Discussion Paper No. 769

A CHARACTERIZATION OF THE UNIFORM RULE WITH SEVERAL COMMODITIES AND AGENTS

Shuhei Morimoto
Shigehiro Serizawa
Stephen Ching

February 2010

The Institute of Social and Economic Research
Osaka University
6-1 Mihogaoka, Ibaraki, Osaka 567-0047, Japan

A Characterization of the Uniform Rule with Several Commodities and Agents*

Shuhei Morimoto
Graduate School of Economics, Osaka University
1-7, Machikaneyama, Toyonaka, Osaka 560-0043, Japan
E-mail: gge015ms@mail2.econ.osaka-u.ac.jp

Shigehiro Serizawa
Institute of Social and Economic Research, Osaka University
6-1, Mihogaoka, Ibaraki, 567-0047, Japan
E-mail: serizawa@iser.osaka-u.ac.jp

and

Stephen Ching
School of Economics and Finance, The University of Hong Kong
Pokfulam Road, Hong Kong
E-mail: steve.ching@hku.hk

August 25, 2009

Abstract

We consider the problem of allocating infinitely divisible commodities among a group of agents. Especially, we focus on the case where there are several commodities to be allocated, and agents have continuous, strictly convex, and separable preferences. In this paper, we establish that the uniform rule is the only rule satisfying *strategy-proofness*, *unanimity*, *symmetry*, and *nonbossiness*.

Keywords: strategy-proofness, several infinitely divisible commodities, uniform rule, separable preferences

JEL Classification Numbers: C72, D71

*We would like to thank Hans Peters for his valuable ideas at the initial stage of this research. We are also very grateful to William Thomson for his many detailed and helpful comments. We also acknowledge the useful comments by Masaki Aoyagi, Shinji Ohseto, Yuji Fujinaka, Kazuhiko Hashimoto, Kentaro Hatsumi, Hiroki Saitoh, Shohei Takagi, Atsushi Tsuneki, and participants at the 2008 Spring Annual Meeting of the Japanese Economic Association in Tohoku University, and the Ninth International Meeting of the Society for Social Choice and Welfare in Concordia University. Morimoto acknowledges financial support from the Research Fellowships for Young Scientists of the Japan Society for the Promotion of Science. All remaining errors are our own.

1 Introduction

We consider the problem of allocating infinitely divisible commodities among a group of agents. We assume that each agent has continuous, strictly convex, and "separable" preference. A preference is *separable* if the preference over consumption of each commodity is not affected by the consumption levels of the other commodities. A *rule* is a function which chooses a feasible allocation for each preference profile.

Preferences are usually private information. Agents may strategically misrepresent their preferences to obtain assignments they prefer. As a result, the chosen allocations may not be socially desirable relative to the agents' true preferences. Thus, it is important for a rule to give agents the incentive to represent their preferences truthfully. The condition is called *strategy-proofness*.[1] Our purpose is to identify the class of strategy-proof rules which yield socially desirable allocations.

In the one-commodity case, it is well-known that the so-called "uniform rule" is strategy-proof. For this rule, agents are allowed to choose their consumption subject to a common upper or lower bound, which is chosen so as to attain feasibility. In this article, we characterize a multiple-commodity version of the uniform rule by strategy-proofness and the following three axioms. First is *unanimity*, a weak condition of efficiency. It says that if the sum of the peak amounts of each commodity is equal to the supply of the commodity, then each agent's assignment should be equal to his own peak vector. Second is *symmetry*, a weak condition of fairness. It says that two agents with the same preferences should receive assignments between which they are indifferent. Third is *nonbossiness* (Satterthwaite and Sonnenschein, 1981). It says that when an agent's preferences change, if his assignment remains the same, then the chosen allocation should remain the same. We establish that *on the class of continuous, strictly convex, and separable preferences, a rule satisfies strategy-proofness, unanimity, symmetry, and nonbossiness if and only if it is the uniform rule.* This result extends to the class of continuous, strictly convex, and "multidimensional single-peaked" preferences.[2]

Sprumont (1991) gave the first axiomatic characterization of the uniform rule, a rule introduced by Benassy (1982) in a general equilibrium model with fixed prices. For the one-commodity case, he showed that the uniform rule is the only rule satisfying strategy-proofness, Pareto-efficiency, and anonymity[3] (alternatively, no-envy[4]). Ching (1994) strengthened Sprumont's (1991) characterization by replacing

[1]Strategy-proofness requires that it is a weakly dominant strategy for each agent to reveal his true preference.

[2]A preference is *multidimensional single-peaked* if there is an ideal consumption point $p \equiv (p_1, \dots, p_m)$, called the *peak*, and for any two distinct consumption bundles $x \equiv (x_1, \dots, x_m)$ and $y \equiv (y_1, \dots, y_m)$, whenever x_ℓ is between y_ℓ and p_ℓ for each dimension $\ell = 1, \dots, m$, x is strictly preferred to y. The domain of continuous, strictly convex, and separable preferences is a subclass of the multidimensional single-peaked domain.

[3] *Anonymity* requires that if two agent's preferences are switched, then their assignments should be switched too.

[4]*No-envy* requires that no agent should prefer anyone else's assignment to his own.

anonymity with the weaker condition of symmetry.[5]

However, it is desirable to analyze the uniform rule allocating several commodities so that it can be applied to Benassy's (1982) general equilibrium model with fixed prices. Consider an economy with money and several (nonmonetary) commodities in which money consumption is not bounded. Agents have preferences which are continuous, separable, linear with respect to money, and strictly convex in commodities. If the prices are exogenously fixed, then preferences only on consumption of commodities are induced, and they are continuous, strictly convex, and separable. Our result can be applied to this class of economies.

When preferences are single-peaked, Moulin (1980) first characterized the class of strategy-proof voting schemes in one-dimensional public alternative model. He established that a rule satisfies strategy-proofness and unanimity if and only if it is a so called "generalized median voter scheme". Border and Jordan (1983) established that when the set of alternatives is multidimensional and each agent has continuous, strictly convex, separable, and star-shaped[6] preferences, a rule is strategy-proof and unanimous if and only if it can be decomposed into a product of one-dimensional rules, each of which is a generalized median voter scheme. In the same way as Border and Jordan (1983) generalized Moulin's (1980) result to a model with several public alternatives, we generalize Sprumont's (1991) result to a model with several private commodities.

Amorós (2002) also analyzed this situation. Assuming that there are only two agents, he showed that a rule on the class of multidimensional single-peaked preferences satisfies strategy-proofness, same-sideness,[7] and no-envy (alternatively, strong symmetry[8]) if and only if it is the uniform rule.[9] In the two-agent case, since knowing one agent's consumption implies knowing the other agent's consumption, the model is like a public alternatives model. Thus, the model of Amorós (2002) can be treated as that of Border and Jordan (1983), and his result can be derived from theirs,[10] although his proof differs from theirs. On the other hand, when there

---

[5]The single-commodity allotment problem has been analyzed from a wide variety of viewpoints. See, for example, Thomson (1994a,b, 1995, 1997), Otten, Peters and Volij (1996), Barberà, Jackson and Neme (1997), Ching and Serizawa (1998), Massó and Neme (2001, 2007), and Serizawa (2006).

[6]A preference is *star-shaped* if there is an ideal consumption point p such that for any bundle x differing from p, and any real number $a \in (0,1)$, $a \cdot p + (1-a) \cdot x$ is strictly preferred to x and p is strictly preferred to $a \cdot p + (1-a) \cdot x$. Note that, in our model, if a preference is continuous and strictly convex, then it is star-shaped.

[7]*Same-sideness* requires that for each commodity, if the sum of the peak amounts of the commodity is greater (smaller) than, or equal to, the supply of the commodity, then each agent's assignment of the commodity should be smaller (greater) than, or equal to, his own peak amount of the commodity. It is a necessary condition for Pareto-efficiency.

[8]*Strong symmetry* requires that two agents with the same preferences should receive the same assignments.

[9]Sasaki (2003) also showed that the uniform rule is the most efficient rule among all strategy-proof rules in the two-agent and multiple-commodity model.

[10]Since same-sideness implies unanimity, Border and Jordan's (1983) result implies that a rule satisfies strategy-proofness and same-sideness if and only if it can be decomposed into a product of one-dimensional rules, each of which is a generalized median voter scheme. No-envy (alternatively, strong symmetry) implies that the one-dimensional rule for each commodity is the uniform rule.

are more than two agents, the result of Border and Jordan (1983) cannot be applied. Accordingly, we need to devise more complex proof although we owe some techniques to Sprumont's (1991), Ching's (1994), and Border and Jordan's (1983). Besides, unanimity is weaker than same-sideness, and symmetry is weaker than strong symmetry or no-envy. Thus, the result of Amorós (2002) is a corollary of our result.

This paper is organized as follows. Section 2 explains the model and the main result. Section 3 is devoted to the proof of the result in Section 2. Section 4 provides concluding remark.

2 The model and the results

Let $M \equiv \{1, \ldots, m\}$ be a set of infinitely divisible commodities. Let $N \equiv \{1, \ldots, n\}$ be a set of agents. Assume that $2 \leq n < \infty$. For each commodity $\ell \in M$, there is an amount $W_\ell \in \mathbb{R}_{++}$ to be allocated. Let $W \equiv (W_1, \ldots, W_m) \in \mathbb{R}^m_{++}$. For each $i \in N$, agent i's **consumption set** is $X \equiv \{x^i \in \mathbb{R}^m_+ \mid \text{ for each } \ell \in M, 0 \leq x^i_\ell \leq W_\ell\}$, and agent i's **bundle** is a vector $x^i \equiv (x^i_\ell)_{\ell \in M} \in X$. For each $\ell \in M$, let $X_\ell \equiv [0, W_\ell]$ and $X_{-\ell} \equiv \prod_{\ell' \neq \ell}[0, W_{\ell'}]$.

