FREQUENCY DOMAIN FORMULATION OF LINEARIZED NAVIER-STOKES EQUATIONS

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ABSTRACT. A naturally parallelizable formulation is considered for solving linearized time-dependent Navier-Stokes equations. The evolution problem is first converted into a complex valued elliptic system by Fourier transformation. Existence and uniqueness are then given for the resulting problems for each frequency. Stability and regularity depending on frequency are analyzed. Next, standard finite element methods are used to approximate solutions for the transformed elliptic systems. Finally, time-dependent solutions are constructed by Fourier inversion with a full estimate of errors generated in the truncation in the Fourier transformation, quadrature rules, and finite element approximations.

1. INTRODUCTION

The domain Ω will be assumed to be a bounded Lipschitz domain in $\mathbb{R}^N, N = 2, 3$, with the boundary Γ. Set $J = (0, T)$. We consider the following linearized Navier-Stokes equations:

$$\frac{\partial u}{\partial t} - \mu\Delta u + (\mathcal{U} \cdot \nabla)u + \nabla p = f, \qquad \Omega \times J, \tag{1.1a}$$

$$\nabla \cdot u = 0, \qquad \Omega \times J, \tag{1.1b}$$

$$u = 0, \qquad \Gamma \times J, \tag{1.1c}$$

$$u(x, 0) = 0 \qquad \text{for all } x \in \Omega, \tag{1.1d}$$

where $\mathcal{U}$ is independent of t and the positive constant μ denotes the dynamic viscosity.

A motivation for our problem (1.1a) comes from the governing equations of Oseen's flow at low Reynolds numbers in which $\mathcal{U}$ is regarded as a constant vector [2, 12, 18, 23, 24]. Notice that the usual linearized Navier-Stokes equations contain an additional term $(u \cdot \nabla)\mathcal{U}$ in (1.1a) which is obtained from those governed by fully nonlinear Navier-Stokes equations up to a first-order approximation.

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Instead of solving Problem (1.1) in the original space-time formulation, we first take Fourier transforms of (1.1) in the time variable, then solve the Fourier-transformed complex-valued elliptic problems for each frequency of interest simultaneously, and then lastly obtain the time snap informations by inverse transformation.

The Fourier transform $\widehat{u}(\cdot,\omega)$ of a function $u(\cdot,t)$ in time is defined by

$$\widehat{u}(\cdot,\omega) = \int_{-\infty}^{\infty} u(\cdot,t)e^{-i\omega t}\,dt$$

and the Fourier inversion formula is given by

$$u(\cdot,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{u}(\cdot,\omega)e^{i\omega t}\,d\omega.$$

Note that if we consider a real function $u(x,t)$, its Fourier transform satisfies the conjugate relation.

$$\widehat{u}(\cdot,-\omega) = \overline{\widehat{u}(\cdot,\omega)} \text{ for all } \omega \in \mathbb{R}.$$

In this case the Fourier inversion formula takes the form

$$u(\cdot,t) = \frac{1}{\pi}\,\mathrm{Re}\left(\int_{0}^{\infty} \widehat{u}(\cdot,\omega)e^{i\omega t}\,d\omega\right).$$

For negative t, extend $u(\cdot,t)$ and $f(\cdot,t)$ by zeros. Thus, we consider the Fourier transformed elliptic problems:

$$i\omega\widehat{u} - \mu\Delta\widehat{u} + (\mathcal{U}\cdot\nabla)\widehat{u} + \nabla\widehat{p} = \widehat{f}, \qquad \Omega, \tag{1.2a}$$

$$\nabla\cdot\widehat{u} = 0, \qquad \Omega, \tag{1.2b}$$

$$\widehat{u} = 0, \qquad \Gamma. \tag{1.2c}$$

Problems (1.2) drags an attention since it is a complex-valued problem depending on frequency. Indeed, in order to have the inverse Fourier transforms, one must see asymptotic behaviors of $\widehat{u}(\cdot,\omega)$ and $\widehat{p}(\cdot,\omega)$ as ω tends to 0 and ∞. In reality sources have finite frequency spectra, and therefore those asymptotic behaviors as ω tends to ∞ may be negligible. However, the behavior near $\omega = 0$ is important.

A motivation of solving time-dependent problems in the frequency-domain formulation is as follows: Numerical solutions of time-dependent problems are obtained in the original space-time formulation. Along the discretized time axis various marching algorithms have been used, for example, backward-Euler, forward-Euler, Crank-Nicolson methods, and so on. In these algorithms finding solutions at one time step requires the knowledge of the solution at all previous time steps. These methods are thus not naturally parallelizable to obtain $u(x,t)$ for large t. Of course there are certain parallel algorithms in the axis of time. Also, in these ten years, there have been remarkable progresses on the parallelization in the space variables for time-independent and time-dependent problems among researchers;

for instance see [15, 5, 6, 16, 20, 25, 21] and recent publications in major numerical analysis journals. In these methods, they require heavy communication costs among processors.

Our method is to solve Fourier-transformed problems for discrete number of frequencies ω's of interest, and to take the discrete inverse Fourier transform. This method is a very natural parallel algorithm since when one solves the Fourier-transformed (elliptic) problem at one frequency, there is no need of solution informations on other frequencies. Thus there is no communication cost if one assigns a Fourier-transformed elliptic problem to a virtual processor.

Since Hermann von Helmholtz (1860) it has been well-known that wave equations become Helmholtz-type equations in the space-frequency domain, which have eigensolutions with Dirichlet or Neumann boundary conditions with real eigen frequencies. This is not the case with absorbing boundary conditions (see, for instance, [7, 13, 17], and for more references on absorbing boundary conditions see J. Comp. Phys. and other major numerical analysis journals). Indeed, with absorbing boundary conditions the Helmholtz-type equations turn out to be uniquely solvable for real frequencies [10, 11, 26]. Thus in this case a natural parallelization is possible. Parabolic problems are more tractable in the space-frequency formulation than hyperbolic ones [27]. Equations (1.1) have both parabolic and hyperbolic characteristics and the analysis is much more complicated.

