ON SUMMABILITY OF EIGENFUNCTION EXPANSIONS OF PIECEWISE SMOOTH FUNCTIONS

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Abstract

We consider two forms of eigenfunction expansions associated with an arbitrary elliptic differential operator with constant coefficients and order $m$, that is the multiple Fourier series and integrals. For the multiple Fourier integrals we prove the convergence of the Riesz means of order $s > (N - 3)/2$ of piecewise smooth functions of $N \geq 2$ variables. The same result is proved in the case of the $N \geq 3$ dimensional multiple Fourier series.
1 Introduction

Let \( A(D) \) be an arbitrary elliptic differential expression with constant coefficients and order \( m \):

\[
A(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha.
\]

If we consider \( A(D) \) on \( L_2(\mathbb{R}^N) \) with the domain of definition \( C_0^\infty(\mathbb{R}^N) \), i.e. infinitely differentiable functions with a compact support, then we will have essentially self-adjoint operator (see, for example [1] or [2]). Let us denote a unique self-adjoint extension of this operator by \( A_R \). According to the spectral theorem, to every function \( f(x) \in L_2(\mathbb{R}^N) \) we may associate an eigenfunction expansion, using the corresponding spectral family. In this case it coincides with the Fourier expansion and has the following form:

\[
E_\lambda f(x) = (2\pi)^{-N/2} \int_{A(\xi) < \lambda} \hat{f}(\xi)e^{ix\xi}d\xi,
\]

(1)

where \( \hat{f}(\xi) \) is the Fourier transform of \( f(x) \) and \( A(\xi) \) is the symbol of the expression \( A(D) \), i.e.

\[
A(\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha.
\]

If we consider the elliptic differential expression \( A(D) \) on \( L_2(T^N) \),

\[
T^N = \{ x \in \mathbb{R}^N, -\pi \leq x_j \leq \pi, j = 1, 2, ..., N \},
\]

with the domain of definition \( C^\infty(T^N) \), i.e. infinitely differentiable functions, which are \( 2\pi \)-periodical on each variable \( x_j \), then again we will have an essentially self-adjoint operator. Now we denote the unique self-adjoint extension of \( A(D) \) by \( A_T \). In this case the spectral resolution of any function \( q \in L_2(T^N) \) coincides with the multiple Fourier series and has the following form

\[
\sigma_\lambda q(x) = \sum_{A(\alpha) < \lambda} q_\alpha e^{inx},
\]

(2)

where \( q_\alpha \) - the Fourier coefficients of \( q(x) \).

In this paper we will investigate the Riesz summability of eigenfunction expansions of piecewise smooth functions. The Riesz means of \( E_\lambda f \) are defined for any real number \( s \) as

\[
E_\lambda^s f(x) = (2\pi)^{-N} \int_{A(\xi) < \lambda} (1 - \frac{A(\xi)}{\lambda})^s \hat{f}(\xi)e^{ix\xi}d\xi.
\]

Note, here we allow the index \( s \) to be negative. The Riesz means of \( \sigma_\lambda q \) of order \( s \geq 0 \) are defined in the same way

\[
\sigma_\lambda^s q(x) = \sum_{A(\alpha) < \lambda} (1 - \frac{A(\alpha)}{\lambda})^s q_\alpha e^{inx}.
\]

Definition 1.

A function \( f(x) \) is said to be piecewise smooth if it has the following form

\[
f(x) = \chi_D(x)g(x), \quad g(x) \in C^\infty(\mathbb{R}^N),
\]
where $\chi_D(x)$ is a characteristic function of an open domain $D$ with a smooth boundary $\Gamma$ (in the case of the multiple Fourier series we assume that $D \subset T^N$).

In the one dimensional case, $N = 1$, the Riemann localization principle states that both the Fourier integrals (1) and series (2) of a piecewise smooth function converge at every point where the function under consideration is smooth. In 1956 V.A.II’in [3] proved that this principle remains true in case of $N = 2$ too. Namely, he proved that the eigenfunction expansion of a piecewise smooth function associated with the Laplace operator in an arbitrary two-dimensional domain converge uniformly on every compact subset where the function is smooth.

When $N > 2$ this principle is not valid any more. The expansions (1) and (2) may diverge even at points, where the function being expanded is smooth. Indeed, as a simple explicit example we note that the spherical partial integrals of a characteristic function of a ball in $R^3$ diverges at its center, staying bounded and even diverges to infinity in $R^N$ when $N > 3$. This result for the expansions (2) in three dimensional case is due to Taylor [12].

Eigenfunction expansions of piecewise smooth functions associated with elliptic operators of second order have been intensively investigated by many authors for last ten years (see [6-17]).

We first pay attention to one particular result of work [17]. In case of the Laplace operator the authors proved uniformly convergence of the Riesz means of order $s > (N - 3)/2$ in every compact set disjoint from the discontinuities of a piecewise smooth function being expanded. So if $N = 2$, then we have $s > -1/2$, i.e. the index $s$ is allowed to be negative.

But for $N \geq 3$ the above result follows from the localization principle for the Laplace operator. Indeed, for the expansion (1) the localization principle holds in Liouville spaces $L^1_l(R^N)$ when $l + s > (N - 1)/2$ (see [2]). On the other hand a piecewise smooth function is in Nikolskii space $H^1_l(R^N)$ and according to the imbedding theorem it is also in the spaces $L^1_l(R^N)$ for any $l < 1$.

Thus if $s > (N - 3)/2$ then $1 - s + (N - 3)/2 < 1$ and one can choose $l > 1 - s + (N - 3)/2$ i.e. $l + s > (N - 1)/2$.

We also note, that in most of papers [4-16] a piecewise smooth function was defined as in Definition 1 or as a linear combination of such functions. In the above work [17] for elliptic operators of second order on $N$- dimensional smooth manifolds it was considered the eigenfunction expansions of piecewise smooth functions, which have not a simple jump across $\Gamma$ as in Definition 1, but have integrable singularities of order $\alpha > -1$ along the surface $\Gamma$. For more on the eigenfunction expansions of singular piecewise functions see Taylor [13].