Each agent $i \in N$ has a preference relation R^i on X. A preference R^i is a complete and transitive binary relation on X. Let P^i be the strict preference relation associated with R^i, and I^i the indifference relation. Given a preference R^i and a bundle $x \in X$, the **upper contour set** of R^i at x is $UC(R^i, x) \equiv \{y \in X \mid y \; R^i \; x\}$, and the **lower contour set** of R^i at x is $LC(R^i, x) \equiv \{y \in X \mid x \; R^i \; y\}$. A preference R^i is **continuous** on X if $UC(R^i, x)$ and $LC(R^i, x)$ are both closed for each $x \in X$. A preference R^i is **strictly convex** on X if for each $x \in X$, each pair $\{y, z\} \subset UC(R^i, x)$, and each $a \in (0, 1)$, $y \neq z$ implies $ay + (1-a)z \; P^i \; x$. We assume that preferences are continuous and strictly convex.[11] Given a preference R^i, let $p(R^i) \equiv \{x \in X \mid \text{ for each } y \in X, x \; R^i \; y\}$ be the set of preferred consumptions according to R^i. Since R^i is continuous and strictly convex, $p(R^i)$ is a singleton. We call $p(R^i)$ the **peak of $\boldsymbol{R^i}$** and write $p(R^i) \equiv (p_\ell(R^i))_{\ell \in M}$. We also define two additional properties of preferences.

Definition 1. A preference relation R^i on X is **separable** if for each $\ell \in M$, each $x^i_\ell, \hat{x}^i_\ell \in X_\ell$, and each $x^i_{-\ell}, \hat{x}^i_{-\ell} \in X_{-\ell}$, $(x^i_\ell, x^i_{-\ell}) \; R^i \; (\hat{x}^i_\ell, x^i_{-\ell})$ if and only if $(x^i_\ell, \hat{x}^i_{-\ell}) \; R^i \; (\hat{x}^i_\ell, \hat{x}^i_{-\ell})$.

Definition 2. A preference relation R^i on X is **multidimensional single-peaked** if $p(R^i)$ is a singleton, and for each $x^i, \hat{x}^i \in X$ such that $x^i \neq \hat{x}^i$, whenever for each $\ell \in M$, either $p_\ell(R^i) \geq x^i_\ell \geq \hat{x}^i_\ell$ or $p_\ell(R^i) \leq x^i_\ell \leq \hat{x}^i_\ell$, we have $x^i \; P^i \; \hat{x}^i$.

Let $\mathcal{R}$ denote the class of continuous, strictly convex, and separable preference relations on X. Any such relation is multidimensional single-peaked.

A **feasible allocation** is a list $x \equiv (x^i)_{i \in N} \in X^n$ such that $\sum_{i \in N} x^i = W$. Note that free disposal is not assumed. Let $Z \equiv \{(x^1, \ldots, x^n) \in X^n \mid \sum_{i \in N} x^i = W\}$ be

[11] If R^i is continuous and strictly convex on X, then for each $x \in X$, the set $UC(R^i, x)$ is strictly convex. The converse is not necessarily true.

the set of feasible allocations.

A **preference profile** is a list $R \equiv (R^1, \dots, R^n) \in \mathcal{R}^n$. An allocation rule, or simply a **rule**, is a function $f : \mathcal{R}^n \to Z$. Let R^{-i} be a list of preferences for all agents except for agent i, that is, $R^{-i} \equiv (R^j)_{j \in N \setminus \{i\}}$. We sometimes write the profile $(R^1, \dots, R^{i-1}, \bar{R}^i, R^{i+1}, \dots, R^n)$ as $(\bar{R}^i, R^{-i})$. Let $f^i(R) \equiv (f^i_1(R), \dots, f^i_m(R))$ be the bundle assigned to agent i by f when the preference profile is R.

We now introduce the axioms. Let f be a rule. First is an incentive property: no agent should obtain an assignment he prefers by misrepresenting his preferences.

Strategy-proofness: For each $R \in \mathcal{R}^n$, each $i \in N$, and each $\hat{R}^i \in \mathcal{R}$, $f^i(R)\ R^i\ f^i(\hat{R}^i, R^{-i})$.

Our next three axioms are related to efficiency. An allocation $x \in Z$ is **Pareto-efficient for** $\boldsymbol{R}$ if there is no $y \in Z$ such that, for each $i \in N$, $y^i\ R^i\ x^i$, and for some $j \in N$, $y^j\ P^j\ x^j$. For each $R \in \mathcal{R}^n$, let $P(R)$ be the set of Pareto-efficient allocations for R.

Pareto-efficiency: For each $R \in \mathcal{R}^n$, $f(R) \in P(R)$.

Second, for each commodity, if the sum of the peak amounts of the commodity is greater than, or equal to, the supply of the commodity, then each agent's assignment of the commodity should be smaller than, or equal to, his own peak amount of the commodity, and conversely.

Same-sideness: For each $R \in \mathcal{R}^n$ and each $\ell \in M$,
(i) if $\sum_{i \in N} p_\ell(R^i) \geq W_\ell$, then for each $i \in N$, $f^i_\ell(R) \leq p_\ell(R^i)$, and
(ii) if $\sum_{i \in N} p_\ell(R^i) \leq W_\ell$, then for each $i \in N$, $f^i_\ell(R) \geq p_\ell(R^i)$.

In the one-commodity case, *same-sideness* is equivalent to *Pareto-efficiency*.[12] In the multiple-commodity case, *Pareto-efficiency* implies *same-sideness*, but the converse is not necessarily true. Example 1 illustrates this fact.

Example 1. Let $N \equiv \{1, 2\}$ and $M \equiv \{1, 2\}$. Let f be the rule defined as follows.[13] For each $R \in \mathcal{R}^2$, each $i \in \{1, 2\}$, and each $\ell \in \{1, 2\}$,

$$f^i_\ell(R) \equiv \begin{cases} \frac{p_\ell(R^i) \cdot W_\ell}{p_\ell(R^1) + p_\ell(R^2)} & \text{if } p_\ell(R^1) + p_\ell(R^2) > 0 \\ \frac{W_\ell}{2} & \text{otherwise.} \end{cases}$$

Then, f satisfies *same-sideness*. Let $R \in \mathcal{R}^2$ be such that for each $\ell \in \{1, 2\}$, $p_\ell(R^1) = p_\ell(R^2) = W_\ell$, and there is a bundle $y^1 \in Z$ such that $y^2 = W - y^1$ and for each $i \in \{1, 2\}$, $y^i\ P^i\ (\frac{W_1}{2}, \frac{W_2}{2})$. In this case, $f^1(R) = f^2(R) = (\frac{W_1}{2}, \frac{W_2}{2})$. Then $f(R)$ is Pareto-dominated by y, contradicting *Pareto-efficiency*.

Third, if the sum of the peak amounts of each commodity is equal to the supply of the commodity, then each agent's assignment should be equal to his own peak vector.

Unanimity: For each $R \in \mathcal{R}^n$, if for each $\ell \in M$, $\sum_{i \in N} p_\ell(R^i) = W_\ell$, then for each $i \in N$, $f^i(R) = p(R^i)$.

[12] See Sprumont (1991)

[13] This rule is called *Proportional Rule*.

Obviously, *same-sideness* implies *unanimity.* It is the weakest of our three axioms related to efficiency.

Our next four axioms are related to fairness. First, no agent should prefer anyone else's assignment to his own. Second, if two agents' preferences are switched, then their assignments should be switched too.

No-envy (Foley, 1967)**:** For each $R \in \mathcal{R}^n$ and each $i, j \in N$, $f^i(R)\ R^i\ f^j(R)$.

Anonymity: For each $R \in \mathcal{R}^n$, each $i, j \in N$, and each $\hat{R}^i, \hat{R}^j \in \mathcal{R}$, if $\hat{R}^i = R^j$ and $\hat{R}^j = R^i$, then $f^i(\hat{R}^i, \hat{R}^j, R^{-i,j}) = f^j(R)$ and $f^j(\hat{R}^i, \hat{R}^j, R^{-i,j}) = f^i(R)$.

Third, two agents with the same preferences should receive the same assignments.

Strong symmetry: For each $R \in \mathcal{R}^n$ and each $i, j \in N$, if $R^i = R^j$, then $f^i(R) = f^j(R)$.

Note that *anonymity* implies *strong symmetry.*

Fourth, two agents with the same preferences should receive assignments between which they are indifferent.

Symmetry: For each $R \in \mathcal{R}^n$ and each $i, j \in N$, if $R^i = R^j$, then $f^i(R)\ I^i\ f^j(R)$.

No-envy and *strong symmetry* both imply symmetry. *Symmetry* is the weakest of our four axioms related to fairness. In the one-commodity case, for any rule satisfying *Pareto-efficiency, strong symmetry* is equivalent to *symmetry.*[14] However, in the multiple-commodity case, *same-sideness* and *symmetry* do not imply *strong symmetry.*

Our final axiom says that when an agent's preferences change, if his assignment remains the same, then the chosen allocation should remain the same.

Nonbossiness: For each $R \in \mathcal{R}^n$, each $i \in N$, and each $\hat{R}^i \in \mathcal{R}$, if $f^i(R) = f^i(\hat{R}^i, R^{-i})$, then $f(R) = f(\hat{R}^i, R^{-i})$.

Remark 1. If there are only two agents, then any rule is *nonbossy.*

Next, we introduce a rule that is central to our paper. For each commodity, agents are allowed to choose their consumption subject to a common upper or lower bound, which is chosen so as to attain feasibility.

Uniform rule, *U*: For each $R \in \mathcal{R}^n$, each $\ell \in M$, and each $i \in N$,

$$U^i_\ell(R) = \begin{cases} \min\{p_\ell(R^i), \lambda_\ell(R)\} & \text{if} \sum_{j \in N} p_\ell(R^j) \geq W_\ell \\ \max\{p_\ell(R^i), \lambda_\ell(R)\} & \text{if} \sum_{j \in N} p_\ell(R^j) \leq W_\ell, \end{cases}$$

where $\lambda_\ell(R)$ solves $\sum_{j \in N} U^j_\ell(R) = W_\ell$.

Example 2 illustrates the definition.

Example 2. Let $N \equiv \{1, 2, 3, 4\}$, $M \equiv \{1, 2\}$, and $(W_1, W_2) \equiv (10, 20)$. Let $R \in \mathcal{R}^4$ be such that $p(R^1) = (3, 5)$, $p(R^2) = p(R^3) = (2, 2)$, and $p(R^4) = (5, 6)$.

