Solution to Homework Set VI

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1. Assume $x^* := \mathbb{E}[X] < \infty$. Since $U : \mathbb{R} \mapsto \mathbb{R}$ is concave, there exist $m, b \in \mathbb{R}$ such that $mx^* + b = U(x^*)$ and $mx + b \geq U(x)$ for all $x \in \mathbb{R}$. It follows that

$$\mathbb{E}[U(X)] = \int_\Omega U(X) d\mathbb{P} \leq \int_\Omega (mX + b) d\mathbb{P} = m\mathbb{E}[X] + b = mx^* + b = U(x^*) = U(E[X]).$$

2. As shown in class, the Bellman equation for $H(v,t)$ is

$$\begin{aligned} \partial_t H - \frac{\mu^2}{2\sigma^2}\frac{H_v^2}{H_{vv}} + e^{-\rho t} g^*(-e^{\rho t} H_v) = 0,\ x > 0, t < T; \\ H(v,T) = 0,\ v > 0; \\ H(0,t) = 0,\ t < T, \end{aligned} \tag{1}$$

where $g^*(\cdot)$ is the Legendre transform of $-U(x) = -x^\gamma$ with $0 < \gamma < 1$, i.e.

$$g^*(x^*) := \sup_{x \in \mathbb{R}} \{x^* x + x^\gamma\} = \begin{cases} \infty & \text{if } x^* > 0, \\ (1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}}(-x^*)^{\frac{\gamma}{\gamma-1}} & \text{if } x^* \leq 0. \end{cases}$$

Take $H(v,t) = u(t)v^\gamma$. Then by dirct calculations, (1) becomes the ODE

$$\begin{aligned} u_t - \frac{\mu^2}{2\sigma^2}\frac{\gamma}{\gamma-1} u + (1-\gamma) e^{\frac{\rho t}{\gamma-1}} u^{\frac{\gamma}{\gamma-1}} = 0,\ t < T; \\ u(T) = 0. \end{aligned} \tag{2}$$

Set $w(t) = (u(t))^{\frac{1}{1-\gamma}}$. Then by dirct calculations, (2) becomes

$$\begin{aligned} w_t - \frac{\mu^2}{2\sigma^2}\frac{\gamma}{(\gamma-1)^2} w + e^{\frac{\rho t}{\gamma-1}} = 0,\ t < T; \\ w(T) = 0. \end{aligned} \tag{3}$$

We can solve the ODE (3) explicitly and have

$$w(t) = \frac{2\sigma^2(\gamma-1)^2}{2\sigma^2(\gamma-1)\rho + \mu^2\gamma}\left(e^{\frac{\rho T}{\gamma-1}} e^{\frac{\mu^2\gamma}{2\sigma^2(\gamma-1)^2}(T-t)} - e^{\frac{\rho t}{\gamma-1}}\right),\ t \leq T.$$

We can therefore conclude

$$\begin{aligned} H(v,t) &= u(t)v^\gamma = (w(t))^{1-\gamma} v^\gamma \\ &= \left[\frac{2\sigma^2(\gamma-1)^2}{2\sigma^2(\gamma-1)\rho + \mu^2\gamma}\left(e^{\frac{\rho T}{\gamma-1}} e^{\frac{\mu^2\gamma}{2\sigma^2(\gamma-1)^2}(T-t)} - e^{\frac{\rho t}{\gamma-1}}\right)\right]^{1-\gamma} v^\gamma. \end{aligned}$$

3. In deriving the Bellman equation (1) in problem **2.**, we used the formula for optimal control

$$u_1^* = -\frac{\mu H_v}{\sigma^2 v H_{vv}} > 0.$$

Now suppose $u_1^*(t,v) \leq 1$ on some region $F \subseteq [0,T] \times [0,\infty)$. Then the Bellman equation (1) is satisfied on F. As shown in problem **2.**, the solution is of the form $H(v,t) = u(t)v^\gamma$ on F. However, this implies that for any $(t,v) \in F$,

$$u_1^*(t,v) = -\frac{\mu H_v(v,t)}{\sigma^2 v H_{vv}(v,t)} = -\frac{\mu\gamma u(t) v^{\gamma-1}}{\sigma^2 v \gamma(\gamma-1) u(t) v^{\gamma-2}} = \frac{\mu}{\sigma^2(1-\gamma)} > 1,$$

under the assumption that $\sigma^2(1-\gamma) < \mu$. This contradicts $u^* \leq 1$ on F. Thus, we conclude that in the case where $\sigma^2(1-\gamma) < \mu$, the optimal control u_1^* should be taken as $u_1^* \equiv 1$. Then, the corresponding Bellman equation becomes