In §2, we show that the equation (1.2) has the unique solution $\widehat{u}(\cdot,\omega)$ for $\omega > 0$, and regularity and stability results are proved for such solutions. In §3 we treat finite element procedures for (1.2) and present error estimates.

2. Continuous Problems

2.1. **Notations and Preliminaries.** All the functions and the inner products are taken in the complex field. But, they are considered in the real field for the time-dependent problems.

For positive integer $N = 2, 3$, denote by $[L^2(\Omega)]^N$ and $[L^2(\Gamma)]^N$ the spaces of square integrable vector functions on Ω and on Γ, respectively. Corresponding inner products and norms will be designated by $(\cdot,\cdot), \langle\cdot,\cdot\rangle_\Gamma$ and $\|\cdot\|, |\cdot|_\Gamma$, respectively. As usual,

$$L_0^2(\Omega) = \{f \in L^2(\Omega) : \int_\Omega f(x)\,dx = 0\}.$$

Let $[H^m(\Omega)]^N$ and $[H^m(\Gamma)]^N$, for nonnegative integer m, denote the usual vector Sobolev spaces with norms $\|\cdot\|_m$ and $|\cdot|_{m,\Gamma}$, and $[H_0^m(\Omega)]^N$ the completion of $[C_0^\infty(\Omega)]^N$ in the norm of $[H^m(\Omega)]^N$. Also notations $|\cdot|_m$ for seminorms will be used for the space $H^m(\Omega)$; see [1, 9, 14] for more details of function spaces and related norms.

By $\langle\cdot,\cdot\rangle$ we will mean the duality pairing. Let X be a Banach space and X' the dual space of X with the corresponding dual norm:

$$\|l\|_{X'} = \sup_{v\in X} \frac{\langle l, v\rangle}{\|v\|_X}.$$

In what follows an assumption on $\mathcal{U}$ will be made:

$$\|\mathcal{U}\|_\infty + \|\nabla\cdot\mathcal{U}\|_\infty < \infty. \tag{2.1}$$

2.2. **Variational Formulation.** Set

$$X = [H_0^1(\Omega)]^N \qquad \text{and } M = L_0^2(\Omega).$$

Let the sesquilinear forms $a_\omega : X\times X \to \mathbb{C}$ and $b : X\times M \to \mathbb{C}$ defined by

$$\begin{aligned} a_\omega(\widehat{u},\widehat{v}) &= i\omega(\widehat{u},\widehat{v}) + (\mu\nabla\widehat{u},\nabla\widehat{v}) + ((\mathcal{U}\cdot\nabla)\widehat{u},\widehat{v}), \\ b(\widehat{v},\widehat{p}) &= -(\widehat{p},\nabla\cdot\widehat{v}). \end{aligned}$$

Problem (1.2) then reads

$$a_\omega(\widehat{u},\widehat{v}) + b(\widehat{v},\widehat{p}) = \langle\widehat{f},\widehat{v}\rangle \qquad \text{for all } \widehat{v}\in X, \tag{2.2a}$$
$$b(\widehat{u},\widehat{q}) = 0 \qquad \text{for all } \widehat{q}\in M. \tag{2.2b}$$

2.3. **Existence and Uniqueness.**

Lemma 2.1 (Continuity). $a_\omega : X\times X\to\mathbb{C}$ is continuous. More precisely,

$$|a_\omega(\widehat{u},\widehat{v})| \le C(1+\omega)\|\widehat{u}\|_1\|\widehat{v}\|_1 \qquad \text{for all } \widehat{u},\widehat{v}\in X. \tag{2.3}$$

where C depends only upon $\mu + \|\mathcal{U}\|_\infty$.

Proof. This follows by direct calculations:

$$\begin{aligned} |a_\omega(\widehat{u},\widehat{v})| &\le \omega\|\widehat{u}\|\,\|\widehat{v}\| + \mu\|\widehat{u}\|_1\,\|\widehat{v}\|_1 + \|\mathcal{U}\|_\infty\|\widehat{u}\|_1\,\|\widehat{v}\| \\ &\le \{\omega+\mu+\|\mathcal{U}\|_\infty\}\|\widehat{u}\|_1\,\|\widehat{v}\|_1. \end{aligned}$$

□

Remark 2.1. The above estimate (2.3) implies that

$$\|a_\omega\| \le C(1+\omega).$$

In what follows C will denote a generic positive constant which may differ from place to place.

Recall Poincaré lemma:

$$\|\nabla v\| \ge C_p\|v\| \qquad \text{for all } v\in[H_0^1(\Omega)]^N, \tag{2.4}$$

where C_p depends only upon Ω.

Lemma 2.2 (Coercivity). Suppose that

$$\mu C_p^2 - \frac{1}{2}\|\nabla \cdot \mathcal{U}\|_\infty > 0. \tag{2.5}$$

Then, $a_\omega : X \times X \to \mathbb{C}$ is coercive. Moreover,

$$|a_\omega(\widehat{v}, \widehat{v})| \geq C_1 \|\widehat{v}\|_1^2 \qquad \text{for all } \widehat{v} \in X, \tag{2.6}$$

where C_1 depends only upon μ, $\|\nabla \cdot \mathcal{U}\|_\infty$ and Ω.