[14] See Ching (1994).

Then, $\sum_{i\in N} p_1(R^i) > W_1$ and $\sum_{i\in N} p_2(R^i) < W_2$. We calculate $\lambda_1(R) = 3$ and $\lambda_2(R) = 4.5$. Then, $U^1(R) = (3,5)$, $U^2(R) = U^3(R) = (2,4.5)$, and $U^4(R) = (3,6)$.

Next is our main result, a characterization of the uniform rule.

Theorem. *A rule defined on the domain of continuous, strictly convex, and separable preferences satisfies strategy-proofness, unanimity, symmetry, and nonbossiness if and only if it is the uniform rule.*

We remark that our result can be extended to the domain of continuous, strictly convex, and multidimensional single-peaked preferences. The following is a corollary of the Theorem.

Corollary. *A rule defined on the domain of continuous, strictly convex, and multidimensional single-peaked preferences satisfies strategy-proofness, unanimity, symmetry, and nonbossiness if and only if it is the uniform rule.*

The proofs of the Theorem and Corollary are in Section 3.

Amorós (2002) showed that when there are only two agents, the uniform rule is the only rule satisfying *strategy-proofness*, *same-sideness*, and *no-envy* (alternatively, *strong symmetry*). In the two-agent case, by Remark 1, any rule is *nonbossy*. As we mentioned above, *unanimity* is a necessary condition for *same-sideness*, and *symmetry* is weaker than *no-envy* or *strong symmetry*. Thus, we obtain his result as a corollary of our Theorem.

3 Proof of the Theorem

We devote this section to the proof of the Theorem. It is easy to check that the uniform rule is *strategy-proof*, *unanimous*, and *symmetric*.[15] Furthermore, we can easily verify that the single-commodity uniform rule is *nonbossy*. Since the uniform rule assigns commodities by applying the single-commodity uniform rule commodity by commodity, it too is *nonbossy*. Thus, the *if* part of the Theorem holds. We turn to the *only if* part.

For each $\ell \in M$, let $Z_{-\ell} \equiv \{(x^1_{-\ell}, \dots, x^n_{-\ell}) \in (X_{-\ell})^n \mid \text{for each } \ell' \neq \ell, \sum_{i\in N} x^i_{\ell'} = W_{\ell'}\}$ be the set of feasible allocations except for commodity ℓ. Given $\ell \in M$, $x_{-\ell} \in Z_{-\ell}$, and $i \in N$, let $\bar{\mathcal{R}}^i(x_{-\ell}) \equiv \{R^i \in \mathcal{R} \mid \text{for each } \ell' \neq \ell, p_{\ell'}(R^i) = x^i_{\ell'}\}$ and $\bar{\mathcal{R}}^N(x_{-\ell}) \equiv \prod_{i\in N} \bar{\mathcal{R}}^i(x_{-\ell})$. Then, all preference profiles in $\bar{\mathcal{R}}^N(x_{-\ell})$ are the same except for commodity ℓ, that is, for each $R \in \bar{\mathcal{R}}^N(x_{-\ell})$ and each $\ell' \neq \ell$, $\sum_{i\in N} p_{\ell'}(R^i) = W_{\ell'}$. For simplicity, given $\ell \in M$, $R \in \mathcal{R}^n$, and $i \in N$, we write $p_{-\ell}(R^i) \equiv (p_{\ell'}(R^i))_{\ell'\neq\ell}$, $f^i_{-\ell}(R) \equiv (f^i_{\ell'}(R))_{\ell'\neq\ell}$, and $f_{-\ell}(R) \equiv (f_{\ell'}(R))_{\ell'\neq\ell}$.

Let $\ell \in M$ and $x_{-\ell} \in Z_{-\ell}$. We first restrict the domain of rules to the domain $\bar{\mathcal{R}}^N(x_{-\ell})$. We show that on this domain, only the uniform rule satisfies the axioms of the Theorem. Second, we extend the result to the entire domain $\mathcal{R}^n$. This proof technique is similar to that of Border and Jordan (1983). However, they study a model with public alternative, whereas we study a model with private commodities

[15]See Amorós (2002) for the formal proofs.

model. As we explain later, owing to this difference, the naive application of their proof techniques would cause problems that we have to overcome.

Our proof of our Theorem is by means of seven Lemmas. Lemma 1 says that *strategy-proofness*, *unanimity*, and *nonbossiness* imply *same-sideness*. In the one-commodity case, if a rule is *strategy-proof*, *unanimous*, and *nonbossy*, then it is *same-sided* (Serizawa, 2006). This implication also holds in the multiple-commodity case. However, the proofs differ.

Lemma 1. *If a rule is strategy-proof, unanimous, and nonbossy, then it is same-sided.*

Proof. Let f be a rule satisfying the hypotheses. Let $R \in \mathcal{R}^n$ and $\ell \in M$. Assume that $\sum_{h\in N} p_\ell(R^h) \leq W_\ell$. The opposite case can be treated symmetrically. By contradiction, suppose that there is $i \in N$ such that $f^i_\ell(R) < p_\ell(R^i)$. Since $\sum_{h\in N} p_\ell(R^h) \leq W_\ell$, there is $j \in N \setminus \{i\}$ such that $f^j_\ell(R) > p_\ell(R^j)$. Without loss of generality, let $i = 1$ and $j = 2$. For each $k \in N \setminus \{1, 2\}$, let $\hat{R}^k \in \mathcal{R}$ be such that $p(\hat{R}^k) = f^k(R)$. Then, by *strategy-proofness*, $f^3(\hat{R}^3, R^{-3}) = f^3(R)$. By *nonbossiness*, $f(\hat{R}^3, R^{-3}) = f(R)$. Repeating this argument for $k = 4, \ldots, n$, we obtain $f(R^{1,2}, \hat{R}^{-1,2}) = f(R)$.

There are two cases.

Case 1: $p_\ell(R^1) - f^1_\ell(R) \geq f^2_\ell(R) - p_\ell(R^2)$.

Let $(\hat{R}^1, \hat{R}^2) \in \mathcal{R}^2$ be such that (i) $p_\ell(\hat{R}^1) = f^1_\ell(R) + f^2_\ell(R) - p_\ell(R^2)$ and $p_\ell(\hat{R}^2) = p_\ell(R^2)$, (ii) for each $i \in \{1, 2\}$, $p_{-\ell}(\hat{R}^i) = f^i_{-\ell}(R)$, and (iii) for each $i \in \{1, 2\}$, $UC(\hat{R}^i, f^i(R)) \subset UC(R^i, f^i(R))$ and $UC(\hat{R}^i, f^i(R)) \cap LC(R^i, f^i(R)) = \{f^i(R)\}$ (Figure 1).[16] Then, by *strategy-proofness*, $f^1(R^2, \hat{R}^{-2}) = f^1(R^{1,2}, \hat{R}^{-1,2})$. By *nonbossiness*, $f(R^2, \hat{R}^{-2}) = f(R^{1,2}, \hat{R}^{-1,2})$. Since $f(R^{1,2}, \hat{R}^{-1,2}) = f(R)$, we have $f(R^2, \hat{R}^{-2}) = f(R)$. Similarly, by *strategy-proofness* and *nonbossiness*, $f(\hat{R}) = f(R)$. However, by feasibility, for each $\ell' \in M$, $\sum_{k\in N} p_{\ell'}(\hat{R}^k) = \sum_{k\in N} f^k_{\ell'}(R) = W_{\ell'}$. This contradicts *unanimity*.

Case 2: $p_\ell(R^1) - f^1_\ell(R) < f^2_\ell(R) - p_\ell(R^2)$.

Similarly to Case 1, we derive a contradiction to *unanimity* by using preferences $(\hat{R}^1, \hat{R}^2) \in \mathcal{R}^2$ such that (i) $p_\ell(\hat{R}^1) = p_\ell(R^1)$ and $p_\ell(\hat{R}^2) = f^2_\ell(R) - p_\ell(R^1) + f^1_\ell(R)$, (ii) for each $i \in \{1, 2\}$, $p_{-\ell}(\hat{R}^i) = f^i_{-\ell}(R)$, and (iii) for each $i \in \{1, 2\}$, $UC(\hat{R}^i, f^i(R)) \subset UC(R^i, f^i(R))$ and $UC(\hat{R}^i, f^i(R)) \cap LC(R^i, f^i(R)) = \{f^i(R)\}$. □

We now introduce two additional properties of rules. First, if an agent's preferences change but his peak amounts remain the same, then his assignment should remain the same. Second, if all agents' preferences change but the peak profiles remain the same, then the chosen allocation should remain the same.

Own peak-onlyness: For each $R \in \mathcal{R}^n$, each $i \in N$, and each $\hat{R}^i \in \mathcal{R}$, if $p(\hat{R}^i) = p(R^i)$, then $f^i(R) = f^i(\hat{R}^i, R^{-i})$.

Peak-onlyness: For each $R \in \mathcal{R}^n$ and each $\hat{R} \in \mathcal{R}^n$, if for each $i \in N$, $p(\hat{R}^i) = p(R^i)$, then $f(R) = f(\hat{R})$.

[16] The condition (iii) means that $\hat{R}^i$ is a Maskin monotonic transformation of R^i at $f^i(R)$. This notion was first defined by Maskin (1999).

Note that *peak-onlyness* implies *own peak-onlyness*. In the one-commodity case, if a rule is *strategy-proof* and *Pareto-efficient*, then it is *own peak-only* (Sprumont, 1991). In the multiple-commodity case, if a rule is *strategy-proof*, *unanimous*, and *nonbossy*, then for each $\ell \in M$ and each $x_{-\ell} \in Z_{-\ell}$, we can establish the same property on the domain $\bar{\mathcal{R}}^N(x_{-\ell})$. Furthermore, we can also show that if a rule is *nonbossy* and *own peak-only*, then it is *peak-only*.