$$\begin{aligned} 0 &= H_t + \max_{u_1\in[0,1]} \left\{ u_1 \mu v H_v + \frac{(u_1\sigma v)^2}{2} H_{vv} \right\} + \max_{u_2>0}\{e^{-\rho t}U(u_2) - u_2 H_v\} \\ &= H_t + \frac{\sigma^2}{2} v^2 H_{vv} + e^{-\rho t} g^*(-e^{\rho t} H_v), \end{aligned} \tag{4}$$

where g^* is given as in problem **2.**. Taking $H(v,t) = u(t)v^\gamma$ and $w(t) = (u(t))^{\frac{1}{1-\gamma}}$, we may proceed as what we did in problem **2.** and derive from (4) the following ODE

$$0 = w_t + \gamma\left(\frac{-\sigma^2}{2} - \frac{\mu}{\gamma-1}\right) w + e^{\frac{\rho t}{\gamma-1}}. \tag{5}$$

Note that $H(v,T) = 0$ implies $w(T) = 0$. We can then solve (5) explicitly and obtain

$$w(t) = \frac{2(\gamma-1)}{-\gamma\sigma^2(\gamma-1) - 2\gamma\mu + 2\rho}\left[\exp\left\{\frac{\rho}{\gamma-1}T + \gamma(\frac{-\sigma^2}{2} - \frac{\mu}{\gamma-1})(T-t)\right\} - e^{\frac{\rho}{\gamma-1}t}\right].$$

Thus,

$$\begin{aligned} H(v,t) &= u(t)v^\gamma = (w(t))^{1-\gamma} v^\gamma \\ &= \left(\frac{2(\gamma-1)}{-\gamma\sigma^2(\gamma-1) - 2\gamma\mu + 2\rho}\left[\exp\left\{\frac{\rho}{\gamma-1}T + \gamma(\frac{-\sigma^2}{2} - \frac{\mu}{\gamma-1})(T-t)\right\} - e^{\frac{\rho}{\gamma-1}t}\right]\right)^{1-\gamma} v^\gamma. \end{aligned}$$

Now, we intend to verify that $H(v,t) = M(v,t)$, where

$$M(v,t) := \max_{u_1\in[0,1], u_2>0} \mathbb{E}\left[\int_t^T e^{-\rho s} U(u_2(s))ds \mid V(t) = v\right].$$

Note that $H(v,t)$ is the maximum expected utility when we set $u_1 \equiv 1$. It follows that $H(v,t) \leq M(v,t)$. Thus, we only need to show the opposite inequality. Take arbitrary controls $u_1(\cdot)$ and $u_2(\cdot)$ such that

$$0 \leq u_1(s) \leq 1 \text{ and } u_2(s) > 0, \text{ a.s.}$$

Then the corresponding wealth process $V(\cdot)$ satisfies

$$dV(t) = u_1(t)V(t)[\mu dt + \sigma dB(t)] - u_2(t)dt,$$

where $B(\cdot)$ is a Brownian motion. Now, applying Itô's rule to the process $H(V_\cdot, \cdot)$, we get

$$\begin{aligned} H(V_{T\wedge\tau_n}, T\wedge\tau_n) = H(v,t) &+ \int_t^{T\wedge\tau_n} \left(H_t + (u_1\mu V - u_2)H_v + \frac{1}{2}(u_1\sigma V)^2 H_{vv} \right)(V_s, s)ds \\ &+ \int_t^{T\wedge\tau_n} (u_1\sigma V H_v)(V_s, s)dB(s), \end{aligned}$$

where $\tau_n := \inf\{s \geq t : |V_s| > n\}$, $n \in \mathbb{N}$. It follows that for each $n \in \mathbb{N}$

$$\mathbb{E}[H(V_{T\wedge\tau_n}, T\wedge\tau_n)] = H(v,t) + \mathbb{E}\left[\int_t^{T\wedge\tau_n} \left(H_t + (u_1\mu V - u_2)H_v + \frac{1}{2}(u_1\sigma V)^2 H_{vv} \right)(V_s, s)ds \right]. \tag{6}$$

Under the condition $\sigma^2(1-\gamma) < \mu$, we know that $H(v,t)$ solves (4). Then we see from (4) that we must have

$$0 \geq H_t + u_1\mu V H_v + \frac{1}{2}(u_1\sigma V)^2 H_{vv} + e^{-\rho t}U(u_2) - u_2 H_v.$$

This, together with (6), implies that for each $n \in \mathbb{N}$

$$H(v,t) \geq \mathbb{E}[H(V_{T\wedge\tau_n}, T\wedge\tau_n)] + \mathbb{E}\left[\int_t^{T\wedge\tau_n} e^{-\rho t}U(u_2(s))ds \right].$$