Proof. Let $\delta = \mu C_p^2 - \frac{1}{2}\|\nabla \cdot \mathcal{U}\|_\infty > 0$. By integrations by parts,

$$\begin{aligned} a_\omega(\widehat{v}, \widehat{v}) &= i\omega\|\widehat{v}\|^2 + ((\mathcal{U} \cdot \nabla)\widehat{v}, \widehat{v}) + (\mu \nabla \widehat{v}, \nabla \widehat{v}) \\ &= i\omega\|\widehat{v}\|^2 - \frac{1}{2}((\nabla \cdot \mathcal{U})\widehat{v}, \widehat{v}) + \|\mu^{1/2}\nabla \widehat{v}\|^2. \end{aligned}$$

Due to Poincaré lemma (2.4), we see that

$$\begin{aligned} \mu\|\nabla \widehat{v}\|^2 - \frac{1}{2}((\nabla \cdot \mathcal{U})\widehat{v}, \widehat{v}) &\geq \mu\|\nabla \widehat{v}\|^2 - \frac{1}{2C_p^2}\|\nabla \cdot \mathcal{U}\|_\infty \|\nabla \widehat{v}\|^2 \\ &= \frac{\delta}{C_p^2}\|\nabla \widehat{v}\|^2, \end{aligned}$$

which implies

$$\mu\|\nabla \widehat{v}\|^2 - \frac{1}{2}((\nabla \cdot \mathcal{U})\widehat{v}, \widehat{v}) \geq C_1\|\widehat{v}\|_1^2,$$

with $C_1 > 0$ independent of $\widehat{v}$. Therefore,

$$|a_\omega(\widehat{v}, \widehat{v})| \geq \frac{1}{\sqrt{2}}\left[\omega\|\widehat{v}\|^2 + C_1\|\widehat{v}\|_1^2\right] \geq C_1\|\widehat{v}\|_1^2.$$

This completes the proof. □

In what follows we shall always assume the condition (2.5). In Oseen's flow, $\mathcal{U}$ is constant and μ is large so that (2.5) holds.

Remark 2.2. The coercivity is independent of ω.

Now turn to estimate the velocity and pressure in appropriate norms. Let us introduce the bounded linear operators

$$A_\omega : X \to X',$$

and

$$B : X \to M'$$

defined by

$$\begin{aligned} \langle A_\omega \widehat{u}, v\rangle &:= a_\omega(\widehat{u}, v) \quad \text{for all } \widehat{u}, v \in X, \\ \langle Bv, \widehat{p}\rangle &:= b(v, \widehat{p}) \quad \text{for all } v \in X, \quad \widehat{p} \in M. \end{aligned}$$

Here, $X' = [H^{-1}(\Omega)]^N$ and $M' = L_0^2(\Omega)'$. Then we have

$$\begin{aligned} \|A_\omega\|_{\mathcal{L}(X;X')} &= \|a_\omega\|, \\ \|B\|_{\mathcal{L}(X;M')} &= \|b\|. \end{aligned}$$

Let $B' : M \to X'$ be the dual operator of B. Problem (2.2) is then equivalent to

$$A_\omega \widehat{u} + B'\widehat{p} = \widehat{f}, \quad X', \tag{2.7a}$$

$$B\widehat{u} = 0, \quad M'. \tag{2.7b}$$

Set $V = \{v \in [H_0^1(\Omega)]^N;\ Bv = 0\}$. Define $\pi \in \mathcal{L}(X'; V')$ by

$$\langle \pi f, v\rangle := \langle f, v\rangle \quad \text{for } f \in X', v \in V.$$

It is immediate to see that $\|\pi \widehat{f}\|_{V'} \le \|\widehat{f}\|_{X'}$. Denote the polar set

$$V^\circ := \{g \in X' : \langle g, v\rangle = 0 \quad \text{for all } v \in V\}.$$

We recall the following result [3, 14].

Lemma 2.3. The following properties are equivalent:

(i) **[Inf-sup condition (B-B condition)]** there exists a positive constant β such that

$$\inf_{q\in M} \sup_{v\in X} \frac{|b(v,q)|}{\|v\|_X \, \|q\|_M} \ge \beta;$$

(ii) the operator B' is an isomorphism from M onto V° with

$$\|B'q\|_{X'} \ge \beta \|q\|_M \qquad \text{for all } q \in M. \tag{2.8}$$

Note that due to Lemmas 2.1 and 2.2, Lax-Milgram lemma implies that the problem to find $\widehat{u} \in V$ such that

$$a_\omega(\widehat{u}, v) = \langle \widehat{f}, v\rangle \quad \text{for all } \ v \in V$$

has a unique solution $\widehat{u}$ which solves (2.2). Moreover, we have the following estimate:

$$\|\widehat{u}\|_1 \le \widetilde{C} \|\widehat{f}\|_{-1}, \qquad \widetilde{C} = \frac{1}{C_1}. \tag{2.9}$$

Indeed, we have

$$\|\widehat{f}\|_{-1} \|\widehat{u}\|_1 \ge |\langle \widehat{f}, \widehat{u}\rangle| \ge |a_\omega(\widehat{u}, \widehat{u})| \ge C_1 \|\widehat{u}\|_1^2.$$

It is easy to check the above Inf-sup condition and therefore by Lemma 2.3 B' is an isomorphism from M onto V°. Also, notice that

$$\langle \widehat{f} - A_\omega \widehat{u}, v\rangle = 0 \quad \text{for all } v \in V,$$

which implies that $\widehat{f} - A_\omega \widehat{u} \in V^\circ$. Therefore there is a unique $\widehat{p} \in M$ such that $B'\widehat{p} = \widehat{f} - A_\omega \widehat{u}$. Note that $(\widehat{u}, \widehat{p}) \in X \times M$ is a unique solution of Problem (2.2).