Lemma 2. *Let f be a strategy-proof, unanimous, and nonbossy rule. Then, for each $\ell \in M$, each $x_{-\ell} \in Z_{-\ell}$, each $R \in \bar{\mathcal{R}}^N(x_{-\ell})$, and each $\hat{R} \in \bar{\mathcal{R}}^N(x_{-\ell})$ such that for each $i \in N$, $p(\hat{R}^i) = p(R^i)$, we have $f(\hat{R}) = f(R)$.*

Proof. Let $\ell \in M$, $x_{-\ell} \in Z_{-\ell}$, $R \in \bar{\mathcal{R}}^N(x_{-\ell})$, and $\hat{R} \in \bar{\mathcal{R}}^N(x_{-\ell})$. Assume that for each $i \in N$, $p(\hat{R}^i) = p(R^i)$. First, we show $f^1(R) = f^1(\hat{R}^1, R^{-1})$. By *same-sideness* (Lemma 1) and $p(R^1) = p(\hat{R}^1)$, we have $f^1_{-\ell}(R) = x^1_{-\ell} = f^1_{-\ell}(\hat{R}^1, R^{-1})$. Assume that $\sum_{i \in N} p_\ell(R^i) \leq W_\ell$. The opposite case can be treated symmetrically. We first show $f^1_\ell(R) \geq f^1_\ell(\hat{R}^1, R^{-1})$. By contradiction, suppose that $f^1_\ell(R) < f^1_\ell(\hat{R}^1, R^{-1})$. Then, by *same-sideness* (Lemma 1), $p_\ell(\hat{R}^1) = p_\ell(R^1) \leq f^1_\ell(R) < f^1_\ell(\hat{R}^1, R^{-1})$. Thus $f^1(R)\ \hat{P}^1\ f^1(\hat{R}^1, R^{-1})$, contradicting *strategy-proofness*. Thus, $f^1_\ell(R) \geq f^1_\ell(\hat{R}^1, R^{-1})$. Similarly, we can show $f^1_\ell(R) \leq f^1_\ell(\hat{R}^1, R^{-1})$. Hence, $f^1(R) = f^1(\hat{R}^1, R^{-1})$. By *nonbossiness*, $f(R) = f(\hat{R}^1, R^{-1})$. Repeating this argument for $k = 2, \ldots, n$, we get $f(\hat{R}) = f(R)$. □

We introduce two more properties of rules. First, for any commodity and any agent, if his peak amount of the commodity is smaller (greater) than his assignment of the commodity and his new peak amount of the commodity is smaller (greater) than, or equal to, his initial assignment of the commodity, then his assignment of the commodity should not change. Second, for any commodity and any group of agents, if for any agent in the group, the same assumption holds, then the chosen allocation of the commodity should not change.

Own uncompromisingness: For each $\ell \in M$, each $R \in \mathcal{R}^n$, each $i \in N$, and each $\hat{R}^i \in \mathcal{R}$,
if $p_\ell(R^i) < f^i_\ell(R)$ and $p_\ell(\hat{R}^i) \leq f^i_\ell(R)$, then, $f^i_\ell(\hat{R}^i, R^{-i}) = f^i_\ell(R)$,
if $p_\ell(R^i) > f^i_\ell(R)$ and $p_\ell(\hat{R}^i) \geq f^i_\ell(R)$, then, $f^i_\ell(\hat{R}^i, R^{-i}) = f^i_\ell(R)$.

Group uncompromisingness: For each $\ell \in M$, each $R \in \mathcal{R}^n$, each $\hat{N} \subseteq N$, and each $\hat{R}^{\hat{N}} \in \mathcal{R}^{|\hat{N}|}$,[17]
if for each $i \in \hat{N}$, $p_\ell(R^i) < f^i_\ell(R)$ and $p_\ell(\hat{R}^i) \leq f^i_\ell(R)$, then, $f_\ell(\hat{R}^{\hat{N}}, R^{-\hat{N}}) = f_\ell(R)$,
if for each $i \in \hat{N}$, $p_\ell(R^i) > f^i_\ell(R)$ and $p_\ell(\hat{R}^i) \geq f^i_\ell(R)$, then, $f_\ell(\hat{R}^{\hat{N}}, R^{-\hat{N}}) = f_\ell(R)$.

In the one-commodity case, if a rule is *strategy-proof* and *Pareto-efficient*, then it satisfies *own uncompromisingness* (Ching, 1994). In the multiple-commodity case, if a rule is *strategy-proof*, *unanimous*, and *nonbossy*, then for each $\ell \in M$ and each $x_{-\ell} \in Z_{-\ell}$, we can establish the same property on the domain $\bar{\mathcal{R}}^N(x_{-\ell})$. Furthermore, we can also show that *nonbossiness* and *own uncompromisingness* imply *group uncompromisingness*.

[17] $|A|$ denotes the cardinality of set A.

Lemma 3. *Let f be a strategy-proof, unanimous, and nonbossy rule. Then, for each $\ell \in M$, and each $x_{-\ell} \in Z_{-\ell}$, we have*
(i) Own uncompromisingness:
for each $R \in \bar{\mathcal{R}}^N(x_{-\ell})$, each $i \in N$, and each $\hat{R}^i \in \bar{\mathcal{R}}^i(x_{-\ell})$,
if $p_\ell(R^i) < f^i_\ell(R)$ and $p_\ell(\hat{R}^i) \leq f^i_\ell(R)$, then, $f^i(\hat{R}^i, R^{-i}) = f^i(R)$,
if $p_\ell(R^i) > f^i_\ell(R)$ and $p_\ell(\hat{R}^i) \geq f^i_\ell(R)$, then, $f^i(\hat{R}^i, R^{-i}) = f^i(R)$,
(ii) Group uncompromisingness:
for each $R \in \bar{\mathcal{R}}^N(x_{-\ell})$, each $\hat{N} \subseteq N$, and each $\hat{R}^{\hat{N}} \in \prod_{i \in \hat{N}} \bar{\mathcal{R}}^i(x_{-\ell})$,
if for each $i \in \hat{N}$, $p_\ell(R^i) < f^i_\ell(R)$ and $p_\ell(\hat{R}^i) \leq f^i_\ell(R)$, then, $f(\hat{R}^{\hat{N}}, R^{-\hat{N}}) = f(R)$,
if for each $i \in \hat{N}$, $p_\ell(R^i) > f^i_\ell(R)$ and $p_\ell(\hat{R}^i) \geq f^i_\ell(R)$, then, $f(\hat{R}^{\hat{N}}, R^{-\hat{N}}) = f(R)$.

Proof. Let $\ell \in M$, $x_{-\ell} \in Z_{-\ell}$, and $R \in \bar{\mathcal{R}}^N(x_{-\ell})$.

Proof of (i). Let $i \in N$ and $\hat{R}^i \in \bar{\mathcal{R}}^i(x_{-\ell})$. Assume that $p_\ell(R^i) < f^i_\ell(R)$ and $p_\ell(\hat{R}^i) \leq f^i_\ell(R)$. The opposite case can be treated symmetrically. We show $f^i(\hat{R}^i, R^{-i}) = f^i(R)$. By *same-sideness* (Lemma 1), $f^i_{-\ell}(\hat{R}^i, R^{-i}) = f^i_{-\ell}(R)$. Also, by *same-sideness* (Lemma 1) and $p_\ell(R^i) < f^i_\ell(R)$, for each $j \in N \setminus \{i\}$, we have $p_\ell(R^j) \leq f^j_\ell(R)$. Since $p_\ell(\hat{R}^i) \leq f^i_\ell(R)$, by feasibility, $p_\ell(\hat{R}^i) + \sum_{j \neq i} p_\ell(R^j) \leq \sum_{j \in N} f^j_\ell(R) = W_\ell$. Thus, by *same-sideness* (Lemma 1), $f^i_\ell(\hat{R}^i, R^{-i}) \geq p_\ell(\hat{R}^i)$. By contradiction, suppose that $f^i_\ell(\hat{R}^i, R^{-i}) \neq f^i_\ell(R)$. There are two cases.
Case 1: $f^i_\ell(\hat{R}^i, R^{-i}) > f^i_\ell(R)$.
In this case, $p_\ell(\hat{R}^i) \leq f^i_\ell(R) < f^i_\ell(\hat{R}^i, R^{-i})$. Thus, $f^i(R)\ \hat{P}^i\ f^i(\hat{R}^i, R^{-i})$, contradicting *strategy-proofness*.
Case 2: $f^i_\ell(\hat{R}^i, R^{-i}) < f^i_\ell(R)$.
Let $\bar{R}^i \in \mathcal{R}$ be such that $p(\bar{R}^i) = p(R^i)$ and $f^i(\hat{R}^i, R^{-i})\ \bar{P}^i\ f^i(R)$. Then, by *peak-onlyness* (Lemma 2), $f^i(\bar{R}^i, R^{-i}) = f^i(R)$. Thus, $f^i(\hat{R}^i, R^{-i})\ \bar{P}^i\ f^i(\bar{R}^i, R^{-i})$, contradicting *strategy-proofness*. □

Proof of (ii). Let $\hat{N} \subseteq N$ and $\hat{R}^{\hat{N}} \in \prod_{i \in \hat{N}} \bar{\mathcal{R}}^i(x_{-\ell})$. Assume that for each $i \in \hat{N}$, $p_\ell(R^i) < f^i_\ell(R)$ and $p_\ell(\hat{R}^i) \leq f^i_\ell(R)$. The opposite case can be treated symmetrically. Without loss of generality, let $\hat{N} \equiv \{1, \ldots, \hat{n}\}$.