Elliptic operators of higher order than 2 were considered by Sh. Alimov [4,5]. In [4] the author proved the result of II’in [3] for expansions (1), associated with an arbitrary elliptic operator of order $m$ with constant coefficients in $R^2$. As we mentioned above, when $N > 2$ expansions of piecewise smooth functions may diverge even at the points where the function under consideration is smooth. In [5] for the Fourier expansions (1) of piecewise smooth function associated with an elliptic operator of order $m$ the sets of uniform convergence are described.
2 The Main Results

In this section we formulate the main results of the paper. Let us denote by $A_0(\xi)$ the principal symbol of the differential expression $A(D)$:

$$A_0(\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha.$$ 

Observe that $A_0(\xi)$ is a homogeneous polynomial. We introduce the set

$$\Lambda = \{ \xi \in \mathbb{R}^N, \; A_0(\xi) < 1 \}. \quad (3)$$

**Theorem 1.**

Let $N \geq 3$ and the set $\Lambda$ be convex. Then for every piecewise smooth function $f$ with the surface of discontinuity $\Gamma$ the Riesz means $E_\lambda^s f(x)$ of order $s = (N - 3)/2$ are uniformly bounded on each compact set $K \subset \mathbb{R}^N \setminus \Gamma$.

If $s > (N - 3)/2$ then

$$\lim_{\lambda \to +\infty} (2\pi)^{-N} \int_{A(\xi) > \lambda} (1 - \frac{A(\xi)}{\lambda})^s \hat{f}(\xi) e^{ix\xi} d\xi = f(x)$$

uniformly on $x \in K \subset \mathbb{R}^N \setminus \Gamma$.

The second part of this theorem for the Laplace operator was proved by many authors (see [10], [17] and references therein). But, as we mentioned above, in this case it is a simple consequence of the known localization principle.

In the proof of this theorem we use a result of B. Randol [18] on the oscillatory integrals over convex sets. For this reason we need the convexity of the set (3). However instead of this we may demand convexity of the domain $D$ from Definition 1 and consider an arbitrary elliptic operators $A(D)$.

**Theorem 2.**

Let the domain $D$ from Definition 1 be convex with an analytic surface $\Gamma$, $N \geq 3$ and $A(\xi)$ be an arbitrary elliptic polynomial with constant coefficients. Then the statements of Theorem 1 hold.

If $N = 2$ then we have the same result with [17] for negative order of the Riesz means $E_\lambda^s$ associated with an arbitrary homogeneous elliptic operator of order $m$.

**Theorem 3.**

(i) Let $N = 2$ and $A(D)$ be an arbitrary homogeneous elliptic differential operator. Then for any piecewise smooth function $f$ with the surface of discontinuity $\Gamma$ the Riesz means $E_\lambda^s f(x)$ of order $s > -1/2$, defined as in (1), uniformly converge on $x \in K \subset \mathbb{R}^2 \setminus \Gamma$.

(ii) If $s = -1/2$ then the corresponding Riesz means are uniformly bounded on each compact set $K \subset \mathbb{R}^2 \setminus \Gamma$.

For simplicity in Theorem 3 we consider homogeneous operators. However, we believe that a similar result holds also for non homogeneous operators, since the proof of Theorem 3 is based
on an estimate of B. Randol [19] and this estimate for the Fourier transform of the indicator function of a planar set is true in the general case. We also note, that this result for the eigenfunction expansions (1) i.e. for the case of $s = 0$, was proved by Alimov [4].

In the proof of Theorem 3 we use the following result on localization principle. Since, it seems to us, it has some interest itself, we formulate it as a separate proposition.

**Proposition 1.**

Let $N = 2$ and $A(D)$ be an arbitrary homogeneous elliptic differential operator. Let $s > -\frac{1}{2}$ and $l + s \geq \frac{1}{2}$. Then for the Riesz means $E^s_\lambda$ of any function $f \in L^1_2(R^2)$ the localization principle holds, i.e. uniformly on $x \in K \subset R^2 \setminus \text{supp} f$

$$\lim_{\lambda \to +\infty} E^s_\lambda f(x) = 0 .$$

For $s \geq 0$ this is a known result on localization (see [2] and references therein).

Now we turn to the multiple Fourier series (2). The surface $\Gamma$ is said to be *strictly convex* if the Gaussian curvature is positive at every point of this surface.

**Theorem 4.**

Let $N \geq 3$ and the surface $\Gamma$ from Definition 1 be strictly convex. Let $A(D)$ be an arbitrary elliptic differential expression. Then for every piecewise smooth function $q$ with the surface of discontinuity $\Gamma$ the Riesz means $\sigma^s_\lambda q(x)$ of order $s > (N-3)/2$ uniformly converge on $x \in K \subset T^N \setminus \Gamma$:

$$\lim_{\lambda \to \infty} \sum_{A(n) < \lambda} (1 - \frac{A(n)}{\lambda})^s q_n e^{inx} = q(x).$$

If $s = (N-3)/2$ then the corresponding Riesz means are uniformly bounded on each compact set $K \subset T^N \setminus \Gamma$.

In this form the result for the Laplace operator is proved by Alimov [7] without any conditions on the surface $\Gamma$.

### 3 On the Randol estimate

We recall, that $A_0(\xi)$ is the principal symbol of the operator $A(D)$. Denote by $B_\lambda(\xi), \lambda > 1$, the following polynomial

$$B_\lambda(\xi) = A_0(\xi) + \sum_{|\alpha| < m} a_\alpha \xi^\alpha \lambda^{-m-|\alpha|}.$$ 

Observe that $B_\lambda(\xi)$ is a polynomial of order $m$ on $\varepsilon = \frac{1}{\lambda}$ and if $\varepsilon = 0$, then we have $A_0(\xi)$.

In the present section we investigate the oscillatory integral

$$v(x, \lambda) = \int_{B_\lambda(\xi) = 1} F(\xi) e^{i\lambda x \xi} d\sigma(\xi) ,$$

with a smooth function $F(\xi)$. Throughout the paper by $d\sigma(\cdot)$ we denote the area element on a surface.
Lemma 1.

Let $\Lambda$, defined by (3), be convex and $H(\alpha)$ be the Gaussian curvature of the surface $\partial \Lambda$ at $\alpha$. Then for any $\lambda > 1$ uniformly on $x \in K \subset R^N \setminus \{0\}$ the estimate

$$|v(x, \lambda)| \leq c\lambda^{-(N-1)/2} H^{-1/2}(\alpha(\theta)),$$

holds. Here $\alpha(\theta)$ is a point from the surface $\partial \Lambda$ where the exterior normal to $\partial \Lambda$ has direction $\theta = \frac{x}{|x|}$.