Now we let $n \to \infty$. For the first expectation, we use the dominated convergence theorem and get

$$\lim_{n\to\infty} \mathbb{E}[H(V_{T\wedge\tau_n}, T\wedge\tau_n)] = \mathbb{E}[H(V_T, T)] = 0.$$

Note that we may apply the dominated convergence theorem here because

$$H(v,t) \leq Cv, \text{ for some constant } C > 0,$$

which can be seen from the formula for $H(v,t)$ we derived above and $0 < \gamma < 1$. For the second expectation, we apply the monotone convergence theorem. Finally, we obtain

$$H(v,t) \geq \mathbb{E}\left[\int_t^T e^{-\rho t}U(u_2(s))ds \right]$$

Now by the arbitrariness of u_1 and u_2,

$$H(v,t) \geq \max_{u_1\in[0,1], u_2>0} \mathbb{E}\left[\int_t^T e^{-\rho t}U(u_2(s))ds \right] = M(v,t).$$

4. (a) As mentioned in class, g_β is the solution to

$$\frac{\partial g_\beta}{\partial t} = \mu \frac{\partial g_\beta}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 g_\beta}{\partial x^2}, \; x, t > 0, \tag{7}$$

with $\lim_{t\to 0} g_\beta(x,t) = \beta + x$ for $x > 0$ and $\lim_{x\to 0} g_\beta(x,t) = 0$ for $t > 0$. Now that we know $p(x,t) := \mathbb{P}_x(\tau < t)$ is the solution to (7) with $\lim_{t\to 0} p(x,t) = 0$ for $x > 0$ and $\lim_{x\to 0} p(x,t) = 1$ for $t > 0$, we conclude that

$$w(x,t) := \mathbb{E}[x + \mu t + \sigma B(t) \mid \tau > t] = g_\beta(x,t) - \beta\mathbb{P}_x(\tau > t) = g_\beta(x,t) - \beta + \beta p(x,t)$$

is the solution to (7) with

$$\lim_{t\to 0} w(x,t) = x \text{ for } x > 0; \; \lim_{x\to 0} w(x,t) = 0 \text{ for } t > 0. \tag{8}$$

Now consider $h_1(x,t) := x + \mu t$ and $h_2(x,t) := \int_0^t p(x,s)ds$. It can be easily checked that h_1 is the solution to (7) with $\lim_{t\to 0} h_1(x,t) = x$ for $x > 0$ and $\lim_{x\to 0} h_1(x,t) = \mu t$ for $t > 0$. Moreover, observe that

$$\begin{aligned}
\frac{\partial h_2}{\partial t}(x,t) &= p(x,t) = \int_0^t \frac{\partial p}{\partial t}(x,s)ds = \int_0^t \mu\frac{\partial p}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 p}{\partial x^2}ds \\
&= \mu\frac{\partial}{\partial x}\int_0^t p(x,s)ds + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}\int_0^t p(x,s)ds \\
&= \mu\frac{\partial h_2}{\partial x}(x,t) + \frac{1}{2}\sigma^2\frac{\partial^2 h_2}{\partial x^2}(x,t).
\end{aligned}$$

Thus, h_2 is the solution to (7) with $\lim_{t\to 0} h_2(x,t) = 0$ for $x > 0$ and $\lim_{x\to 0} h_2(x,t) = t$ for $t > 0$. We can therefore conclude that $h(x,t) := h_1(x,t) - \mu h_2(x,t)$ is the solution to (7) with $\lim_{t\to 0} h(x,t) = x$ for $x > 0$ and $\lim_{x\to 0} h(x,t) = 0$ for $t > 0$. Then by (8), we have $w(x,t) = h(x,t)$. It follows that

$$g_\beta(x,t) = \beta[1 - p(x,t)] + h(x,t) = \beta[1 - p(x,t)] + x + \mu t - \mu\int_0^t p(x,s)ds.$$

(b) From the result in (a), for any $t, x > 0$, we have

$$\begin{aligned}
\frac{\partial g_\beta}{\partial x}(x,t) &= -\beta\frac{\partial p}{\partial x}(x,t) + 1 - \mu\int_0^t \frac{\partial p}{\partial x}(x,s)ds, \\
\frac{\partial^2 g_\beta}{\partial x^2}(x,t) &= -\beta\frac{\partial^2 p}{\partial x^2}(x,t) - \mu\int_0^t \frac{\partial^2 p}{\partial x^2}(x,s)ds.
\end{aligned} \tag{9}$$

Note that $p(x,t)$ is by definition decreasing in x and increasing in t. This implies $\frac{\partial p}{\partial x} \leq 0$, $\frac{\partial p}{\partial t} \geq 0$, and thus $\frac{\partial^2 p}{\partial x^2} = \frac{2}{\sigma^2}(\frac{\partial p}{\partial t} - \mu\frac{\partial p}{\partial x}) \geq 0$. Now, we can see from (9) that

$$\frac{\partial g_\beta}{\partial x} > 0, \text{ and } \frac{\partial g_\beta}{\partial x^2} \leq 0, \text{ as } \beta > 0.$$

We therefore conclude that $g_\beta(x,t)$ is a concave increasing function of $x > 0$, provided $\beta > 0$.