By *own uncompromisingness* (Lemma 3-i), $f^1(\hat{R}^1, R^{-1}) = f^1(R)$. By *nonbossiness*, $f(\hat{R}^1, R^{-1}) = f(R)$. Repeating the same argument for $k = 2, \ldots, \hat{n}$, we have $f(\hat{R}^{\hat{N}}, R^{-\hat{N}}) = f(R)$. □

We define a class of preferences, which we call **almost indifferent to all commodities except for commodity ℓ**. Given $\ell \in M$, $x^i_\ell \in \{0, W_\ell\}$, $x^i_{-\ell} \in X_{-\ell}$, and $d \in (0, \frac{W_\ell}{2n})$, let $\mathcal{R}^V(x^i_\ell, x^i_{-\ell}, d) \subset \mathcal{R}$ be the set of preferences R^i such that (i) $p(R^i) = (x^i_\ell, x^i_{-\ell})$, and (ii) for each $y_\ell \in [0, W_\ell - d]$ and each $y_{-\ell} \in X_{-\ell}$, we have

$$\text{if } x^i_\ell = 0, \text{ then } (y_\ell, y_{-\ell})\ P^i\ (y_\ell + d, p_{-\ell}(R^i)), \tag{1}$$

$$\text{if } x^i_\ell = W_\ell, \text{ then } (y_\ell + d, y_{-\ell})\ P^i\ (y_\ell, p_{-\ell}(R^i)). \tag{2}$$

See Figure 2 for an illustration of such a preference relation. Note that for each $\ell \in M$, each $x^i_\ell \in \{0, W_\ell\}$, each $x^i_{-\ell} \in X_{-\ell}$, and each $d \in (0, \frac{W_\ell}{2n})$, the set $\mathcal{R}^V(x^i_\ell, x^i_{-\ell}, d)$

is nonempty.[18] Given $\ell \in M$, $x^i_\ell \in \{0, W_\ell\}$, and $x^i_{-\ell} \in X_{-\ell}$, for sufficiently small $d > 0$, all preferences in $\mathcal{R}^V(x^i_\ell, x^i_{-\ell}, d)$ almost never depend on the consumption of commodities other than commodity ℓ.

Lemma 4 says that for a *strategy-proof* rule, and for any commodity, when an agent's preference changes, if both his old and his new preferences are almost indifferent to all commodities except for the commodity, then his assignment of the commodity changes little.

Lemma 4. *Let f be a strategy-proof rule. Let $\ell \in M$, $d \in (0, \frac{W_\ell}{2n})$, $i \in N$, $x^i_\ell \in \{0, W_\ell\}$, $\bar{x}^i_{-\ell}, \tilde{x}^i_{-\ell} \in X_{-\ell}$, $\bar{R}^i \in \mathcal{R}^V(x^i_\ell, \bar{x}^i_{-\ell}, d)$, $\tilde{R}^i \in \mathcal{R}^V(x^i_\ell, \tilde{x}^i_{-\ell}, d)$, and $R^{-i} \in \mathcal{R}^{n-1}$. Then,*

$$f^i_\ell(\bar{R}^i, R^{-i}) - d < f^i_\ell(\tilde{R}^i, R^{-i}) < f^i_\ell(\bar{R}^i, R^{-i}) + d. \tag{3}$$

Proof. Assume that $x^i_\ell = 0$. By a similar argument, we can also show that (3) holds when $x^i_\ell = W_\ell$. The proof is in two steps.

Step 1. $f^i_\ell(\tilde{R}^i, R^{-i}) < f^i_\ell(\bar{R}^i, R^{-i}) + d$.

Proof. There are two cases.
Case 1-1: $f^i_\ell(\bar{R}^i, R^{-i}) > W_\ell - d$.
In this case, $f^i_\ell(\bar{R}^i, R^{-i}) + d > W_\ell \geq f^i_\ell(\tilde{R}^i, R^{-i})$.
Case 1-2 (Figure 3)**:** $f^i_\ell(\bar{R}^i, R^{-i}) \leq W_\ell - d$.
By contradiction, suppose that

$$f^i_\ell(\tilde{R}^i, R^{-i}) \geq f^i_\ell(\bar{R}^i, R^{-i}) + d. \tag{4}$$

Then, since $\tilde{R}^i \in \mathcal{R}^V(x^i_\ell, \tilde{x}^i_{-\ell}, d)$, by (1), we have

$$f^i(\bar{R}^i, R^{-i}) \; \tilde{P}^i \; \left(f^i_\ell(\bar{R}^i, R^{-i}) + d, \; p_{-\ell}(\tilde{R}^i)\right).$$

By (4),

$$\left(f^i_\ell(\bar{R}^i, R^{-i}) + d, \; p_{-\ell}(\tilde{R}^i)\right) \; \tilde{R}^i \; f^i(\tilde{R}^i, R^{-i}).$$

This implies

$$f^i(\bar{R}^i, R^{-i}) \; \tilde{P}^i \; f^i(\tilde{R}^i, R^{-i}),$$

contradicting *strategy-proofness*. □

Step 2. $f^i_\ell(\bar{R}^i, R^{-i}) - d < f^i_\ell(\tilde{R}^i, R^{-i})$.

Proof. There are two cases.
Case 2-1: $f^i_\ell(\bar{R}^i, R^{-i}) < d$.
In this case, $f^i_\ell(\bar{R}^i, R^{-i}) - d < 0 \leq f^i_\ell(\tilde{R}^i, R^{-i})$.
Case 2-2 (Figure 4)**:** $f^i_\ell(\bar{R}^i, R^{-i}) \geq d$.
By contradiction, suppose that

$$f^i_\ell(\bar{R}^i, R^{-i}) - d \geq f^i_\ell(\tilde{R}^i, R^{-i}). \tag{5}$$

[18] We can show that such a preference exists by constructing a separable and quadratic preference. See Fact in Appendix.

Then, since $\bar{R}^i \in \mathcal{R}^V(x^i_\ell, \bar{x}^i_{-\ell}, d)$, by (1), we have

$$f^i(\tilde{R}^i, R^{-i})\ \bar{P}^i\ \left(f^i_\ell(\tilde{R}^i, R^{-i}) + d,\ p_{-\ell}(\bar{R}^i)\right).$$

By (5),

$$\left(f^i_\ell(\tilde{R}^i, R^{-i}) + d,\ p_{-\ell}(\bar{R}^i)\right)\ \bar{R}^i\ f^i(\bar{R}^i, R^{-i}).$$

This implies

$$f^i(\tilde{R}^i, R^{-i})\ \bar{P}^i\ f^i(\bar{R}^i, R^{-i}),$$

contradicting *strategy-proofness.* □

Lemma 5 says that for a *symmetric* rule, and for any commodity, when two agents have a same preference, if the preference is almost indifferent to all commodities except for the commodity, then their assignments of the commodity differ little.

Lemma 5. *Let f be a symmetric rule. Let $\ell \in M$, $d \in (0, \frac{W_\ell}{2n})$, $x_\ell \in \{0, W_\ell\}$, $x_{-\ell} \in X_{-\ell}$, $\tilde{R}^0 \in \mathcal{R}^V(x_\ell, x_{-\ell}, d)$, $i, j \in N$, $\tilde{R}^i = \tilde{R}^0 = \tilde{R}^j$, and $R^{-i,j} \in \mathcal{R}^{n-2}$. Then,*

$$f^i_\ell(\tilde{R}^{i,j}, R^{-i,j}) - d < f^j_\ell(\tilde{R}^{i,j}, R^{-i,j}) < f^i_\ell(\tilde{R}^{i,j}, R^{-i,j}) + d. \tag{6}$$

Proof. Assume that $x^i_\ell = 0$. By a similar argument, we can also show that (6) holds when $x^i_\ell = W_\ell$. The proof is in two steps.

Step 1. $f^j_\ell(\tilde{R}^{i,j}, R^{-i,j}) < f^i_\ell(\tilde{R}^{i,j}, R^{-i,j}) + d$.

Proof. There are two cases.
Case 1-1: $f^i_\ell(\tilde{R}^{i,j}, R^{-i,j}) > W_\ell - d$.

In this case, $f^i_\ell(\tilde{R}^{i,j}, R^{-i,j}) + d > W_\ell \geq f^j_\ell(\tilde{R}^{i,j}, R^{-i,j})$.

Case 1-2 (Figure 5)**:** $f^i_\ell(\tilde{R}^{i,j}, R^{-i,j}) \leq W_\ell - d$.

By contradiction, suppose that

$$f^j_\ell(\tilde{R}^{i,j}, R^{-i,j}) \geq f^i_\ell(\tilde{R}^{i,j}, R^{-i,j}) + d. \tag{7}$$

Then, since $\tilde{R}^i \in \mathcal{R}^V(x_\ell, x_{-\ell}, d)$, by (1), we have

$$f^i(\tilde{R}^{i,j}, R^{-i,j})\ \tilde{P}^i\ \left(f^i_\ell(\tilde{R}^{i,j}, R^{-i,j}) + d,\ p_{-\ell}(\tilde{R}^i)\right).$$

By (7),

$$\left(f^i_\ell(\tilde{R}^{i,j}, R^{-i,j}) + d,\ p_{-\ell}(\tilde{R}^i)\right)\ \tilde{R}^i\ f^j(\tilde{R}^{i,j}, R^{-i,j}).$$

This implies

$$f^i(\tilde{R}^{i,j}, R^{-i,j})\ \tilde{P}^i\ f^j(\tilde{R}^{i,j}, R^{-i,j}),$$

contradicting *symmetry.* □

Step 2. $f^i_\ell(\tilde{R}^{i,j}, R^{-i,j}) - d < f^j_\ell(\tilde{R}^{i,j}, R^{-i,j})$.

Proof. There are two cases.
Case 2-1: $f^i_\ell(\tilde{R}^{i,j}, R^{-i,j}) < d$.

In this case, $f^i_\ell(\tilde{R}^{i,j}, R^{-i,j}) - d < 0 \leq f^j_\ell(\tilde{R}^{i,j}, R^{-i,j})$.

Case 2-2 (Figure 6)**:** $f^i_\ell(\tilde{R}^{i,j}, R^{-i,j}) \geq d$.

By contradiction, suppose that

$$f^i_\ell(\tilde{R}^{i,j}, R^{-i,j}) - d \geq f^j_\ell(\tilde{R}^{i,j}, R^{-i,j}). \tag{8}$$

Then, since $\tilde{R}^i \in \mathcal{R}^V(x_\ell, x_{-\ell}, d)$, by (1), we have

$$f^j(\tilde{R}^{i,j}, R^{-i,j})\ \tilde{P}^i\ \left(f^j_\ell(\tilde{R}^{i,j}, R^{-i,j}) + d,\ p_{-\ell}(\tilde{R}^i)\right).$$

By (8),

$$\left(f^j_\ell(\tilde{R}^{i,j}, R^{-i,j}) + d,\ p_{-\ell}(\tilde{R}^i)\right)\ \tilde{R}^i\ f^i(\tilde{R}^{i,j}, R^{-i,j}).$$

This implies

$$f^j(\tilde{R}^{i,j}, R^{-i,j})\ \tilde{P}^i\ f^i(\tilde{R}^{i,j}, R^{-i,j}),$$

contradicting *symmetry*. □

Next Lemma follows from Lemmas 4 and 5.