Proof. Suppose that $\epsilon > 0$, and let $\chi(\xi)$ be a nonnegative $C^\infty$ function in $R^N$, which is supported in the ball $|\xi| \leq \epsilon$, and which is identically 1 in a neighborhood of the origin. For each $\theta \in S^{N-1}$, introduce functions $\chi_1(\theta, \xi)$ and $\chi_2(\theta, \xi)$ on the level surface $\partial C \equiv \{\xi \in R^N; B_\Lambda(\xi) = 1\}$, by defining $\chi_1(\theta, \xi)$ to be the restriction to $\partial C$ of $\chi(\xi - \alpha(\theta))$ and $\chi_2(\theta, \xi)$ to be the restriction to $\partial C$ of $\chi(\xi - \alpha(-\theta))$. Then if $\epsilon$ is sufficiently small, by the method of stationary phase, one has

$$v(x, \lambda) = \sum_{j=1}^2 \int_{\partial C} F(\xi) \chi_j(\theta, \xi) e^{i\lambda x \xi} d\sigma(\xi) + R(\lambda|x|, \theta),$$

where $R(r, \theta) = O(r^-k)$, $r > 1$, uniformly in $\theta$ for any fixed $k$. Although, according to the stationary phase method, one should consider neighborhoods of slightly different points than $\alpha(\theta)$ and $\alpha(-\theta)$, since we have the level surface $\partial C$ and not $\partial \Lambda$, but when $\lambda$ is large enough this does not make any difference.

Both integrals can be estimated in the same way. Consider, for example, the first of them, which we denote by $v_1(x, \lambda)$.

Introduce polar coordinates $(\rho, \phi)$ in the $(N - 1)$-dimensional hyperplane tangent to $\partial C$ at $\alpha(\theta)$, placing the origin at $\alpha(\theta)$. Then

$$v_1(x, \lambda) = \int_{S^{N-2}} \rho^{N-2} f(\theta, \phi, \rho, \lambda) e^{i\lambda x \rho} d\sigma(\phi),$$

where $f(\theta, \phi, \rho, \lambda)$ is for each $\theta$ infinitely differentiable with support in the ball $\rho \leq \epsilon$, and the partial derivatives of $f(\theta, \phi, \rho, \lambda)$ with respect to $\rho$ can be bounded independently of $\theta, \phi$ and $\lambda$. The phase function $E(\theta, \phi, \rho, \lambda)$ has the Taylor power series on $\epsilon = \frac{1}{\lambda}$ of the form $d(\theta) - \psi(\theta, \phi, \rho) + \lambda^{-1} \varphi(\theta, \phi, \rho, \lambda)$, where $d(\theta)$ is the distance from the origin in $R^N$ to the support plane to $\partial C$ at $\alpha(\theta)$, the summand $\psi(\theta, \phi, \rho)$ corresponds to the level surface $\partial \Lambda$ according to the implicit function theorem and the function $\varphi(\theta, \phi, \rho, \lambda)$ appears because of the perturbation.
of the surface $\partial \Lambda$ and it is a $C^\infty$ function, the partial derivatives of $\varphi(\theta, \phi, \rho, \lambda)$ with respect to $\rho$ are bounded independently of $\theta, \phi$ and $\lambda$. If we define

$$f_1(\theta, \phi, \rho, \lambda) = f(\theta, \phi, \rho, \lambda)e^{ix|\varphi(\theta, \phi, \rho, \lambda)}$$

then the partial derivatives of $f_1$ have the same properties as $f$, and the phase function for the oscillatory integral $v_1(x, \lambda)$ will be the same as in [18], i.e. $d(\theta) - \psi(\theta, \phi, \rho)$, therefore following Randol (see Estimate (3) and Lemma 2 in [18]) we have

$$|v_1(x, \lambda)| \leq c\lambda^{-(n-1)/2}H^{-1/2}(\alpha(\theta)).$$

A similar estimate is valid for the second integral in (5). Observe that, since the polynomial $A_0(\xi)$ is homogeneous, $H(\alpha(\theta)) = H(\alpha(-\theta))$. Lemma 1 is proved.

Now we want to show that the integral

$$I = \int_{S_{N-1}} \frac{d\sigma(\theta)}{H^{1/2}(\alpha(\theta))}$$

is bounded. First change the variables: $\eta : S_{N-1} \to \partial \Lambda; \eta = \alpha(\theta).$ Observe, $\alpha(\theta)$ is the inverse of the Gaussian normal map. The fact that $\alpha(\theta)$ is 1-1 mapping follows immediately from the convexity of $\Lambda$, and the analyticity of the surface $\partial \Lambda.$ The Jacobian $J(\eta)$ for the map $\theta = \alpha^{-1}(\eta)$, i.e. for the Gaussian normal map is the Gaussian curvature and it means $J(\eta) = H(\eta)$ (see, e.g., Chapter 6, Theorem 8.4 in [24]). Thus, following Randol [18, Proof of Theorem 1], we will have

$$I = \int_{\partial \Lambda} \frac{J(\eta)d\sigma(\eta)}{H^{1/2}(\eta)} = \int_{\partial \Lambda} H^{1/2}(\eta)d\sigma(\eta) < \text{const.} \quad (6)$$

We should note, that in fact, in [18], using the analyticity of the surface $\partial \Lambda$, it is proved that $H^{-1/2}(\alpha(\theta))$ is in $L_p(S_{N-1})$ for some $p > 2$.

### 4 Proofs of Theorems 1, 2 and 4

We prove all four theorems by the method of Il’in [3], based on representability of a piecewise smooth function as a double layer potential.

