Curvature behaviours at extraordinary points of subdivision surfaces

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Abstract

During the development of subdivision surface methods one of the important questions has been the degree of continuity of the limit surface. In particular whether continuity of curvature can be achieved at the extraordinary points. However, there are several different curvature behaviours, not just two, and this note demonstrates them by examples.

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1. Introduction

The curvature behaviour around an extraordinary point of a subdivision surface has been the subject of much analysis. It is often thought that there are just two behaviours that are of interest; C2 or not C2, but it transpires that there are at least seven different behaviours which may occur. The acceptability of each of these for various graphical and design purposes remains an open question, and so this note describes each of the known behaviours, so that discussion of their acceptability can proceed.

The possibilities so far known are:

1. divergent curvature;
2. mildly divergent curvature;
3. local spike;
4. well defined but discontinuous curvature;
5. locally zero curvature;
6. bounded but undefined curvature;
7. full C2 continuity.

This note describes each of them in turn, giving for each at least one known subdivision scheme with that particular behaviour and an example as a closed form univariate equation. These examples are contrived illustrations, not necessarily related to the subdivision scheme (which may be much more complex), but devised to illustrate the behaviour simply.

For each of the interesting cases the graph of the example is shown, together with the graph of the second derivative. The differences in the latter are far more visible than the differences in shape of the former.

We use the standard notation of $\lambda$ and $\mu$ to denote the subdominant and the subsubdominant eigenvalues, respectively, where $\lambda$ comes from the odd component of the configuration and $\mu$ from the even component. We consider only C1 subdivisions, and assume that the coordinate system is chosen so that the origin is at the point we are considering and that the tangent to the curve lies along the $x$-axis.

For bivariate cases, we assume that the tangent plane to the surface is taken as the $xy$-plane, and we look at an $x-z$ section.

2. The behaviours

2.1. Divergent curvature

This situation occurs in a modified Chaikin subdivision when the new points are created not at 1/4 and 3/4 of each edge, but at places near the middle, and also in the standard\textsuperscript{1} Catmull–Clark surface formulation [2] for $n > 4$, $\omega = 0$.

The eigenanalysis then has $|\mu| > \lambda^2$, and so we find that estimators of curvature are multiplied by a constant factor at each step in the subdivision.

\textsuperscript{1} The original Catmull–Clark paper described two variants, differing in the weights used for the new point corresponding to the extraordinary point.
The limit curve for $x > 0$ has a behaviour in the neighbourhood of $x = 0$ analogous to:

$$y = ax^\rho$$

where $\rho$ has a fractional value between 1 and 2.

Fig. 1 shows the graph of $y = |x|^{1.75}$ and its second derivative.

### 2.2. Mildly divergent curvature

This situation occurs in the four-point scheme. It was shown in Ref. [4] that this scheme does not have a second derivative anywhere unless the initial configuration happened to have sufficient consecutive points lying on a cubic. At each subdivision, estimators of curvature have a constant term added to them.

This behaviour happens despite the fact that $\mu = \lambda^2$, and is related to the fact that there are two eigenvalues of this value, with the same eigenvector, giving a geometric degeneracy.

The limit curve for $x > 0$ has a behaviour analogous to:

$$y = ax^2 \log x$$

in the neighbourhood of $x = 0$.

It is interesting to note that the equation of the thin plate spline [6] has basis functions of exactly this form, and one would be tempted to deduce that this behaviour can therefore be regarded as smooth for practical purposes.

Fig. 2 shows the graph of $y = x^2 \log|x|$ and its second derivative.

### 2.3. Large local ‘spike’ of curvature

This situation occurs in the four-point ternary interpolating curve scheme described in Ref. [7]. It is technically continuous in curvature, but has a large local spike of curvature at the control point.

$\mu = \lambda^2$, but there is another eigenvalue only just smaller than $\mu$ and with a very similar eigenvector.

The limit curve for $x > 0$ has a behaviour in the neighbourhood of $x = 0$ analogous to:

$$y = \frac{ax^2 - a(1 - 2\epsilon)x^{2+\epsilon}}{\epsilon}$$

where $\epsilon$ is small.