Lemma 6. *Let f be a strategy-proof and symmetric rule. Let $\hat{N} \subsetneq N$, $i \in N \setminus \hat{N}$, and $K = N \setminus (\hat{N} \cup \{i\})$. Let $\ell \in M$, $d \in (0, \frac{W_\ell}{2n})$, $x^i_\ell \in \{0, W_\ell\}$, $\bar{x}^i_{-\ell}, \tilde{x}^i_{-\ell} \in X_{-\ell}$, $\bar{R}^i \in \mathcal{R}^V(x^i_\ell, \bar{x}^i_{-\ell}, d)$, $\tilde{R}^i \in \mathcal{R}^V(x^i_\ell, \tilde{x}^i_{-\ell}, d)$, and $R^K \in \mathcal{R}^{|K|}$. For each $j \in \hat{N}$, let $\tilde{R}^j = \tilde{R}^i$. Then,*

$$f^i_\ell(R^K, \tilde{R}^{\hat{N}}, \bar{R}^i) - d < f^i_\ell(R^K, \tilde{R}^{\hat{N}}, \tilde{R}^i) < f^i_\ell(R^K, \tilde{R}^{\hat{N}}, \bar{R}^i) + d, \text{ and} \tag{9}$$

$$\text{for each } j \in \hat{N},\ f^i_\ell(R^K, \tilde{R}^{\hat{N}}, \bar{R}^i) - 2 \cdot d < f^j_\ell(R^K, \tilde{R}^{\hat{N}}, \tilde{R}^i) < f^i_\ell(R^K, \tilde{R}^{\hat{N}}, \bar{R}^i) + 2 \cdot d. \tag{10}$$

Proof. Since $\bar{R}^i \in \mathcal{R}^V(x^i_\ell, \bar{x}^i_{-\ell}, d)$, $\tilde{R}^i \in \mathcal{R}^V(x^i_\ell, \tilde{x}^i_{-\ell}, d)$, and $(R^K, \tilde{R}^{\hat{N}}) \in \mathcal{R}^{n-1}$, Lemma 4 implies (9). Next, we show (10). Let $j \in \hat{N}$. Since $\tilde{R}^j = \tilde{R}^i$, $\tilde{R}^j \in \mathcal{R}^V(x^i_\ell, \tilde{x}^i_{-\ell}, d)$, and $(R^K, \tilde{R}^{\hat{N}\setminus\{j\}}) \in \mathcal{R}^{n-2}$, Lemma 5 implies that

$$f^i_\ell(R^K, \tilde{R}^{\hat{N}}, \tilde{R}^i) - d < f^j_\ell(R^K, \tilde{R}^{\hat{N}}, \tilde{R}^i) < f^i_\ell(R^K, \tilde{R}^{\hat{N}}, \tilde{R}^i) + d.$$

Now, (10) follows from (9). □

Consider a rule f satisfying the axioms of the Theorem on the domain $\mathcal{R}^n$. Given $\ell \in M$ and $x_{-\ell} \in Z_{-\ell}$, when a preference profile belongs to $\bar{\mathcal{R}}^N(x_{-\ell})$, by *same-sideness* (Lemma 1), each agent receives his own peak amounts of all commodities except for commodity ℓ (or, including commodity ℓ in the case where the sum of the peak amounts of commodity ℓ is also equal to the supply of commodity ℓ). Accordingly, on $\bar{\mathcal{R}}^N(x_{-\ell})$, f induces a rule for commodity ℓ. Lemma 7 says that this induced rule is the single-commodity uniform rule. Although we borrow some techniques from Sprumont (1991) and Ching (1994), as we discuss below, we cannot directly apply their proofs to obtain Lemma 7.

Let $\ell \in M$ and $x_{-\ell} \in Z_{-\ell}$. For each $i \in N$ and each $R^i \in \bar{\mathcal{R}}^i(x_{-\ell})$, let $\bar{R}(R^i)$ be a preference relation on $[0, W_\ell]$ such that for each $x^i_\ell, y^i_\ell \in [0, W_\ell]$, $x^i_\ell \; \bar{R}(R^i) \; y^i_\ell$ if and only if $(x^i_\ell, x^i_{-\ell}) \; R^i \; (y^i_\ell, x^i_{-\ell})$. Let $\bar{P}(R^i)$ be the strict relation associated with $\bar{R}(R^i)$, and $\bar{I}(R^i)$ the indifference relation. Let $\bar{\mathcal{R}}_\ell \equiv \{\bar{R}(R^i) \mid R^i \in \mathcal{R}^i(x_{-\ell})\}$. Then, any preference in $\bar{\mathcal{R}}_\ell$ is continuous and single-peaked on $[0, W_\ell]$.

Consider the restriction $\bar{f}$ of a rule f to $\bar{\mathcal{R}}^N(x_{-\ell})$. If f is *strategy-proof*, then for each $R \in \bar{\mathcal{R}}^N(x_{-\ell})$, each $i \in N$, and each $\hat{R}^i \in \bar{\mathcal{R}}^i(x_{-\ell})$, we have $\bar{f}^i_\ell(R) \; \bar{R}(R^i) \; \bar{f}^i_\ell(\hat{R}^i, R^{-i})$. Thus $\bar{f}_\ell$ is *strategy-proof.* When f is *same-sided*, for each $R \in \bar{\mathcal{R}}^N(x_{-\ell})$, if $\sum_{j \in N} p_\ell(R_j) \geq W_\ell$, then for all $i \in N$, $\bar{f}^i_\ell(R) \leq p_\ell(R_i)$, and if $\sum_{j \in N} p_\ell(R_j) \leq W_\ell$, then for all $i \in N$, $\bar{f}^i_\ell(R) \geq p_\ell(R_i)$. Thus $\bar{f}_\ell$ is *Pareto-efficient.*

Assume that $\bar{f}_\ell$ is *symmetric* for commodity ℓ, that is, for each $R \in \bar{\mathcal{R}}^N(x_{-\ell})$ and each $i, j \in N$, if $\bar{R}(R^i) = \bar{R}(R^j)$, then $\bar{f}^i_\ell(R) \; \bar{I}(R^i) \; \bar{f}^j_\ell(R)$. Then, since the single-commodity uniform rule is the only rule satisfying *strategy-proofness*, *Pareto-efficiency*, and *symmetry* (Sprumont, 1991; Ching, 1994), $\bar{f}_\ell$ is the single-commodity uniform rule.

Therefore, if *symmetry* implied *symmetry* for commodity ℓ, then we could directly apply their result to obtain Lemma 7. However, *symmetry* does not imply *symmetry* for commodity ℓ. We illustrate this point in the next Example.

Example 3. Let $N \equiv \{1, 2\}$ and $M \equiv \{1, 2\}$. Let f be the rule on $\mathcal{R}^2$ defined as follows: for each $R \in \mathcal{R}^2$, if $R^1 = R^2$, $f^1(R) \equiv f^2(R) \equiv (\frac{W_1}{2}, \frac{W_2}{2})$; otherwise $f^1(R) \equiv p(R^1)$, and $f^2(R) \equiv (W_1, W_2) - p(R^1)$. Note that f satisfies *symmetry.*

Let $x_{-1} \equiv (x^1_{-1}, x^2_{-1}) \equiv \left(\frac{2 \cdot W_2}{3}, \frac{W_2}{3}\right)$. Let $\bar{f}$ be the restriction of f to $\bar{\mathcal{R}}^N(x_{-1})$ (Figure 7). Let $R \in \bar{\mathcal{R}}^N(x_{-1})$ be such that $\bar{R}(R^1) = \bar{R}(R^2)$. Then, since $p_2(R^1) = \frac{2 \cdot W_2}{3} \neq \frac{W_2}{3} = p_2(R^2)$, we have $R^1 \neq R^2$. By the definition of $\bar{f}$, $\bar{f}^1_1(R) = p_1(R^1)$ and $\bar{f}^2_1(R) = W_1 - p_1(R^1)$. Thus, $\bar{f}^1_1(R) \; \bar{P}(R^2) \; \bar{f}^2_1(R)$. In fact, $\bar{f}_1$ is dictatorial, and so it violates *symmetry* for commodity 1.

Furthermore, as we discussed in Section 2, in the one-commodity case, any *Pareto-efficient* and *symmetric* rule is *strongly symmetric.* Ching (1994) could apply *strong symmetry* to prove this uniqueness result. However, in the multiple-commodity case, since *same-sideness* and *symmetry* do not imply *strong symmetry*, we cannot apply *strong symmetry* to obtain Lemma 7.

Lemma 7. *Let f be a strategy-proof, unanimous, symmetric, and nonbossy rule. Then, for each $\ell \in M$, each $x_{-\ell} \in Z_{-\ell}$, each $R \in \bar{\mathcal{R}}^N(x_{-\ell})$, and each $i \in N$, $f^i(R) = U^i(R)$.*

Proof. Let $\ell \in M$. By *same-sideness* (Lemma 1), for each $x_{-\ell} \in Z_{-\ell}$, each $R \in \bar{\mathcal{R}}^N(x_{-\ell})$, each $i \in N$, and each $\ell' \neq \ell$, we have $f^i_{\ell'}(R) = x^i_{\ell'} = U^i_{\ell'}(R)$. Thus we only show that for each $x_{-\ell} \in Z_{-\ell}$, each $R \in \bar{\mathcal{R}}^N(x_{-\ell})$, and each $i \in N$, $f^i_\ell(R) = U^i_\ell(R)$.

Let $x_{-\ell} \in Z_{-\ell}$ and $R \in \bar{\mathcal{R}}^N(x_{-\ell})$ be such that $\sum_{i \in N} p_\ell(R^i) = W_\ell$. Then, by the definition of U, for each $i \in N$, $U^i_\ell(R) = p_\ell(R^i)$. By *same-sideness* (Lemma 1), for each $i \in N$, we have $f^i_\ell(R) = p_\ell(R^i)$. Thus, for each $x_{-\ell} \in Z_{-\ell}$, each $R \in \bar{\mathcal{R}}^N(x_{-\ell})$ such that $\sum_{i \in N} p_\ell(R^i) = W_\ell$, and each $i \in N$, we have $f^i_\ell(R) = U^i_\ell(R)$.