Moreover, due to (2.3), (2.8) and (2.9),

$$\begin{aligned}\|\widehat{p}\| &\le \frac{1}{\beta}\|\widehat{f} - A_\omega \widehat{u}\|_{-1} \\ &\le \frac{1}{\beta}\{\|\widehat{f}\|_{-1} + \|A_\omega\|_{\mathcal{L}(X;X')}\|\widehat{u}\|_1\} \\ &\le \frac{1}{\beta}\{\|\widehat{f}\|_{-1} + C(1+\omega)\|\widehat{u}\|_1\} \\ &\le \frac{1}{\beta}\{1 + C\widetilde{C}(1+\omega)\}\|\widehat{f}\|_{-1} \\ &= C_2(1+\omega)\|\widehat{f}\|_{-1}.\end{aligned}$$

Moreover, if we assume that Ω is a convex polygon in $\mathbb{R}^2$ or a C^2–domain in $\mathbb{R}^3$, we have that, for $\|\widehat{f}\| < \infty$, there is a regular solution such that $\widehat{u} \in [H^2(\Omega) \cap H_0^1(\Omega)]^N$, $\widehat{p} \in H^1(\Omega) \cap L_0^2(\Omega)$. Indeed, due to the standard regularity result [4, 8, 19, 22, 28]

$$\|v\|_2 + \|q\|_1 \le C\|F\|$$

for the incompressible Stokes problem

$$\begin{aligned} -\mu\Delta v + \nabla q &= F, & \Omega, \\ \nabla \cdot v &= 0, & \Omega, \\ v &= 0, & \Gamma, \end{aligned}$$

an application of (2.9) yields

$$\begin{aligned}\|\widehat{u}\|_2 + \|\widehat{p}\|_1 &\le C\|\widehat{f} - i\omega\widehat{u} - (\mathcal{U}\cdot\nabla)\widehat{u}\| \\ &\le C(1+\omega)\|\widehat{f}\|.\end{aligned}$$

We summarize the above results in

Theorem 2.1. Suppose Ω to be a bounded Lipschitz domain in $\mathbb{R}^N$, $N = 2, 3$, with the boundary Γ. Assume that $\|\widehat{f}\|_{-1} < \infty$. Then there exists a unique solution $\widehat{u} \in [H_0^1(\Omega)]^N$, $\widehat{p} \in L_0^2(\Omega)$ of Problem (1.2) such that

$$\|\widehat{u}\|_1 + \|\widehat{p}\| \le C(1+\omega)\|\widehat{f}\|_{-1}.$$

Moreover, if Ω is a convex polygon in $\mathbb{R}^2$ or a C^2–domain in $\mathbb{R}^3$, then, for $\|\widehat{f}\| < \infty$, the solution $(\widehat{u}, \widehat{p})$ of Problem (1.2) belongs to $[H^2(\Omega) \cap H_0^1(\Omega)]^N \times [H^1(\Omega) \cap L_0^2(\Omega)]$ with

$$\|\widehat{u}\|_2 + \|\widehat{p}\|_1 \le C(1+\omega)\|\widehat{f}\|.$$

As immediate results of the Theorem 2.1, we have the followings:

Corollary 2.1. Assume that $\|(1+\omega)\widehat{f}(\cdot,\omega)\|_{-1}$ is integrable over $\mathbb{R}$ with respect to ω. Then there exist inverse Fourier transforms of $\widehat{u}$, $\frac{\partial \widehat{u}}{\partial x_i}, 1 \le i \le N$, and $\widehat{p}$.

Corollary 2.2. Assume that $\|(1+\omega)\widehat{f}(\cdot,\omega)\|$ is integrable over $\mathbb{R}$ with respect to ω. Then there exist inverse Fourier transforms of $\widehat{u}$, $\frac{\partial \widehat{u}}{\partial x_i}$, $\frac{\partial^2 \widehat{u}}{\partial x_i x_j}$, $\widehat{p}$, and $\frac{\partial \widehat{p}}{\partial x_i}, 1 \leq i,j \leq N$.

3. Finite Element Approximation

3.1. **Error Estimates for fixed ω.** Recall the Sobolev spaces $X = [H_0^1(\Omega)]^N$ and $M = L_0^2(\Omega)$. Let $h > 0$ denote a discretization parameter tending to zero. For each h, let W_h and Q_h be two finite-dimensional subspaces of $[H^1(\Omega)]^N$ and $L^2(\Omega)$, respectively. Following [14], set $X_h = W_h \cap X$ and $M_h = Q_h \cap M$.

The finite element solution is then given by $(\widehat{u}_h, \widehat{p}_h) \in X_h \times M_h$ such that

$$a_\omega(\widehat{u}_h, v) + b(v, \widehat{p}_h) = \langle \widehat{f}, v\rangle \qquad \text{for all } v \in X_h, \tag{3.1a}$$

$$b(\widehat{u}_h, q) = 0 \qquad \text{for all } q \in M_h. \tag{3.1b}$$

By Divergence Theorem, $\int_\Omega \nabla \cdot \widehat{u}_h dx = \int_\Gamma \nu \cdot \widehat{u}_h d\sigma = 0$, which implies that $\nabla \cdot \widehat{u}_h \in L_0^2(\Omega)$. Here ν denotes the unit outward normal vector to Ω. Let $q \in Q_h$ be arbitrary. Since $q - \frac{1}{|\Omega|}\int_\Omega q dx \in M_h$,

$$b(\widehat{u}_h, q) = (\nabla \cdot \widehat{u}_h, q) = (\nabla \cdot \widehat{u}_h, q - \frac{1}{|\Omega|}\int_\Omega q dx) + (\nabla \cdot \widehat{u}_h, 1)\frac{1}{|\Omega|}\int_\Omega q dx = 0.$$

Therefore, problem (3.1) is equivalent to

$$a_\omega(\widehat{u}_h, v) + b(v, \widehat{p}_h) = \langle \widehat{f}, v\rangle \qquad \text{for all } v \in X_h, \tag{3.2a}$$

$$b(\widehat{u}_h, q) = 0 \qquad \text{for all } q \in Q_h. \tag{3.2b}$$

Set $V_h = \{v \in X_h : b(v,q) = 0, \text{ for all } q \in Q_h\}$. We then consider the following problem associated with (3.1): *Find $\widehat{u}_h \in V_h$ such that*

$$a_\omega(\widehat{u}_h, v) = \langle \widehat{f}, v\rangle \qquad \text{for all } v \in V_h. \tag{3.3}$$

We shall assume the uniform inf-sup condition such that for each $q_h \in M_h$ there exists a $v_h \in X_h$ satisfying

$$b(v_h, q_h) = \|q_h\|^2, \tag{3.4a}$$

$$|v_h|_1 \leq C\|q_h\|, \tag{3.4b}$$

where $C > 0$ is independent of h, q_h and v_h.