To a first order in $\epsilon$ the second derivative is $(2a - (2 - \epsilon)ax^\epsilon)/\epsilon$ which is approximately $a$ for $x > 0$ and $2a/\epsilon$ at $x = 0$.

Case 2 can usefully be thought of as the limit as $\epsilon \to 0$ of this behaviour.

Fig. 3 shows the graph of $y = 5x^2 - 4|x|^{2.25}$ and its second derivative.

### 2.4. Well-defined but discontinuous curvature

This occurs in the Chaikin scheme [3], where all estimators of curvature on the two sides of the midpoint of an original edge converge cleanly and unambiguously, but to different values on the two sides. It also occurs in the Doo–Sabin surface scheme [3], where the estimator of radial curvature is a function of radial direction.

In the eigenanalysis we find that an eigenvector corresponding to the eigenvalue $\mu$ gives curvature estimators differing on the two sides. Typically there is also another $\mu$ eigenvector and the space spanned by the two contains a vector which gives normal curvature behaviour.

In the univariate case we expect the odd part to contribute the discontinuity and the even part to contribute to continuous curvature.
The limit curve has a behaviour analogous to
\[ y = ax^2 + bx|x| \]
in the neighbourhood of \( x = 0 \).

In fact this issue is more or less orthogonal to the issue covered by cases 1, 2 and 3, since in each of those cases it is quite possible for the estimators on the two sides to be equal at each level of subdivision. In such cases we could argue, as Ball and Storry [1] did, that the curvature is continuous even when it is unbounded or undefined.

Fig. 4 shows the graph of
\[ y = x^2 + 2x|x|(1 - |x|)/11 \]
and its second derivative.

2.5. Zero curvature

This occurs in a modified Chaikin subdivision when the new points are created not at 1/4 and 3/4 of each edge, but at places nearer the ends, and also in the original2 Catmull Clark surface formulation [2] for \( n > 4 \), \( \omega = 0 \).

The eigenanalysis then has \( |\mu| < \lambda^2 \) and so we find that estimators of curvature are multiplied by a constant factor < 1 at each step in the subdivision.

In the surface case, with isolated singularities, ringed by analytic pieces, it is probably safe to deduce that the behaviour is analogous to
\[ z = ax^\rho \]
where \( \rho \) has a value greater than 2, but in the curve case we find that we would have to deduce that the curvature was zero at every rational point. These are dense, but the limit curve is still capable of getting round corners. We must have something analogous to the second integral of the Cantor function, zero at all rationals and unit-valued at all other abscissae. We can hardly claim that this is continuous in curvature.

Recently surface forms have been proposed [9] which do not have analytic rings, but are essentially fractal. Flat-spots on these cannot safely be called C2 without very careful analysis of the behaviour at the regular points.

Fig. 5 shows the graph of \( y = |x|^{2.25} \) and its second derivative.

2.6. Bounded but undefined curvature

This situation occurs in the Catmull–Clark surface scheme as modified in Ref. [10].

Again \( \mu = \lambda^2 \). The complication arises due to the fact that the subsubdominant eigenvector defines surface rings which do not lie within a quadratic surface. The scheme

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2 See footnote to case 1.
does not have quadratic precision, and so the region round the extraordinary point cannot have constant curvature. However, the slightly varying curvatures in each ring are echoed exactly in the next ring in, and in the next, and indefinitely at finer and finer scales.

A normal section through the limit surface and passing through the extraordinary point has a behaviour analogous to

\[ y = x^2(1 + 0.1 \cos(2\pi \log_2 |x|)) \]

in the neighbourhood of \( x = 0 \).

We cannot call this behaviour continuous curvature, but if \( b \ll a \) the small amount of undefinedness may not be important for any practical purpose.

**Fig. 6** shows the graph of \( y = x^2(1 + 0.1 \cos(2\pi \log_2 x)) \) and its second derivative.

### 2.7. Full C2 continuity

This can be achieved by an interpolating six-point scheme mentioned in Ref. [4] and, of course, by the schemes derived from the cubic and higher B-splines.

The limit curve has a behaviour analogous to

\[ y = ax^2 \]

in the neighbourhood of \( x = 0 \).