Next, we show that for each $x_{-\ell} \in Z_{-\ell}$, each $R \in \bar{\mathcal{R}}^N(x_{-\ell})$ such that $\sum_{i \in N} p_\ell(R^i) < W_\ell$, and each $i \in N$, we have $f^i_\ell(R) = U^i_\ell(R)$.

We introduce some notations. Given a preference profile $R \in \mathcal{R}^n$, let $\pi \equiv (\pi_1, \dots, \pi_n)$ be the permutation of N such that $p_\ell(R^{\pi_1(R)}) \geq \cdots \geq p_\ell(R^{\pi_n(R)})$. We simply write $\pi_1, \dots, \pi_n$ when we can omit R as an argument without confusion. Let $\mathcal{R}_\ell^N(0)$ be the set of preference profiles R such that $\sum_{i \in N} p_\ell(R^i) < W_\ell$ and for each $i \in N$, $p_\ell(R^i) \leq \frac{W_\ell}{n}$. Given $k \in \{1, \dots, n-1\}$, let $\mathcal{R}_\ell^N(k)$ be the set of preference profiles R such that $\sum_{i \in N} p_\ell(R^i) < W_\ell$ and

$$
\begin{aligned}
p_\ell(R^{\pi_1}) &> \frac{W_\ell}{n}, \\
p_\ell(R^{\pi_2}) &> \frac{W_\ell - p_\ell(R^{\pi_1})}{n-1}, \\
&\cdots \\
p_\ell(R^{\pi_k}) &> \frac{W_\ell - \sum_{i=1}^{k-1} p_\ell(R^{\pi_i})}{n-k+1}, \text{ and} \\
p_\ell(R^{\pi_j}) &\leq \frac{W_\ell - \sum_{i=1}^{k} p_\ell(R^{\pi_i})}{n-k} \quad \text{for each } j \in \{k+1, \dots, n\}.
\end{aligned}
$$

Note that $\bigcup_{k=0}^{n-1} \mathcal{R}_\ell^N(k) = \{R \in \mathcal{R}^n \mid \sum_{i \in N} p_\ell(R^i) < W_\ell\}$. For each $k \in \{0, 1, \dots, n-1\}$, let $\mathcal{R}_\ell^N(k, 0)$ be the subdomain of $\mathcal{R}_\ell^N(k)$ such that for each $j \in \{k+1, \dots, n\}$, $p_\ell(R^{\pi_j}) = 0$.

For each $k \in \{0, 1, \dots, n-1\}$ and each $x_{-\ell} \in Z_{-\ell}$, let $\bar{\mathcal{R}}^N(k, x_{-\ell}) \equiv \mathcal{R}_\ell^N(k) \cap \bar{\mathcal{R}}^N(x_{-\ell})$. Note that for each $x_{-\ell} \in Z_{-\ell}$, $\bigcup_{k=0}^{n-1} \bar{\mathcal{R}}^N(k, x_{-\ell}) = \{R \in \bar{\mathcal{R}}^N(x_{-\ell}) \mid \sum_{i \in N} p_\ell(R^i) < W_\ell\}$.

By induction, we will show that for each $k \in \{0, 1, \dots, n-1\}$, each $x_{-\ell} \in Z_{-\ell}$, each $R \in \bar{\mathcal{R}}^N(k, x_{-\ell})$, and each $i \in N$, we have $f_\ell^i(R) = U_\ell^i(R)$.

Step 1. *For each $x_{-\ell} \in Z_{-\ell}$, each $R \in \bar{\mathcal{R}}^N(0, x_{-\ell})$, and each $i \in N$, we have $f_\ell^i(R) = U_\ell^i(R)$.*

Proof. Let $x_{-\ell} \in Z_{-\ell}$ and $R \in \bar{\mathcal{R}}^N(0, x_{-\ell})$. In this case, by the definition of U, for each $i \in N$, $U_\ell^i(R) = \frac{W_\ell}{n}$. We show that for each $i \in N$, $f_\ell^i(R) = \frac{W_\ell}{n}$. Let $\hat{R} \in \bar{\mathcal{R}}^N(x_{-\ell})$ be such that for each $i \in N$, $p_\ell(\hat{R}^i) = 0$. If for each $i \in N$, $f_\ell^i(\hat{R}) = \frac{W_\ell}{n}$, then, by *group uncompromisingness* (Lemma 3-ii), $f(\hat{R}) = f(R)$. Thus, we only have to show that for each $i \in N$, $f_\ell^i(\hat{R}) = \frac{W_\ell}{n}$.

By contradiction, suppose that there is $j \in N$ such that $f_\ell^j(\hat{R}) > \frac{W_\ell}{n}$. Without loss of generality, let $j = 1$. Denote $e_1 \equiv f_\ell^1(\hat{R}) - \frac{W_\ell}{n}$. For each $k \in N \setminus \{1\}$, let $e_k \equiv \frac{e_{k-1}}{n-k+2}$. Then, (i) for each $k \in N \setminus \{1\}$, $e_k > 0$ and $e_k < e_{k-1}$, (ii) for each $k \in \{2, 3, \dots, n\}$,

$$e_k = \frac{e_1}{\prod_{t=0}^{k-2}(n-t)},$$

and (iii) for each $k \in \{2, 3, \dots, n\}$,

$$\frac{e_{k-1}}{n-k+1} - e_k = \frac{e_1}{\prod_{t=0}^{k-1}(n-t)} \geq \frac{e_1}{n!}. \tag{11}$$

Let $d \equiv \frac{e_1}{2 \cdot n!}$. For each $i \in N$, let $\bar{R}^i \in \mathcal{R}^V(0, p_{-\ell}(R^i), d)$, and $\bar{R} \equiv (\bar{R}^1, \dots, \bar{R}^n)$. Then, by *peak-onlyness* (Lemma 2), $f(\bar{R}) = f(\hat{R})$.

Let $\tilde{x}_{-\ell} \in X_{-\ell}$, $\tilde{R}_0 \in \mathcal{R}^V(0, \tilde{x}_{-\ell}, d)$, and $\tilde{R}_0^1 \equiv \tilde{R}_0$.

Step 1-1. $f_\ell^1(\tilde{R}_0^1, \bar{R}^{-1}) \geq \frac{W_\ell}{n} + e_1 = f_\ell^1(\hat{R})$.

Proof. Suppose on the contrary that $f_\ell^1(\tilde{R}_0^1, \bar{R}^{-1}) < \frac{W_\ell}{n} + e_1$. Since $f(\bar{R}) = f(\hat{R})$, we have $f_\ell^1(\bar{R}) = \frac{W_\ell}{n} + e_1$. Let $\check{R}^1 \in \mathcal{R}$ be such that $p(\check{R}^1) = p(\bar{R}^1)$ and $f^1(\tilde{R}_0^1, \bar{R}^{-1}) \; \check{P}^1 \; f^1(\bar{R})$. Then, by *peak-onlyness* (Lemma 2), $f^1(\check{R}^1, \bar{R}^{-1}) = f^1(\bar{R})$. Thus, $f^1(\tilde{R}_0^1, \bar{R}^{-1}) \; \check{P}^1 \; f^1(\check{R}^1, \bar{R}^{-1})$, contradicting *strategy-proofness*. $\square$

Given $\hat{N} \subset N$, let $\tilde{R}_0^{\hat{N}}$ be such that for each $i \in \hat{N}$, $\tilde{R}_0^i = \tilde{R}_0$.

Step 1-2. *For each $k \in \{1, 2, \ldots, n\}$,*
(a) if k is even, then there is $\hat{N} \subset N$ such that $|\hat{N}| = k$ and for each $i \in \hat{N}$, we have $f_\ell^i(\tilde{R}_0^{\hat{N}}, \bar{R}^{-\hat{N}}) \leq \frac{W_\ell}{n} - e_k$, and
(b) if k is odd, then there is $\hat{N} \subset N$ such that $|\hat{N}| = k$ and for each $i \in \hat{N}$, we have $f_\ell^i(\tilde{R}_0^{\hat{N}}, \bar{R}^{-\hat{N}}) \geq \frac{W_\ell}{n} + e_k$.

Proof. The proof proceeds by induction on k. Let $k \in \{1, 2, \ldots, n\}$. When $k = 1$, by Step 1-1, we have already proven that (b) holds. Assume that $k \geq 2$.
Case (a) (Figure 8)**:** k is even.

Our induction hypothesis is that there is $\bar{N} \subset N$ such that $|\bar{N}| = k - 1$, and for each $i \in \bar{N}$, $f_\ell^i(\tilde{R}_0^{\bar{N}}, \bar{R}^{-\bar{N}}) \geq \frac{W_\ell}{n} + e_{k-1}$.

Suppose on the contrary that for each $j \in N \setminus \bar{N}$, $f_\ell^j(\tilde{R}_0^{\bar{N}}, \bar{R}^{-\bar{N}}) > \frac{W_\ell}{n} - \frac{e_{k-1}}{n-k+1}$. Then,

$$
\begin{aligned}
W_\ell &= \sum_{i \in N} f_\ell^i(\tilde{R}_0^{\bar{N}}, \bar{R}^{-\bar{N}}) \qquad \text{(by feasibility)} \\
&> (k-1) \cdot \left(\frac{W_\ell}{n} + e_{k-1} \right) + (n-k+1) \cdot \left(\frac{W_\ell}{n} - \frac{e_{k-1}}{n-k+1} \right) \\
&= W_\ell + (k-2) \cdot e_{k-1} \\
&\geq W_\ell, \qquad \text{(by } k \geq 2 \text{ and } e_{k-1} > 0)
\end{aligned}
$$

which is a contradiction.