We shall also assume that V_h satisfy the following properties (for such finite element spaces, we refer, for instance, [14, 29].):

A1: There exist a positive constant C and π_h independent of h and v such that

$$\|v - \pi_h v\| + h|v - \pi_h v|_1 \leq Ch^k\|v\|_k, \qquad \text{for all } v \in [H^k(\Omega)]^N,$$

for $k = 1, 2$.

A2: $\int_\Omega \nabla \cdot (v - \pi_h v) q\, dx = 0 \qquad$ for all $q \in Q_h$.

A3: There exist a positive constant C and ρ_h independent of h and q such that $\|q - \rho_h q\| \leq Ch|q|_1 \qquad$ for all $q \in H^1(\Omega)$.

One then has

Theorem 3.1. Suppose that Ω is a convex polygon in $\mathbb{R}^2$ or a C^2–domain in $\mathbb{R}^3$. If $\|(1+\omega)\widehat{f}(\cdot,\omega)\| < \infty$, then the solution $(\widehat{u}_h, \widehat{p}_h) = (\widehat{u}_h(\cdot,\omega), \widehat{p}_h(\cdot,\omega))$ of (3.1) satisfies the estimate:

$$|\widehat{u} - \widehat{u}_h|_1 + \|\widehat{p} - \widehat{p}_h\| \;\le\; C(1+\omega)h(\|\widehat{u}\|_2 + \|\widehat{p}\|_1), \tag{3.5a}$$

$$\|\widehat{u} - \widehat{u}_h\| \;\le\; C(1+\omega)^3 h^2(\|\widehat{u}\|_2 + \|\widehat{p}\|_1). \tag{3.5b}$$

A proof of this theorem follows from a standard argument. See [14, 29] for details.

3.2. **Total Error Estimate.** We are now in a position to start to estimate the total errors generated in the truncation in the Fourier transformation, quadrature rules, and finite element approximations.

We begin with the following lemma:

Lemma 3.1. Suppose that

$$\int_0^\infty s^{2k}\|f(\cdot,s)\|^2\,ds \;<\; \infty,$$

$k = 0, 1, 2, \ldots, m$, for some nonnegative integer m. Let u and p be the solution of Problem (1.1). Then we have the following estimates: for $k = 0, 1, 2, \ldots, m$

$$\|t^k u\|^2_{L^2((0,\infty);[H^1(\Omega)]^N)} + \|t^k p\|^2_{L^2((0,\infty);L^2(\Omega))} \le C\sum_{j=0}^{k}\int_0^\infty t^{2j}\|f(\cdot,t)\|^2\,dt. \tag{3.6}$$

Proof. Multiply (1.1a) by $u(\cdot,t)$ and integrate over Ω to get, for all $\varepsilon > 0$,

$$\frac{d}{dt}\frac{1}{2}\|u(\cdot,t)\|^2 + \mu\|\nabla u(\cdot,t)\|^2 - \frac{1}{2}\|\nabla\cdot\mathcal{U}\|_\infty\|u\|^2 \;\le\; \frac{1}{4\varepsilon}\|f(\cdot,t)\|^2 + \varepsilon\|u(\cdot,t)\|^2.$$

Due to (2.5), a proper choice of $\varepsilon > 0$ leads to

$$\frac{d}{dt}\|u(\cdot,t)\|^2 + \|u(\cdot,t)\|_1^2 \le C\|f(\cdot,t)\|^2, \tag{3.7}$$

from which by integrating the above with respect to t over $[0,T]$ for any positive T, we get

$$\|u(\cdot,T)\|^2 + \int_0^T \|u(\cdot,t)\|_1^2 dt \le C\int_0^T \|f(\cdot,t)\|^2 dt,$$

which of course implies

$$\int_0^\infty \|u(\cdot,t)\|_1^2 dt \le C\int_0^\infty \|f(\cdot,t)\|^2 dt. \tag{3.8}$$

Next, multiply (1.1a) by $u_t(\cdot,t)$ to get, for all $\varepsilon > 0$,

$$\|u_t(\cdot,t)\|^2 + \frac{\mu}{2}\frac{d}{dt}\|\nabla u(\cdot,t)\|^2 + \int_\Omega (\mathcal{U}\cdot\nabla)u\, u_t dx \quad \leq \quad \frac{1}{4\varepsilon}\|f(\cdot,t)\|^2 + \varepsilon\|u_t(\cdot,t)\|^2,$$

from which we have

$$\begin{aligned}\|u_t(\cdot,t)\|^2 + \frac{\mu}{2}\frac{d}{dt}\|\nabla u(\cdot,t)\|^2 \quad &\leq \quad \frac{1}{4\varepsilon}\|f\|^2 + \varepsilon\|u_t\|^2 \\ &\qquad + \|\mathcal{U}\|_\infty\left\{\frac{1}{4\varepsilon}\|u\|_1^2 + \varepsilon\|u_t\|^2\right\}.\end{aligned}$$

Again a choice of sufficiently small $\varepsilon > 0$ in the above inequality will yields

$$\|u_t(\cdot,t)\|^2 + \frac{d}{dt}\|\nabla u(\cdot,t)\|^2 \quad \leq \quad C\left[\|f(\cdot,t)\|^2 + \|u(\cdot,t)\|_1^2\right].$$

Integrating the last inequality with respect to t over $[0,\infty)$ and using (3.8), we have the estimate on u_t:

$$\int_0^\infty \|u_t(\cdot,t)\|^2 dt \leq C\int_0^\infty \|f(\cdot,t)\|^2 dt. \tag{3.9}$$

Since

$$\begin{aligned}\|p\| \quad &\leq \quad C\|\nabla p\|_{-1} \\ &= \quad C\|f - u_t + \mu\Delta u - \mathcal{U}\cdot\nabla u\|_{-1} \\ &\leq \quad C\left[\|f\|_{-1} + \|u\|_1 + \|u_t\|_{-1}\right],\end{aligned}$$

from (3.8) and (3.9) we have (3.6) for $k = 0$.