Let $\zeta(\xi)$ be a $C^\infty$ function, $\zeta(\xi) = 0$ if $|\xi| \leq 1/2$ and $\zeta(\xi) = 1$ when $|\xi| \geq 1$. Define a function $P(x)$ as

$$P(x) = (2\pi)^{-N} \int_{RN} \frac{\zeta(\xi)}{|\xi|^2} e^{ix\xi} d\xi,$$

where the integral converges in the sense of the distributions theory. A simple calculation shows, that if $N \geq 3$, then

$$P(x) = \alpha_N |x|^{2-N} + \nu_N(x), \; \nu_N \in C^\infty(R^N),$$

and if $N = 2$, then

$$P(x) = \alpha_2 \log \frac{1}{|x|} + \nu_2(x), \; \nu_2 \in C^\infty(R^2),$$

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with some constants \( \alpha_2 \) and \( \alpha_N \). Moreover, since for an arbitrary multi-index \( \beta \)
\[
x^\beta P(x) = (2\pi)^{-N} \int_{\mathbb{R}^N} e^{inx} \left( i \frac{\partial}{\partial \xi} \right)^\beta \frac{\zeta(\xi)}{|\xi|^2} d\xi ,
\]
for any fixed \( k \) we may state
\[
P(x) = O(|x|^{-k}) , \quad |x| \to \infty .
\]

Let \( f \) be a piecewise smooth function from Definition 1. Denote by \( g(x) \) the jump of \( f \) across
the surface \( \Gamma \). Consider a double layer potential
\[
f_1(x) = \int_{\Gamma} \frac{i}{\gamma} P(x - y) g(y) d\sigma(y) ,
\]
where \( \gamma = \gamma(y) \)-an interior normal to \( \Gamma \) at point \( y \). It is well known (see, e.g. [20]), that \( f_1(x) \)
has a jump, proportional to \( g(x) \), across the surface \( \Gamma \). Therefore the function \( f(x) - \beta_N f_1(x) \)
with the appropriate constant \( \beta_N \) is continuous on \( \mathbb{R}^N \) and belongs to Liouville space \( L^{1_2}(\mathbb{R}^N) \).
Hence for the Riesz means \( E^s f \) of order \( s > (N - 3)/2 \) of the function \( f(x) - \beta_N f_1(x) \) one has
the localization (see, e.g. [2]).

Thus, according to the equality
\[
E^s f(x) = \beta_N E^s f_1 + E^s [f - \beta_N f_1](x)
\]
it is sufficient to prove the convergence of the Riesz means \( E^s f_1 \).

Since, by the definition,
\[
f_1(x) = (2\pi)^{-N} \int_{\mathbb{R}^N} \frac{\zeta(\xi) e^{inx}}{|\xi|^2} \int_{\Gamma} e^{-iy\xi(-i\xi, \gamma(y))} g(y) d\sigma(y)
\]
the partial integrals \( E_\lambda f_1(x) \) have a simple form
\[
E_\lambda f_1(x) = (2\pi)^{-N} \int_{A(\xi) < \lambda} \frac{\zeta(\xi) e^{inx}}{|\xi|^2} \int_{\Gamma} e^{-iy\xi(-i\xi, \gamma(y))} g(y) d\sigma(y) .
\]

To show the convergence of the Riesz means \( E^s f_1(x) \) we need some estimates for them for
large \( \lambda \). One could obtain these estimates straightforwardly, but it is convenient for us to make
use here of the following Tauberian theorem of L. Hörmander (see [21], Theorem 2.7). Since
we shall not use any interpolation, we formulate this theorem for the Riesz means of real order
only.

**Theorem (L. Hörmander).**

Let \( \varphi \) be of locally bounded variation and have a support on \( R_+ \) and let \( \varphi(t) = O(t^k) \), for
some \( k \). Assume that
\[
\varphi((t + h)^m) - \varphi(t^m) \leq (1 + t)^a \quad \text{if} \quad 0 < h < 1 ,
\]
where \( a \geq 0 \) and \( m \) is an integer \( > 1 \), and assume that the Fourier-Laplace transform
\[
\Phi(\xi) = \int e^{-it\xi} d\varphi(t^m) , \quad \text{Im} \ \xi \ < \ 0 ,
\]
has an analytic continuation to a neighborhood \( \{ \xi : |\xi| < \delta \} \) of 0 such that \( |\Phi(\xi)| \leq 1, \ |\xi| < \delta; \ \Phi^{(n)}(0) = 0 \) for all integers \( l \geq 0 \).

If
\[
\varphi^*(t) = \int_0^t (1 - u/t)^s d\varphi(u),
\]
then for \( s \geq 0 \)
\[
|\varphi^*(t)| \leq C(s)t^{(a-s)/m}, \quad t \geq 1.
\]

To make use of this theorem we need first of all an estimation for the difference
\[
E_{(\mu+h)}f_1(x) - E_\mu f_1(x) = -\int_{\Gamma} V(x - y, \mu, h) \gamma(y) g(y) d\sigma(y), \tag{7}
\]
where a vector function \( V \) is of the form
\[
V(x, \mu, h) = (2\pi)^{-N} \int_{\mu^m < A(\xi) < (\mu+h)^m} \frac{i\xi e^{ix\xi}}{|\xi|^2} d\xi. \tag{8}
\]
Observe, if \( A(\xi) \geq \mu_0 \) and \( \mu_0 \) is large enough, then \( |\xi| \geq 1 \) and therefore \( \zeta(\xi) = 1 \). For this reason the function \( \zeta(\xi) \) is not in \( V \).

**Lemma 2.**

Let \( A, \) defined by (3), be convex and \( H(\alpha) \) be the Gaussian curvature of \( \partial A = \alpha \). Then for any \( \mu \geq 1 \) and \( 0 < h \leq 1 \), uniformly on \( x \in K \subset \mathbb{R}^N \setminus \{0\} \) the estimate
\[
|V(x, \mu, h)| \leq C \mu^{(N-3)/2} H^{-1/2}(\alpha(\theta)), \tag{9}
\]
holds. Here \( \alpha(\theta) \) is a point from the surface \( \partial A \) where the exterior normal to \( \partial A \) has direction \( \theta = \frac{x}{|x|} \).