**Fig. 7** shows the graph of \( y = x^2 \) and its second derivative.

In the surface case we require that \( z = ax^2 + bxy + cy^2 \) locally (plus higher powers of \( x \) and \( y \) with total power greater than 2), which can be expressed in polar coordinates as

\[ z = r^2\phi(\theta) \]

where \( \phi(\theta) = a + \beta \cos(2\theta) + \gamma \sin(2\theta) \), plus terms with powers of \( r \) (possibly fractional) higher than 2, which can involve higher multiples of \( \theta \).

### 3. Discussion

For graphical purposes, we suggest, curvature behaviour is almost irrelevant. As may be seen from the graphs of the examples above, there is not much difference in shape between them. Even when surfaces are shown as shaded and illuminated, it is hard to see point curvature defects. Even reflection lines, which have discontinuities of first derivative where they cross surface discontinuities of second derivative, may not show up the defect unless a particular edge actually crosses the singular point, and so it may not be essential for a surface to have no such defects to meet stringent aesthetic quality standards.

For analysis by the finite element or related methods, what is important is the boundedness of certain integrals over a complete element, and many subdivision schemes do indeed lead to bounded energy integrals even though the curvatures are unbounded.

For machining of dies and moulds there is a much tighter constraint, that at all points the curvature (expressed in the form of the Dupin indicatrix) should everywhere lie outside the indicatrix of the cutter. In principle this demands bounded curvature everywhere, with the ability to evaluate the bounds. However, actual generation of cutter paths always uses a discrete representation of the cutter path (if only because machine controllers have digital sampled-data controllers) and so a sufficiently narrow spike of high curvature may not lead to the theoretical gouging actually happening in practice.

For the building of shapes by material deposition (layer manufacture), all that is required is that cross-section curves
can be computed. This is no problem, whatever the curvature behaviour, even if cross-sections pass through extraordinary points.

There would seem to be scope for considerably more informed discussion of the actual requirements for curvature behaviours.

References


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Malcolm Sabin offers consultancy in geometric computing through his company, Numerical Geometry Ltd. His early work was in representation of aircraft shapes in the late 1960s, when he developed one of the first parametric surface systems (NMG) at British Aerospace, and he was one of the inventors of subdivision surfaces in the late 1970s. He has also been concerned with NC part-programming systems and CNC control systems software, with scattered data contouring methods, and with mesh generation for finite element analysis. In recent years he has contributed chapters to the Handbook of CAGD, to ACTA Numerica and to the Primus workshop proceedings of the MINGLE project, and has given tutorials on subdivision at the Shape Modelling and Applications conferences. His current research interest is in making subdivision surface techniques as useful and as usable as possible before they become the next defacto standard method for representation of sculptured surfaces.

Neil Dodgson is a senior Lecturer in the University of Cambridge Computer Laboratory, where he teaches courses in computer graphics and information theory. He co-leads the Rainbow Research Group, with interests in graphics, interaction, and display technology. His recent research has been in two areas: 3D modelling, particularly subdivision surfaces, and autostereoscopic 3D displays. Dr. Dodgson completed his BSc in Computer Science and Physics at Massey University in New Zealand, followed by a PhD in Computer Science at the Cambridge Computer Laboratory. He joined the permanent staff of the Lab in 1995 and has subsequently supervised ten PhD students in the areas of graphics and imaging. Dr. Dodgson is a Chartered Electrical Engineer, a Member of the IEE, and a Fellow of Emmanuel College, Cambridge.

Ioannis Ivrissimtzis was born in Greece in 1970. He studied mathematics at the University of Thessaloniki and the University of Southampton. Between 2000 and 2001 he worked as a Research Associate in the University of Cambridge, supported by the European Union, under the aegis of the MINGLE project (HPRN-CT-1999-00117). Currently, he is working as a PostDoctoral Researcher at the Max-Planck-Institute für Informatik in Saarbrücken, Germany. His main research interest is in Multiresolution Geometry, in particular, recursive subdivision, and mesh connectivity encoding. Other mathematical interests include discrete groups (especially Hecke groups), the theory of dessins, cyclotomic fields, algebraic K-Theory, and Logic and Foundations.