Now, multiply (1.1a) by t so that we obtain the equation

$$\frac{\partial(tu)}{\partial t} - \mu\Delta(tu) + (\mathcal{U}\cdot\nabla)(tu) + \nabla(tp) = tf + u.$$

Since $tf(\cdot,t) + u(\cdot,t) \in L^2((0,\infty);[L^2(\Omega)]^N)$, we apply the above argument with u and f replaced by tu and $tf + u$, respectively, to obtain the estimate with $k = 1$ of the lemma. Repeating the above process for $k > 1$ completes the proof. □

We consider restricted sources such that $|\widehat{f}(\cdot,\omega)|$ is square integrable with respect to ω and thus negligible for large $|\omega|$. We then choose a sufficiently large $\omega^* > 0$ so that both $\widehat{u}(\cdot,\omega)$ and $\widehat{f}(\cdot,\omega)$ are negligible for $|\omega| > \omega^*$. Let M be a positive integer and define the discretization parameter $\Delta\omega$ of the frequency-domain by the formula $\Delta\omega = \omega^*/M$, and introduce the mesh points $\omega_{j+\frac{1}{2}} = (j+\frac{1}{2})\Delta\omega$, $j =$

$-M, \ldots, M-1$ on the interval $(-\omega^*, \omega^*)$. Then set

$$\begin{aligned} u_{\omega^*}(x,t) &= \frac{1}{2\pi}\int_{-\omega^*}^{\omega^*} \widehat{u}(x,\omega)e^{i\omega t}d\omega, \\ u_{\omega^*,\Delta\omega}(x,t) &= \frac{1}{2\pi}\sum_{j=-M}^{M-1} \widehat{u}(x,\omega_{j+1/2})e^{it\omega_{j+1/2}}\Delta\omega, \\ u^h_{\omega^*,\Delta\omega}(x,t) &= \frac{1}{2\pi}\sum_{j=-M}^{M-1} \widehat{u}_h(x,\omega_{j+1/2})e^{it\omega_{j+1/2}}\Delta\omega. \end{aligned}$$

The time-domain solution $u(x,t)$ is then approximated by $u^h_{\omega^*,\Delta\omega}(x,t)$. We try to estimate the convergence of $u^h_{\omega^*,\Delta\omega}(x,t)$ to $u(x,t)$ as follows:

$$\begin{aligned} u(x,t) - u^h_{\omega^*,\Delta\omega}(x,t) &= (u(x,t) - u_{\omega^*}(x,t)) \\ &\quad + (u_{\omega^*}(x,t) - u_{\omega^*,\Delta\omega}(x,t)) \\ &\quad + \left(u_{\omega^*,\Delta\omega}(x,t) - u^h_{\omega^*,\Delta\omega}(x,t)\right) \\ &\equiv E_1(x,t) + E_2(x,t) + E_3(x,t). \end{aligned} \tag{3.10}$$

By Theorem 2.1,

$$\|E_1(\cdot,t)\|_1 \le \frac{1}{2\pi}\int_{|\omega|>\omega^*} \|\widehat{u}(\cdot,\omega)\|_1 d\omega \le C\int_{|\omega|>\omega^*}(1+\omega)\|\widehat{f}(\cdot,\omega)\|_{-1}d\omega,$$

provided that $\int_{|\omega|>\omega^*}(1+\omega)\|\widehat{f}(\cdot,\omega)\|_{-1}d\omega < \infty$. Thus

$$\|E_1(\cdot,t)\|_1 \to 0 \qquad \text{as } \omega^* \to \infty. \tag{3.11}$$

We also have

$$\begin{aligned} \|E_2(\cdot,t)\|_1^2 &\le \frac{C}{4\pi^2}\int_\Omega \left| \int_{-\omega^*}^{\omega^*} \nabla\widehat{u}(x,\omega)e^{i\omega t}d\omega - \sum_{j=-M}^{M-1}\nabla\widehat{u}(x,\omega_{j+1/2})e^{it\omega_{j+1/2}}\Delta\omega \right|^2 dx \\ &\le C(\Delta\omega)^4 \int_\Omega \left\| \frac{\partial^2(\nabla\widehat{u}(x,\omega)e^{i\omega t})}{\partial\omega^2}\right\|^2_{L^2(-\omega^*,\omega^*)} dx \\ &\le C(\Delta\omega)^4 \int_\Omega \left\| -\widehat{t^2\nabla u}(x,\cdot) + 2t\widehat{t\nabla u}(x,\cdot) - t^2\nabla\widehat{u}(x,\cdot)\right\|^2_{L^2(-\omega^*,\omega^*)} dx \\ &\le C(\Delta\omega)^4 \int_\Omega \Big\{ \|\widehat{t^2\nabla u}(x,\cdot)\|^2_{L^2(-\infty,\infty)} \\ &\qquad + t^2\|\widehat{t\nabla u}(x,\cdot)\|^2_{L^2(-\infty,\infty)} + t^4\|\nabla\widehat{u}(x,\cdot)\|^2_{L^2(-\infty,\infty)} \Big\} dx \\ &\le C(\Delta\omega)^4 \Big\{ \|t^2 u\|^2_{L^2((0,\infty);[H^1(\Omega)]^N)} \\ &\qquad + t^2\|tu\|^2_{L^2((0,\infty);[H^1(\Omega)]^N)} + t^4\|u\|^2_{L^2((0,\infty);[H^1(\Omega)]^N)} \Big\}. \end{aligned}$$