**Proof.** According to the equality
\[
\frac{d}{dt} \int_{A(\xi) \leq t} F(\xi) d\xi = \frac{1}{m} t^{1/m-1} \int_{A(\xi) = t} F(\xi) \cos \omega(\xi) d\sigma(\xi), \tag{A}
\]
where \( \omega(\xi) \) is the angle between a vector \( \xi \) and the normal to the surface \( \{ \xi \in \mathbb{R}^N; A(\xi) = t \} \) at the point \( \xi \), we have
\[
V(x, \mu, h) = (2\pi)^{-N} \frac{1}{m} \int_{\mu^m}^{(\mu+h)^m} t^{1/m-1} dt \int_{A(\xi) = t} i\xi e^{ix\xi} \frac{\cos \omega(\xi)}{|\xi|^2} d\sigma(\xi). 
\]
Changing the variable \( \xi \) into \( t^{1/m} \xi \) in the inner integral, one obtains
\[
V(x, \mu, h) = (2\pi)^{-N} \frac{1}{m} \int_{\mu^m}^{(\mu+h)^m} t^{(N-1)/m-1} dt \int_{B_{t^{1/m}}(\xi) = 1} i\xi e^{it^{1/m}x\xi} \frac{\cos \omega(\xi)}{|\xi|^2} d\sigma(\xi). 
\]
If we use Lemma 1 to estimate the inner integral, we will have
\[
|V(x, \mu, h)| \leq C \int_{\mu^m}^{(\mu+h)^m} t^{N-1/2} H^{-1/2}(\alpha(\theta)) dt \leq \text{const} H^{-1/2}(\alpha(\theta)) \mu^{N-3}.
\]
Lemma 2 is proved.

Lemma 3.

Let $\Lambda$ be convex and $\mu \geq 1, 0 < h \leq 1$. Then uniformly on $x \in K \subset R^N \setminus \Gamma$ the estimate

$$E_{(\mu + h)m}f_1(x) - E_{\mu m}f_1(x) = O(\mu^{\frac{N-2}{2}}), \quad (10)$$

holds.

Proof. We first put the estimate (9) into (7). By use of the Heine-Borel lemma the integral over the surface $\Gamma$ can be represented as a finite sum of integrals over parts $S_j$, with a positive measure, of $S^{N-1}$. Each of such integrals we estimate by applying (6). Hence we have the required estimate (10).

Since according to the conditions of the theorem of L. Hörmander the function, being expanded should be equal to zero, at a neighborhood of a point under consideration, we introduce a function $f_0$, which locally coincides with $f_1$.

Fix any compact set $K \subset R^N \setminus \Gamma$. Consider an open set $U$, such that $K \subset U$ and $U \subset R^N \setminus \Gamma$. Let $f_0(x)$ be smooth in $R^N$, $f_0(x) = f_1(x), x \in U$ and let the expansion $E_\lambda f_0(x)$ tend to $f_0(x)$ uniformly. Then for any $s \geq 0$ uniformly on $x \in K$ the equality

$$E_\lambda^s f_0(x) = f_1(x) + o(1), \quad \lambda \to +\infty. \quad (11)$$

holds.

Now we apply to the following function

$$\varphi(\lambda) = E_\lambda f_1(x) - E_\lambda f_0(x)$$

the Tauberian theorem. By use of Lemma 3 we have

$$\varphi((t + h)m) - \varphi(tm) = O(t^{\frac{N-4}{2}}) + O(1) = O(t^{\frac{N-4}{2}}), \quad t \to +\infty, \quad 0 < h \leq 1, \ x \in K.$$ 

Satisfaction of the other conditions of the Tauberian theorem can be verified as in [4]. According to this theorem one has for the Riesz means of order $s \geq 0$ the following estimate

$$\varphi^s(tm) = O(t^{\frac{N-4}{2} - s}).$$

Since the Riesz means of any order $s \geq 0$ of eigenfunction expansions of the function $f_0(x)$ converge uniformly on every compact, we have for $s = (N - 3)/2$

$$E_\lambda^s f_1(x) = \varphi^s(\lambda) + E_\lambda^s f_0(x) = O(1), \quad x \in K \subset R^N \setminus \Gamma.$$ 

Now suppose $s > (n - 3)/2$. Then for large $\lambda$ uniformly on $x \in K \subset R^N \setminus \Gamma$ one has

$$\varphi^s(\lambda) = o(1), \quad \lambda \to +\infty.$$ 

Thus according to (11) we finally have

$$E_\lambda^s f_1(x) - f_1(x) = \varphi^s(\lambda) + E_\lambda^s f_0(x) - f_1(x) = o(1).$$
Theorem 1 is proved. Observe that the main point in the proof of Theorem 1 is the estimate (10). The left-hand side of equality (10) is the "double" oscillatory integral (see (7) and (8)) over the surface Γ and the domain \( G_\mu = \{ \xi \in \mathbb{R}^N : \mu^m < A(\xi) < (\mu + h)^m \} \). To get the estimate (10) we used the Randol estimate for the integral over \( G_\mu \) (for this reason we need the convexity of the set Λ and the analyticity of the surface \( \partial \Lambda \)) and the estimate (6) for the integral over the surface Γ. To prove Theorem 2 we again need the estimate (10). But according to the conditions of this theorem, now the domain \( D \) from Definition 1 is convex and has the analytic surface Γ. This means, one can use the Randol estimate (see Lemma 1) for the integral over the surface Γ. If we do this and apply for the integral over the domain \( G_\mu \) the estimate (6), then we have

**Lemma 4.**

Let \( A(\xi) \) be an arbitrary elliptic polynomial of order \( m \). Let the domain \( D \) from Definition 1 be convex with an analytic boundary Γ and \( \mu \geq 1, 0 < h \leq 1 \). Then uniformly on \( x \in K \subset \mathbb{R}^N \setminus \Gamma \) the estimate

\[
E_{(\mu+h)^m} f_1(x) - E_{\mu^m} f_1(x) = O(\mu^{\frac{N-3}{2}}),
\]

holds.

Now the proof of Theorem 2 proceeds along the same line as the proof of Theorem 1.

We finalize this section by proving Theorem 4 by the Il’in method. Let \( A(\xi) \) be an arbitrary elliptic polynomial. We consider the Riesz means of the multiple Fourier series \( \sigma^\lambda q(x) \) corresponding to the operator \( AT \). Fix a piecewise smooth function \( q(x) \) from Definition 1. Let \( g(y) \) be the jump of \( q \) across the surface Γ. Using the properties of the double layer potential we again, as in the proof of Theorem 1, come to know that it is sufficient to investigate the multiple Fourier series of the function

\[
q_1(x) = \sum_{|n| \geq 1} \frac{1}{|n|^2} e^{inx} \int_\Gamma e^{-iny}(-in, \gamma(y))g(y)d\sigma(y),
\]

where \( \gamma(y) \) is the interior normal to the surface Γ at point \( y \).

The estimate (10) in this case is given in the following lemma.