5. For any $u_2 \leq \mu/\rho$ and $\beta \leq \mu/\rho - u_2$, consider the two functions $g_{\mu/\rho - u_2}(x,t)$ and $h(x,t) := e^{\rho t}\left[\frac{\mu}{\rho} + x - u_2\right]$. Fix $x \geq u_2$. Observe that at $t = 0$,

$$g_{\mu/\rho - u_2}(x,0) = \frac{\mu}{\rho} - u_2 + x = h(x,0).$$

From the formula of $g_\beta(x,t)$ and the properties of $p(x,t)$ in problem **4.**, we have

$$\frac{\partial}{\partial t} g_{\mu/\rho-u_2}(x,t) = -\left(\frac{\mu}{\rho} - u_2\right)\frac{\partial p}{\partial t}(x,t) + \mu - \mu p(x,t) \le \mu, \text{ for all } t \ge 0.$$

On the other hand,

$$\frac{\partial}{\partial t} h(x,t) = \rho e^{\rho t}\left[\frac{\mu}{\rho} + x - u_2\right] = e^{\rho t}[\mu + \rho(x-u_2)] \ge \mu, \text{ for all } t \ge 0.$$

Since $g_{\mu/\rho-u_2}(x,0) = h(x,0)$ and $\frac{\partial}{\partial t} g_{\mu/\rho-u_2}(x,t) \le \mu \le \frac{\partial}{\partial t} h(x,t)$ for all $t \ge 0$, we conclude that

$$g_\beta(x,t) \le g_{\mu/\rho-u_2}(x,t) \le h(x,t) = e^{\rho t}\left[\frac{\mu}{\rho} + x - u_2\right], \text{ for all } t \ge 0.$$

We therefore have $e^{-\rho t} g_\beta(x,t) \le \mu/\rho + (x - u_2)$, as $x \ge u_2$.

6. Set $y = -z > 0$. Define $h(y,t) := H^*(-y,t)$ for $y > 0$, $0 \le t \le T$. Then the PDE for H^*

$$H_t^* + z^2 H_{zz}^* + f^*(t,z) = 0 \text{ for } z < 0,\ 0 \le t \le T, \text{ with } H^*(z,T) = 0$$

becomes the PDE for h

$$h_t + y^2 h_{yy} + g(t,y) = 0 \text{ for } y > 0,\ 0 \le t \le T, \text{ with } h(y,T) = 0, \tag{10}$$

where $g(t,y) := f^*(t,-y)$. Note that the convexity of f^* in z implies the convexity of g in y. Indeed, for any $y_1, y_2 > 0$ and $\lambda \in [0,1]$,

$$\begin{aligned} g\left(t, \lambda y_1 + (1-\lambda) y_2\right) &= f^*\left(t, \lambda(-y_1) + (1-\lambda)(-y_2)\right) \le \lambda f^*(t,-y_1) + (1-\lambda) f^*(t,-y_2) \\ &= \lambda g(t,y_1) + (1-\lambda) g(t,y_2). \end{aligned} \tag{11}$$

By using the Feynmann-Kac representation theorem, the solution of (10) can be expressed as

$$h(y,t) = \mathbb{E}\left[\int_t^T g(s, Y_s) ds\right],$$

where the process Y_s satisfies $dY_s = \sqrt{2} Y_s dB_s$ for $s \ge t$ and $Y_t = y$, which can be solved explicitly as

$$Y_s = y \exp\{(t-s) + \sqrt{2}(B_s - B_t)\} =: y\mathcal{E}(s).$$

Now, for any $y_1, y_2 > 0$ and $\lambda \in [0,1]$, we have, thanks to the convexity of g in y, that

$$\begin{aligned} h(\lambda y_1 + (1-\lambda) y_2, t) &= \mathbb{E}\left[\int_t^T g(s, (\lambda y_1 + (1-\lambda) y_2)\,\mathcal{E}(s)) ds\right] \\ &\le \lambda \mathbb{E}\left[\int_t^T g(s, y_1 \mathcal{E}(s)) ds\right] + (1-\lambda)\mathbb{E}\left[\int_t^T g(s, y_2 \mathcal{E}(s)) ds\right] \\ &= \lambda h(y_1,t) + (1-\lambda) h(y_2,t). \end{aligned}$$

This shows the convexity of h in y. Finally, since $H^*(z,t) = h(-z,t)$, the convexity of h in y implies the convexity of H^* in z, by the same calculation as (11).