Using Lemma 3.1, we have

$$\|E_2(\cdot,t)\|_1 \to 0 \qquad \text{as } \Delta\omega \to 0. \tag{3.12}$$

Finally, from Theorem 3.1 and Theorem 2.1, we have

$$\begin{aligned} \|E_3(\cdot,t)\|_1 &\leq C \left\| \frac{1}{2\pi} \sum_{j=-M}^{M-1} \left(\nabla \widehat{u}_h(\cdot,\omega_{j+1/2}) - \nabla \widehat{u}(\cdot,\omega_{j+1/2}) \right) e^{it\omega_{j+1/2}} \Delta\omega \right\| \\ &\leq C \frac{\Delta\omega}{2\pi} \sum_{j=-M}^{M-1} \left\| \nabla \widehat{u}_h(\cdot,\omega_{j+1/2}) - \nabla \widehat{u}(\cdot,\omega_{j+1/2}) \right\| \\ &\leq C \frac{\Delta\omega}{2\pi} \sum_{j=-M}^{M-1} h \left(1+\omega_{j+1/2}\right)^2 \|\widehat{f}(\cdot,\omega_{j+1/2})\| \\ &\leq Ch \left\| (1+\omega)^2 \widehat{f}(\cdot,\omega) \right\|_{L^2(\mathbb{R};[L^2(\Omega)]^N)}, \end{aligned} \tag{3.13}$$

provided that $\left\| (1+\omega)^2 \widehat{f}(\cdot,\omega) \right\|_{L^2(\mathbb{R};[L^2(\Omega)]^N)} < \infty$.

Let us turn to obtain a full error estimate for $p(x,t)$. Similarly as before, set

$$\begin{aligned} p_{\omega^*}(x,t) &= \frac{1}{2\pi} \int_{-\omega^*}^{\omega^*} \widehat{p}(x,\omega) e^{i\omega t} d\omega, \\ p_{\omega^*,\Delta\omega}(x,t) &= \frac{1}{2\pi} \sum_{j=-M}^{M-1} \widehat{p}(x,\omega_{j+1/2}) e^{it\omega_{j+1/2}} \Delta\omega, \\ p^h_{\omega^*,\Delta\omega}(x,t) &= \frac{1}{2\pi} \sum_{j=-M}^{M-1} \widehat{p}_h(x,\omega_{j+1/2}) e^{it\omega_{j+1/2}} \Delta\omega. \end{aligned}$$

The time domain solution $p(x,t)$ is then approximated by $p^h_{\omega^*,\Delta\omega}(x,t)$. We try to estimate the convergence of $p^h_{\omega^*,\Delta\omega}(x,t)$ to $p(x,t)$ as follows:

$$\begin{aligned} p(x,t) - p^h_{\omega^*,\Delta\omega}(x,t) &= \left(p(x,t) - p_{\omega^*}(x,t)\right) \\ &\quad + \left(p_{\omega^*}(x,t) - p_{\omega^*,\Delta\omega}(x,t)\right) \\ &\quad + \left(p_{\omega^*,\Delta\omega}(x,t) - p^h_{\omega^*,\Delta\omega}(x,t)\right) \\ &\equiv G_1(x,t) + G_2(x,t) + G_3(x,t). \end{aligned} \tag{3.14}$$

By Theorem 2.1,

$$\|G_1(\cdot,t)\| \leq \frac{1}{2\pi} \int_{|\omega|>\omega^*} \|\widehat{p}(\cdot,\omega)\| d\omega \leq C \int_{|\omega|>\omega^*} (1+\omega) \|\widehat{f}(\cdot,\omega)\|_{-1} d\omega,$$

provided that $\int_{|\omega|>\omega^*} (1+\omega)\|\widehat{f}(\cdot,\omega)\|_{-1} d\omega$. Thus

$$\|G_1(\cdot,t)\| \to 0 \qquad \text{as } \omega^* \to \infty. \tag{3.15}$$

We also have

$$\begin{aligned}
\|G_2(\cdot,t)\|^2 &= \frac{1}{4\pi^2}\int_\Omega \left| \int_{-\omega^*}^{\omega^*} \widehat{p}(x,\omega)e^{i\omega t}d\omega - \sum_{j=-M}^{M-1} \widehat{p}(x,\omega_{j+1/2})e^{it\omega_{j+1/2}}\Delta\omega \right|^2 dx \\
&\le C(\Delta\omega)^4 \int_\Omega \left\| \frac{\partial^2(\widehat{p}(x,\omega)e^{i\omega t})}{\partial\omega^2}\right\|^2_{L^2(-\omega^*,\omega^*)} dx \\
&\le C(\Delta\omega)^4 \int_\Omega \left\| -\widehat{t^2p}(x,\cdot) + 2t\widehat{tp}(x,\cdot) - t^2\widehat{p}(x,\cdot)\right\|^2_{L^2(-\omega^*,\omega^*)} dx \\
&\le C(\Delta\omega)^4 \int_\Omega \Big\{ \|\widehat{t^2p}(x,\cdot)\|^2_{L^2(-\infty,\infty)} \\
&\qquad + t^2\|\widehat{tp}(x,\cdot)\|^2_{L^2(-\infty,\infty)} + t^4\|\widehat{p}(x,\cdot)\|^2_{L^2(-\infty,\infty)}\Big\} dx \\
&\le C(\Delta\omega)^4 \Big\{ \|t^2p\|^2_{L^2((0,\infty);[L^2(\Omega)]^N)} \\
&\qquad + t^2\|tp\|^2_{L^2((0,\infty);[L^2(\Omega)]^N)} + t^4\|p\|^2_{L^2((0,\infty);[L^2(\Omega)]^N)}\Big\} .
\end{aligned}$$