**Lemma 5.**

Let \( A(\xi) \) be an arbitrary elliptic polynomial of order \( m \). Let the surface Γ from Definition 1 be strictly convex and \( \mu \geq 1, 0 < h \leq 1 \). Then uniformly on \( x \in K \subset T^N \setminus \Gamma \) the estimate

\[
\sigma_{(\mu+h)^m} q_1(x) - \sigma_{\mu^m} q_1(x) = O(\mu^{\frac{N-3}{2}}),
\]

holds.

**Proof.** Let us denote the left-hand side of (12) by \( \psi(\mu) \). By the definition of the function \( q_1(x) \) one has

\[
\psi(\mu) = \sum_{\mu^m < A(n) < (\mu + h)^m} \frac{1}{|n|^2} e^{inx} \int_\Gamma e^{-iny}(-in, \gamma(y))g(y)d\sigma(y).
\]
Since the surface $\Gamma$ is strictly convex, using the stationary phase method [22], we have
\[ |\int_{\Gamma} e^{-iny(-in, \gamma(y))g(y)d\sigma(y)}| \leq c|n|^{-\frac{n-1}{2}+1}, \quad |n| > 1. \]
Hence
\[ |\psi(\mu)| \leq c \sum_{\mu^m < A(n)<(\mu+h)^m} \frac{1}{|n|^2} |n|^{-\frac{n-1}{2}+1} \leq C\mu^{-\frac{n-1}{2}} \sum_{\mu^m < A(n)<(\mu+h)^m} 1 \leq c\mu^{-\frac{n-1}{2}}, \]
and we get the required estimate (12). Lemma 5 is proved.

Now again the proof of the Theorem 4 can be finalized as the proof of the previous Theorems.

5 Proofs of Theorem 3 and Proposition 1

We start with the proof of the part (i) of Theorem 3 assuming Proposition 1. Part (ii) will be proved at the end of the paper. Let $N = 2$ and $A(D)$ be an arbitrary homogeneous elliptic differential operator of any order $m$. Fix any piecewise smooth function $f(x)$ from Definition 1. Let $g(x)$ be the jump of $f$ across the surface $\Gamma$. According to the Il’in method, since Proposition 1, it is sufficient to investigate the expansion (1) of the following function
\[ f_1(x) = (2\pi)^{-2} \int_{R^2} \zeta(\xi) e^{in\xi} |\xi|^2 d\xi \int_{\Gamma} e^{-iy\xi(-i\xi, \gamma(y))}g(y)d\sigma(y). \]
Observe, in this case the function $f(x) - \beta_2 f_1(x)$ is in the space $H_3^3(R^2)$ and for the corresponding Riesz means of order $s > -\frac{1}{2}$, since Proposition 1, the localization holds. Although the function $f(x) - \beta_2 f_1(x)$ is smooth enough, we could not find for our operator in the current mathematical literature a result about the localization for the negative order Riesz means. Therefore at the end of this section we present a proof of Proposition 1.

The Riesz means of order $s$ of the eigenfunction expansion of $f_1$ have the form
\[ E_\lambda^s f_1(x) = (2\pi)^{-2} \int_{A(\xi)<\lambda} \zeta(\xi) e^{in\xi} |\xi|^2 (1 - \frac{A(\xi)}{\lambda})^s d\xi \int_{\Gamma} e^{-iy\xi(-i\xi, \gamma(y))}g(y)d\sigma(y). \]
One should consider the difference $E_\lambda^s f_1(x) - f_1(x)$ and for any $s > -1/2$ show that when $\lambda \to +\infty$ it tends to zero uniformly on every compact $K \subset R^2 \setminus \Gamma$. But in [4] it is proved that $E_\lambda f_1(x) \to f_1(x)$ when $\lambda \to +\infty$ uniformly on $x \in K \subset R^2 \setminus \Gamma$. Hence it is enough to prove that $E_\lambda^s f_1(x) - E_{\sqrt{\lambda}} f_1(x) \to 0$ for $x \in K \subset R^2 \setminus \Gamma$ when $\lambda \to +\infty$.

We can write
\[
E_\lambda^s f_1(x) - E_{\sqrt{\lambda}} f_1(x) = (2\pi)^{-2} \int_{A(\xi)<\sqrt{\lambda}} \zeta(\xi) e^{in\xi} \left|(1 - \frac{A(\xi)}{\lambda})^s - 1\right| d\xi \int_{\Gamma} e^{-iy\xi(-i\xi, \gamma(y))}g(y)d\sigma(y) \\
+ (2\pi)^{-2} \int_{\sqrt{\lambda}<A(\xi)<\lambda} \zeta(\xi) e^{in\xi} \left|(1 - \frac{A(\xi)}{\lambda})^s - 1\right| d\xi \int_{\Gamma} e^{-iy\xi(-i\xi, \gamma(y))}g(y)d\sigma(y) \\
= I_1(x, \mu) + I_2(x, \mu), \quad \mu = \sqrt{\lambda}.
\]
Note, in $I_2$ there is no function $\zeta(\xi)$, since here $A(\xi)$ is large enough.

The integrand in $I_1$ is small. Therefore it can be estimated straightforwardly:

$$|I_1(x, \mu)| \leq c\mu^{-\frac{m-1}{2}}.$$

Consider $I_2(x, \mu)$. It is convenient to denote $a(\xi) = [A(\xi)]^{1/\mu}$. Note, for any real $t$, $a(t\xi) = |t|a(\xi)$. Using the formula (A) with $m = 1$ one can rewrite the integral $I_2$ as

$$I_2(x, \mu) = -(2\pi)^{-2} \int_{\Gamma} \gamma(y)g(y)d\sigma(y) \int_{a(\xi)=1} \frac{i\xi\cos(\omega(\xi))}{|\xi|^2}d\sigma(\xi) \int_0^\mu e^{i(x-y)\xi t}(1 - \frac{t}{\mu})^s dt.$$

Now we recall one more result of B. Randol [19] on the Fourier transform of the indicator function of a planar set. In fact we formulate this result for the integral over the surface $\partial \Lambda = \{ \xi \in \mathbb{R}^2 : a(\xi) = 1 \}$.

