 0.9843]$ and $K_r = -1.7354$. Unless stated otherwise, the design parameters of the adaptive controller are chosen as:

$$\Gamma = \Gamma^2, \quad \sigma = \Gamma^2, \quad Q = Q_m = 2.\Gamma^2$$

Furthermore, the reference signal is assumed to satisfy $\|r(t)\|_{\mathcal{L}_\infty} \leq 0.1 \text{ rad} = 5.7^\circ$ and $\|r(t)\|_{\mathcal{L}_\infty} \leq 0.5 \text{ rad} = 28.7^\circ$. Since an adaptive controller is volatile to time-delay even in the absence of matched parametric uncertainties, we assume at first $\Theta = 0^2$. Hence, the Lyapunov inequality (16) may be replaced by the respective Lyapunov equality. Due to the choice $Q = Q_m$, this implies

$$P = P_m = \begin{bmatrix} 1.8136 & 0.0341 \\ 0.0341 & 0.1085 \end{bmatrix}.$$

Subsequently, Theorem 2 is used to compute time-delay margins for different learning rates $\Gamma$ and different modification gains $\sigma$. Furthermore, the influence of the level set constants is considered. The optimization problem of Theorem 2 is formulated using the modeling language YALMIP [21] and solved by SDPT3 [17]. The computations have been performed on a desktop PC with Intel Core i5-2500 CPU and 12 GB RAM. On this computer, the solution of Theorem 2 on average took 12 min. without matched uncertainties and 27 min. with matched uncertainties.

Under the preceding assumptions, we first examine the influence of different learning rates $\Gamma$ on the provable time-delay margin. In order to isolate the effect of the learning rate, the modification gain $\sigma$ is chosen such that the leakage in the update law (15) remains constant, i.e. $\Gamma \sigma = \Gamma^2$. Since the choice of the level set constants $\zeta$ and $\hat{\zeta}$ also affects the provable time-delay margin, the optimization problem of Theorem 2 was additionally solved for different values of the level set constant $c$. When considering the initial conditions $c(\zeta) = 0$, $x_m(\zeta) = 0$, $\Theta(\zeta) = 0$ with $\zeta = [-2\tau, 0]$, these initial conditions lie within the level set $\zeta(\hat{\zeta}) \leq \zeta$. Hence, the remaining level set constant $\hat{\zeta}$ is arbitrarily chosen slightly larger than $\zeta$, i.e. $\hat{\zeta} = \zeta + 0.01$. In order to keep the computational complexity small, the multipliers $\lambda_i(x)$ and $\lambda_i(x)$ were restricted to polynomials of degree 2. In Fig. 2(a), the analytically computed time-delay margin (for different choices of the level set constant $c$) is compared to the time-delay margin determined in simulations. As
expected, it may be observed that the time-delay margin decreases with increasing learning rate. Despite of the use of SOS programming, the gap between the analytically computed time-delay margin and the simulation-based time-delay margin remains considerable. Although increasing the degree of the multipliers might reduce conservatism, further research is required to substantially reduce this gap. The influence of the level set constant $c$ may be seen best in Fig. 2(b). At first, the time-delay margin increases with an increasing value of the level set constant $c$. After reaching a maximum, the time-delay margin begins to decrease again. This illustrates that the choice of an appropriate level set constant $c$ is crucial for determining the largest value of the time-delay margin.

Next, the influence of the modification gain on the time-delay margin is to be assessed. Hence, the learning rate $\Gamma$ is kept at the constant value $\Gamma = I_{2 \times 2}$. Once again, multipliers $\lambda_1(x)$, $\lambda_2(x)$ of degree 2 are used and the level set constant $\bar{c}$ is chosen as $\bar{c} = c + 0.01 \cdot \sigma$. For different choices of the level set constant $c$, Fig. 3(a) compares the analytically computed time-delay margin to the one determined in simulations. As expected, the time-delay margin is small for small modification gains $\sigma$. For large modification gains, the time-delay margin however counter-intuitively decreases to zero as well. This is counter-intuitive since an increasing modification gain essentially disables the adaptation and hence, the time-delay margin is expected to increase. The observed behavior may however be explained with the help of Fig. 1: By increasing $\sigma$, the size of the interior indefinite region is increased due to the term $e^T PB_p \dot{u}(t + \bar{\xi})$ in $f(x)$. At the same time, the level set $V(z) \leq c$ remains at the same size. By decreasing the time-delay margin, the influence of $e^T PB_p \dot{u}(t + \bar{\xi})$ is mitigated such that the set containment conditions of Theorem 2 may be satisfied. The counter-intuitive result of Theorem 2 illustrates a drawback of the presented approach. Due to the use of a quadratic Lyapunov function, whose parameters have been determined beforehand, the shape of the Lyapunov function and its derivative are determined. Hence, the remaining degrees of freedom for the optimization are the multipliers and the scaling of the Lyapunov function by the level set constants. Thus, future research has to focus on mitigating the conservatism to arrive at results that (for
all $\Gamma$ and $\sigma$) are comparable to those that have already been achieved in the case of stabilization [6]. In Fig. 3(b), one observes that the dependence of the time-delay margin on the level set constant $c$ is similar to what has been previously observed for varying learning rates $\Gamma$.

Finally, assume that $\Theta^* = [\Theta_1^* \Theta_2^*]$ is chosen such that it reflects 30% matched uncertainty in the coefficients of the matrix $A_p$, i.e.

$$|\Theta_1^*| \leq 4.1 \cdot 10^{-1}, \quad |\Theta_2^*| \leq 6.6 \cdot 10^{-3}. \tag{43}$$

For $Q = 2 \cdot I^{2 \times 2}$, a solution to the Lyapunov inequality (16) is given by:

$$P = \begin{bmatrix} 2.1662 & 0.0548 \\ 0.0548 & 0.1312 \end{bmatrix}. \tag{44}$$

The computation of the time-delay margin is carried out as before, however using $P$ as in (44) and $P_n$ as in (42). For $c = 0.25$, Fig. 4 compares the dependence of the time-delay margin on the learning rate $\Gamma$ with and without 30% matched uncertainty. It may be observed that compared to the case of no matched uncertainties, the provable time-delay margin is reduced by 46.1% at $\Gamma = 10^{-3}$ and by 42.5% at $\Gamma = 8.48$.

V. CONCLUSION

In this paper, a novel approach for the computation of the time-delay margin of MRAC with $\sigma$-modification is presented. In contrast to previous publications, the presented approach allows the computation of a time-delay margin for a single-input LTI plant in the tracking case. Furthermore, the resulting time-delay margin is valid for all matched uncertainties of admissible size. Since level set constants are explicitly used within the approach, the largest tracking error that may occur due to the time-delay can also be estimated. Although the explicit use of level set constants is advantageous, further research is required to quickly determine initial level set constants that solve the optimization problem of Theorem 2. Furthermore, research is required to mitigate the conservatism that is introduced by using a Lyapunov function whose parameters have been determined beforehand. As discussed in [6], another aspect that requires further attention is the SOS optimization itself, since the size of problems that may be solved is limited by the capabilities of today’s LMI solvers.

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