Thus, by using Lemma 3.1, it follows that

$$\|G_2(\cdot,t)\| \to 0 \qquad \text{as } \Delta\omega \to 0. \tag{3.16}$$

Finally, from Theorem 3.1 and Theorem 2.1, we have

$$\begin{aligned}
\|G_3(\cdot,t)\| &= \left\| \frac{1}{2\pi}\sum_{j=-M}^{M-1} \left(\widehat{p}_h(\cdot,\omega_{j+1/2}) - \widehat{p}(\cdot,\omega_{j+1/2})\right) e^{it\omega_{j+1/2}}\Delta\omega \right\| \\
&\le \frac{\Delta\omega}{2\pi}\sum_{j=-M}^{M-1} \left\|\widehat{p}_h(\cdot,\omega_{j+1/2}) - \widehat{p}(\cdot,\omega_{j+1/2})\right\| \\
&\le C\frac{\Delta\omega}{2\pi}\sum_{j=-M}^{M-1} h\left(1+\omega_{j+1/2}\right)^2 \|\widehat{f}(\cdot,\omega_{j+1/2})\| \\
&\le Ch\left\|(1+\omega)^2\,\widehat{f}(\cdot,\omega)\right\|_{L^2(\mathbb{R};[L^2(\Omega)]^N)},
\end{aligned} \tag{3.17}$$

provided that $\left\|(1+\omega)^2\,\widehat{f}(\cdot,\omega)\right\|_{L^2(\mathbb{R};[L^2(\Omega)]^N)} < \infty$.

Combining the estimates (3.11), (3.12), (3.13), (3.15), (3.16), and (3.17), we have the following full error estimates for $u(x,t)$ and $p(x,t)$:

Theorem 3.2. Suppose Ω to be as in Theorem 3.1. Assume that for $k = 0, 1, 2$

$$\int_0^\infty s^{2k}\|f(\cdot,s)\|^2\,ds < \infty,$$

and

$$\left\|(1+\omega)^2\,\widehat{f}(\cdot,\omega)\right\|_{L^2_\omega(\mathbb{R};[L^2(\Omega)]^N)} < \infty. \tag{3.18}$$

Then $u^h_{\omega^*,\Delta\omega}(\cdot,t)$ converges to $u(\cdot,t)$ in $[H^1(\Omega)]^N$ and $p^h_{\omega^*,\Delta\omega}(\cdot,t)$ converges to $p(\cdot,t)$ in $L^2(\Omega)$ for a fixed time t; furthermore, for $t>0$,

$$\begin{aligned}\|u(\cdot,t)-u^h_{\omega^*,\Delta\omega}(\cdot,t)\|_1 &+ \|p(\cdot,t)-p^h_{\omega^*,\Delta\omega}(\cdot,t)\| \\ &\le C\int_{|\omega|>\omega^*}(1+\omega)\|\widehat{f}(\cdot,\omega)\|_{-1}\,d\omega \qquad (3.19)\\ &+C(\Delta\omega)^2\sum_{k=0}^{2}t^{2-k}\sum_{j=0}^{k}\left[\int_0^\infty s^{2j}\|f(\cdot,s)\|^2\,ds\right]^{1/2}\\ &+Ch\left\|(1+\omega)^2\,\widehat{f}(\cdot,\omega)\right\|_{L^2_\omega(\mathbb{R};[L^2(\Omega)]^N)},\end{aligned}$$

with C dependent only on the domain and the coefficient μ and $\|\mathcal{U}\|_\infty+\|\nabla\cdot\mathcal{U}\|_\infty$.

Remark 3.1. In the last estimate (3.19) of Theorem 3.2, by using higher-order quadrature rules the second term may be replaced by

$$(\Delta\omega)^m\sum_{k=0}^{m}t^{m-k}\sum_{j=0}^{k}\left[\int_0^\infty s^{2j}\|f(\cdot,s)\|^2\,ds\right]^{1/2},$$

provided that, for $k=0,1,\cdots,m$, for $m>2$,

$$\int_0^\infty s^{2k}\|f(\cdot,s)\|^2\,ds<\infty.$$

Remark 3.2. Under a slightly stronger assumption on the source term than (3.18), the L^2–norm estimate of the velocity part only is obtained in a slightly improved form by repeating the above arguments which lead to Theorem 3.2. More precisely, assume further that for $k=0,1,2$

$$\int_0^\infty s^{2k}\|f(\cdot,s)\|^2\,ds<\infty,$$

and

$$\left\|(1+\omega)^4\,\widehat{f}(\cdot,\omega)\right\|_{L^2_\omega(\mathbb{R};[L^2(\Omega)]^N)} < \infty.$$

Then $u^h_{\omega^*,\Delta\omega}(\cdot,t)$ converges to $u(\cdot,t)$ in $[H^1(\Omega)]^N$ for a fixed time t; furthermore, for $t>0$, the $L^2(\Omega)$-norm estimate for the velocity can be derived as follows:

$$\begin{aligned}\|u(\cdot,t)-u^h_{\omega^*,\Delta\omega}(\cdot,t)\| &\le C\int_{|\omega|>\omega^*}(1+\omega)\|\widehat{f}(\cdot,\omega)\|_{-1}\,d\omega\\ &+C(\Delta\omega)^2\sum_{k=0}^{2}t^{2-k}\sum_{j=0}^{k}\left[\int_0^\infty s^{2j}\|f(\cdot,s)\|^2\,ds\right]^{1/2}\\ &+Ch^2\left\|(1+\omega)^4\,\widehat{f}(\cdot,\omega)\right\|_{L^2_\omega(\mathbb{R};[L^2(\Omega)]^N)},\end{aligned}$$

with C dependent only on the domain and the coefficient μ, $\|\mathcal{U}\|_\infty+\|\nabla\cdot\mathcal{U}\|_\infty$.

Remark 3.3. Notice that the first and third terms in the right side of (3.19) is independent of time t, while the second term shows the rate of increase in time for fixed $\Delta\omega$. In order to get the full error to be bounded within a certain precision for large t, it is recommended to use finer frequency division $\Delta\omega$ such that $t\Delta\omega$ remains small enough. In marching algorithms for evolution problems, e.g. forward Euler or Crank-Nicolson schemes, the mesh size Δt for time is determined by the mesh size h for space. The larger t becomes, the more time steps we need to compute the solutions. In the Frequency Domain Method, the larger t becomes, the smaller $\Delta\omega$ is needed in order to balance the estimate in the second term in the right side of (3.19) according to the third one which depends on the mesh size h.

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