**Lemma 6.**

Let $\alpha(\theta)$ be a point from the surface $\partial \Lambda$ where the exterior normal to $\partial \Lambda$ has direction $\theta = \frac{x}{|x|}$ and $F(\xi)$ be a smooth function. Then there exists a function $G$, $G(\alpha(\theta)) \in L_{p_0}(S^1)$ for some $p_0 > 2$, such that for any $\lambda > 1$ uniformly on $x \in K \subset \mathbb{R}^2 \setminus \{0\}$ the estimate

$$|\int_{a(\xi)=1} F(\xi)e^{\lambda x\xi}d\sigma(\xi)| \leq c\lambda^{-1/2}G(\alpha(\theta))$$

holds.

We should note first of all, in [19] the explicit form of the function $G$ is given. Next, although in [19] the estimate (14) is proved for $F(\xi) \equiv 1$, the proof remains good for any smooth function $F(\xi)$.

We turn back to the integral (13). Let us denote the integral on $t$ by $T$ and change the variable $t = \mu p$. Then

$$T = \mu \int_{\mu^{-1/2}}^{1} e^{\mu (x-y)\xi \mu}(1 - \mu^m)^s dp.$$

We write

$$T = \mu \int_{\mu^{-1/2}}^{1-\mu^{-1}} + \mu \int_{1-\mu^{-1}}^{1} = T_1 + T_2.$$

We put $T_j$ into the integral (13) and denote the corresponding integral by $I_2^j(x, \mu)$. We hence have

$$I_2^2(x, \mu) = -\mu(2\pi)^{-2} \int_{\Gamma} \gamma(y)g(y)d\sigma(y) \int_{1-\mu^{-1}}^{1} (1 - \mu^m)^s dp \int_{a(\xi)=1} \frac{i\xi\cos(\omega(\xi))}{|\xi|^2} e^{i\mu (x-y)\xi \mu}d\sigma(\xi).$$

Using Lemma 6 one has for $s > -1$ the uniformly on $x \in K \subset \mathbb{R}^2 \setminus \Gamma$ estimate with $\theta = \frac{x-y}{|x-y|}$:

$$|I_2^j(x, \mu)| \leq C\mu^{-\frac{1}{2} + s} \int_{\Gamma} |\gamma(y)g(y)|G(\alpha(\theta))d\sigma(y) \leq c\mu^{-\frac{1}{2} + s}.$$
Let us consider the integral $I_2^1$. Integration by part in $T_1$ gives

$$T_1 = \frac{(1 - (1 - \mu^{-1})^m) \cdot i(x-y)}{\xi} e^{i\mu(x-y)\xi(1-\mu^{-1})}$$

$$- \frac{(1 - \mu^{-m/2}) \cdot i(x-y)}{\xi} e^{i\sqrt{\mu}(x-y)\xi}$$

$$+ \frac{sm}{i(x-y)\xi} \int_{\mu^{-1/2}}^{1-\mu^{-1}} e^{i\mu(x-y)\xi p (1 - p^m) s^{-1} p^{m-1} dp}.$$ 

The corresponding integrals denote accordingly by $I_{1}^{11}, I_{2}^{12}$ and $I_{2}^{13}$. Again using Lemma 6 one has for $x \in K \subset R^2 \setminus \Gamma$ the estimates

$$|I_2^{11}(x,\mu)| \leq c\mu^{-(s+\frac{1}{2})} \quad \text{and} \quad |I_2^{12}(x,\mu)| \leq c\mu^{-\frac{s}{2}}.$$ 

Hence we only have to estimate the integral

$$I_2^{13}(x,\mu) = -\frac{s \cdot m}{(2\pi)^2} \int_{\Gamma} \gamma(y) g(y) d\sigma(y) \int_{\mu^{-1/2}}^{1-\mu^{-1}} (1-p^m) s^{-1} p^{m-1} dp \int_{\omega(\xi)}^{\gamma(\xi)} e^{i\mu(x-y)\xi p} d\sigma(\xi).$$ 

For $x \in K \subset R^2 \setminus \Gamma$ Lemma 6 gives

$$|I_2^{13}(x,\mu)| \leq c_1\mu^{-(s+1/2)}.$$ 

Thus, if $s > -1/2$ then from the estimates of the integrals $I_1, I_2^2$ and $I_2^{13}$ one has

$$|I_1(x,\mu)| + |I_2(x,\mu)| \to 0, \quad \text{when} \quad x \in K \subset R^2 \setminus \Gamma, \text{ and } \mu \to +\infty,$$

and therefore we will have the proof of part (i) of Theorem 3 assuming Proposition 1.

We now turn to the proof of Proposition 1. Let $f(x)$ be any function from the Liouville space $L^1_l(R^2)$ and $l + s \geq \frac{1}{2}$. It is sufficient to consider only $-\frac{1}{2} < s \leq 0$, since otherwise this result is well known (see e.g.,[2]). Therefore we have $l \geq \frac{1}{2}$ and for the eigenfunction expansion $E_\lambda f(x)$ the localization principle holds (see [2]), i.e., uniformly on $x \in K \subset R^2 \setminus \text{supp} f$

$$E_\lambda f(x) = \int_{\omega(\xi)}^{\gamma(\xi)} f(\xi) e^{ix\xi} d\xi \to 0, \quad \text{when} \quad \lambda \to +\infty.$$

Hence to prove Proposition 1 it is enough to show $E_\lambda^x f(x) - E_\lambda f(x) \to 0$ for $x \in K \subset R^2 \setminus \text{supp} f$.

One has

$$E_\lambda^x f(x) - E_\lambda f(x) = \int_{\omega(\xi)}^{\gamma(\xi)} A^{-\frac{l}{R}}(\xi) [(1 - \frac{A(\xi)}{\lambda})^s - 1] A^{\frac{l}{R}}(\xi) f(\xi) e^{ix\xi} d\xi$$

$$= \int_{\text{supp} f} A^{\frac{l}{R}} f(y) \theta_{l,s}(x-y,\lambda) dy,$$

where

$$\theta_{l,s}(x-y,\lambda) = \int_{\omega(\xi)}^{\gamma(\xi)} A^{-\frac{l}{R}}(\xi) [(1 - \frac{A(\xi)}{\lambda})^s - 1] e^{i(x-y)\xi} d\xi.$$
If we apply the Cauchy-Bunyakovskyi inequality, we obtain

$$|E^s_\lambda f(x) - E^\mu f(x)| \leq ||A^{p,\mu}_R f||_{L_2(R^2)} ||\theta_{l,s}||_{L_2(supp \ f)} .$$

(15)

In [23] it is proved, that

$$||A^{p,\mu}_R f||_{L_2(R^2)} \leq C ||f||_{L_2(R^2)} .$$

Hence it is sufficient to estimate the $L_2(supp \ f)$ norm of $\theta_{l,s}$. If $r_0 = \min_{x \in K, y \in supp \ f} |x - y| > 0$ then this norm applying the formula (A) with $m = 1$ and introducing polar coordinates $(\rho, \phi)$ in $R^2$, placing the origin at $x$, can be estimated as

$$||\theta_{l,s}||_{L_2(supp \ f)} \leq \int_{r_0}^\infty \rho d\rho \int_{S^1} d\sigma(y) \int_0^1 \left[ \left( 1 - \left( \frac{\rho}{\mu} \right)^m \right)^s - 1 \right] t^{1-\mu} dt \int_{\sigma(\xi) = 1} \cos(\omega(\xi)) e^{iy\xi t} d\sigma(\xi) ^2$$

$$= \int_{r_0}^\infty \rho d\rho ||J(y, \mu)||^2_{L_2(S^1)} .$$

(16)

In what follows we may assume, that $l < 2$. To estimate the integral $J(y, \mu)$ we put (see the integrals $T_1$ and $T_2$ above)

$$R = \int_0^\mu e^{iy\xi t} \left[ (1 - \left( \frac{\rho}{\mu} \right)^m \right)^s - 1] t^{1-\mu} dt = \mu^{2-l} \int_0^1 e^{iy\xi t} \left[ (1 - p^m)^s - 1] p^{1-l} dp \right.$$

$$= \mu^{2-l} \int_0^{1-\mu^{-1}\rho^{-1}} + \mu^{2-l} \int_{1-\mu^{-1}\rho^{-1}}^1 \cdot R_1 + R_2 .$$

Denote the integral $J$ with $R_j$ by $J_j(y, \mu)$. Applying Lemma 6 one has the estimate

$$||J_2(y, \mu)||^2_{L_2(S^1)} \leq c\rho^{-3-2s} \mu^{1-2(l+s)} .$$

(17)

Let us consider the integral $J_1$. Integration by part in $R_1$ gives

$$R_1 = \frac{\mu^{1-l}}{iy\xi} e^{iy\xi t} \left[ (1 - p^m)^s - 1] p^{1-l} \right]_{p=1-\mu^{-1}\rho^{-1}}$$

$$= \mu^{1-l} \int_0^{1-\mu^{-1}\rho^{-1}} \{ (1 - l)p^{1-l}[(1 - p^m)^s - 1] + ms p^{1-l}(1 - p^m)^s p^{m-1} \} e^{iy\xi t} dp .$$

The corresponding two parts of the integral $J_1$ denote accordingly by $J_{11}$ and $J_{12}$. Again by use of Lemma 6 one has

$$||J_{11}(y, \mu)||^2_{L_2(S^1)} \leq c\rho^{-3-2s} \mu^{1-2(l+s)} .$$

(18)

In the integral $J_{12}$ first apply Lemma 6 then estimate the integral on $p$. Then we have

$$||J_{12}(y, \mu)||^2_{L_2(S^1)} \leq c\rho^{-3-2s} \mu^{1-2(l+s)} .$$

(19)

Thus if $l + s > \frac{1}{2}$ then putting (17)-(19) into (16) we have

$$||\theta_{l,s}||^2_{L_2(supp \ f)} \to 0 , \text{ when } \mu = \sqrt{\lambda} \to +\infty ,$$

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or
\[ E^s_\lambda f(x) - E_\lambda f(x) \to 0 \quad \text{when} \quad \mu = \sqrt{\lambda} \to +\infty \quad \text{and} \quad x \in K \subset R^2 \setminus \text{supp} \, f. \]

If \( l + s = \frac{1}{2} \) and \( x \in K \subset R^2 \setminus \text{supp} \, f \), then from above we have the estimate
\[ |E^s_\lambda f(x)| \leq C ||f||_{L^l(R^2)}. \]

Note this estimate for \( s = 0 \) is well known (see e.g.\[2\]). Now the convergence of the Riesz means \( E^s_\lambda f \) follows from the fact that \( C^0_\infty(R^2) \) is dense in \( L^l(R^2) \). Proposition 1 is proved.

Proof of the part (ii) of Theorem 3. Let \( f(x) \) be any piecewise smooth function and \( f_1(x) \) be defined as above. First observe, that from the estimates of the integrals \( I_1 \) and \( I_2 \) it follows \( E^{-1/2}_\lambda f_1(x) = O(1), \ x \in K \subset R^2 \setminus \Gamma \). Hence it remains to show \( E^{-1/2}_\lambda h(x) = O(1) \) for the function \( h(x) = f(x) - \beta_2 f_1(x) \). Note we could not consider \( s = -1/2 \) in Proposition 1, since in this case the integral on \( \rho \) in (16) does not converge. This integral does converge if we take norm of \( \theta_{l,s} \) not in \( L_2 \) but in \( L_p \) for some \( p > 2 \). Now take \( p_0 > 2 \) from Lemma 6 and put \( q_0 = \frac{p_0}{p_0-1} < 2 \).

As we mentioned above the function \( h(x) \) is smooth enough, so that \( h \in H^{1-1/p}_0(R^2) \subset L^l_{q_0}(R^2) \) for some \( l > 1 \). Let \( f \in L^l_{q_0}(R^2) \). If we consider now \( E^{-1/2}_\lambda f(x) - E_\lambda f(x) \) and apply the Hölder inequality, then instead of (15) we will get for \( x \in K \subset R^2 \setminus \text{supp} \, f \) the estimate
\[ |E^{-1/2}_\lambda f(x) - E_\lambda f(x)| \leq ||A^l_{R} f||_{L^l(R^2)} ||\theta_{l,-1/2}||_{L^l_{q_0}(\text{supp} \, f)}, \ l > 1. \]

Proceeding along the same lines as above we will have that the norm of \( \theta_{l,-1/2} \) tends to zero while \( \lambda \to \infty \). Thus we have the localization principle for \( E^{-1/2}_\lambda f(x) \). Theorem 3 is proved completely.

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References