Thus, there is $j \in N \setminus \bar{N}$ such that $f_\ell^j(\tilde{R}_0^{\bar{N}}, \bar{R}^{-\bar{N}}) \leq \frac{W_\ell}{n} - \frac{e_{k-1}}{n-k+1}$. Let $\hat{N} \equiv \{j\} \cup \bar{N}$ and $\tilde{R}_0^j \equiv \tilde{R}_0$. Let $i \in \hat{N}$. Then,

$$
\begin{aligned}
f_\ell^i(\tilde{R}_0^{\hat{N}}, \bar{R}^{-\hat{N}}) &< f_\ell^j(\tilde{R}_0^{\bar{N}}, \bar{R}^{-\bar{N}}) + 2 \cdot d \qquad \text{(by Lemma 6)} \\
&\leq \frac{W_\ell}{n} - \frac{e_{k-1}}{n-k+1} + 2 \cdot d \\
&= \frac{W_\ell}{n} - \frac{e_{k-1}}{n-k+1} + \frac{e_1}{n!} \qquad \left(\text{by } d = \frac{e_1}{2 \cdot n!} \right) \\
&\leq \frac{W_\ell}{n} - e_k. \qquad \text{(by } k \geq 2 \text{ and (11))}
\end{aligned}
$$

Case (b) (Figure 9)**:** k is odd.

Our induction hypothesis is that there is $\bar{N} \subset N$ such that $|\bar{N}| = k - 1$, and for each $i \in \bar{N}$, $f_\ell^i(\tilde{R}_0^{\bar{N}}, \bar{R}^{-\bar{N}}) \leq \frac{W_\ell}{n} - e_{k-1}$.

Suppose on the contrary that for each $j \in N \setminus \bar{N}$, $f_\ell^j(\tilde{R}_0^{\bar{N}}, \bar{R}^{-\bar{N}}) < \frac{W_\ell}{n} + \frac{e_{k-1}}{n-k+1}$. Then,

$$\begin{aligned} W_\ell &= \sum_{i \in N} f_\ell^i(\tilde{R}_0^{\bar{N}}, \bar{R}^{-\bar{N}}) \qquad \text{(by feasibility)} \\ &< (k-1) \cdot \left(\frac{W_\ell}{n} - e_{k-1} \right) + (n-k+1) \cdot \left(\frac{W_\ell}{n} + \frac{e_{k-1}}{n-k+1} \right) \\ &= W_\ell - (k-2) \cdot e_{k-1} \\ &\leq W_\ell, \qquad \text{(by } k \geq 2 \text{ and } e_{k-1} > 0) \end{aligned}$$

which is a contradiction.

Thus, there is $j \in N \setminus \bar{N}$ such that $f_\ell^j(\tilde{R}_0^{\bar{N}}, \bar{R}^{-\bar{N}}) \geq \frac{W_\ell}{n} + \frac{e_{k-1}}{n-k+1}$. Let $\hat{N} \equiv \{j\} \cup \bar{N}$ and $\tilde{R}_0^j \equiv \tilde{R}_0$. Let $i \in \hat{N}$. Then,

$$\begin{aligned} f_\ell^i(\tilde{R}_0^{\hat{N}}, \bar{R}^{-\hat{N}}) &> f_\ell^j(\tilde{R}_0^{\bar{N}}, \bar{R}^{-\bar{N}}) - 2 \cdot d \qquad \text{(by Lemma 6)} \\ &\geq \frac{W_\ell}{n} + \frac{e_{k-1}}{n-k+1} - 2 \cdot d \\ &= \frac{W_\ell}{n} + \frac{e_{k-1}}{n-k+1} - \frac{e_1}{n!} \qquad \left(\text{by } d = \frac{e_1}{2 \cdot n!} \right) \\ &\geq \frac{W_\ell}{n} + e_k. \qquad \text{(by } k \geq 2 \text{ and (11))} \end{aligned}$$

Thus, Step 1-2 holds. □

Step 1-3. *We derive a contradiction to conclude that for each $i \in N$, $f_\ell^i(\hat{R}) = \frac{W_\ell}{n}$.*

There are two cases.

Case 1: n is even.

Then, by Case (a) of Step 1-2, for each $i \in N$, $f_\ell^i(\tilde{R}_0^N) \leq \frac{W_\ell}{n} - e_n$. Thus,

$$\begin{aligned} W_\ell &= \sum_{i \in N} f_\ell^i(\tilde{R}_0^N) \qquad \text{(by feasibility)} \\ &\leq n \cdot \left(\frac{W_\ell}{n} - e_n \right) \\ &< W_\ell, \qquad \text{(by } e_n > 0) \end{aligned}$$

which is a contradiction.

Case 2: n is odd.

Then, by Case (b) of Step 1-2, for each $i \in N$, $f_\ell^i(\tilde{R}_0^N) \geq \frac{W_\ell}{n} + e_n$. Thus,

$$\begin{aligned} W_\ell &= \sum_{i \in N} f_\ell^i(\tilde{R}_0^N) \qquad \text{(by feasibility)} \\ &\geq n \cdot \left(\frac{W_\ell}{n} + e_n \right) \\ &> W_\ell, \qquad \text{(by } e_n > 0) \end{aligned}$$

which is a contradiction.

Therefore, for each $i \in N$, we have $f^i_\ell(\hat{R}) = \frac{W_\ell}{n}$. □

Step 2. *Let $k \in \{0, 1, \ldots, n-2\}$. Assume that for each $h \in \{0, 1, \ldots, k\}$, each $\hat{x}_{-\ell} \in Z_{-\ell}$, each $\bar{R} \in \bar{\mathcal{R}}^N(h, \hat{x}_{-\ell})$, and each $i \in N$, we have $f^i_\ell(\bar{R}) = U^i_\ell(\bar{R})$. Then, for each $R \in \mathcal{R}^N_\ell(k+1, 0)$ and each $i \in \{\pi_1, \ldots, \pi_{k+1}\}$, we have $f^i_\ell(R) = p_\ell(R^i)$.*

Proof. Let $R \in \mathcal{R}^N_\ell(k+1, 0)$. Without loss of generality, assume that agents are indexed so that $p_\ell(R^1) \geq \cdots \geq p_\ell(R^{k+1})$. Let $\bar{K} \equiv \{1, 2, \ldots, k+1\}$.

Step 2-1. *Let $\hat{x}_{-\ell} \in Z_{-\ell}$ and $\hat{R} \in \mathcal{R}^N(\hat{x}_{-\ell})$ be such that for each $i \in N$, $p_\ell(\hat{R}^i) = p_\ell(R^i)$. Then, for each $i \in \bar{K}$, we have $f^i_\ell(\hat{R}) = p_\ell(\hat{R}^i)$.*

Proof. Let $i \in \bar{K}$. By contradiction, suppose that $f^i_\ell(\hat{R}) \neq p_\ell(\hat{R}^i)$. By *same-sideness* (Lemma 1), $f^i_\ell(\hat{R}) > p_\ell(\hat{R}^i)$.

Let $\tilde{R}^i_0 \in \mathcal{R}$ be such that $p_\ell(\tilde{R}^i_0) = 0$ and $p_{-\ell}(\tilde{R}^i_0) = p_{-\ell}(\hat{R}^i)$. Then, by *own uncompromisingness* (Lemma 3-i), $f^i_\ell(\tilde{R}^i_0, \hat{R}^{-i}) = f^i_\ell(\hat{R})$. Note that for some $h \in \{0, 1, \ldots, k\}$, we have $(\tilde{R}^i_0, \hat{R}^{-i}) \in \bar{\mathcal{R}}^N(h, \hat{x}_{-\ell})$.

By the assumption of Step 2, $f^i_\ell(\tilde{R}^i_0, \hat{R}^{-i}) = U^i_\ell(\tilde{R}^i_0, \hat{R}^{-i})$. Also, by the definition of U and $\hat{R} \in \mathcal{R}^N_\ell(k+1, 0)$, we have $U^i_\ell(\hat{R}) = p_\ell(\hat{R}^i)$. Since $p_\ell(\tilde{R}^i_0) = 0 < p_\ell(\hat{R}^i)$, by the definition of U, we get $U^i_\ell(\tilde{R}^i_0, \hat{R}^{-i}) \leq U^i_\ell(\hat{R})$. Thus, $f^i_\ell(\tilde{R}^i_0, \hat{R}^{-i}) \leq p_\ell(\hat{R}^i) < f^i_\ell(\hat{R}) = f^i_\ell(\tilde{R}^i_0, \hat{R}^{-i})$, which is a contradiction. □

Step 2-2. *For each $i \in \bar{K}$, we have $f^i_\ell(R) = p_\ell(R^i)$.*

Proof. Let $\bar{R} \in \mathcal{R}^n$ be such that for each $i \in N$, (i) $p_\ell(\bar{R}^i) = p_\ell(R^i)$, (ii) $p_{-\ell}(\bar{R}^i) = f^i_{-\ell}(R)$, and (iii) $UC(\bar{R}^i, f^i(R)) \subset UC(R^i, f^i(R))$ and $UC(\bar{R}^i, f^i(R)) \cap LC(R^i, f^i(R)) = \{f^i(R)\}$ (Figure 10).

Then, by *strategy-proofness*, $f^1(\bar{R}^1, R^{-1}) = f^1(R)$. By *nonbossiness*, $f(\bar{R}^1, R^{-1}) = f(R)$. Repeating the same argument for $i = 2, \ldots, n$, we have $f(\bar{R}) = f(R)$.

By feasibility, $f_{-\ell}(R) \in Z_{-\ell}$. Since for each $i \in N$, $p_{-\ell}(\bar{R}^i) = f^i_{-\ell}(R)$, we have $\bar{R} \in \mathcal{R}^N(f_{-\ell}(R))$. By Step 2-1 and for each $i \in N$, $p_\ell(\bar{R}^i) = p_\ell(R^i)$, it follows that for each $i \in \bar{K}$, $f^i_\ell(\bar{R}) = p_\ell(\bar{R}^i) = p_\ell(R^i)$. Since $f(\bar{R}) = f(R)$, for each $i \in \bar{K}$, we have $f^i_\ell(R) = p_\ell(R^i)$. □

Step 3. *For each $k \in \{1, 2, \ldots, n-2\}$, each $x_{-\ell} \in Z_{-\ell}$, each $R \in \bar{\mathcal{R}}^N(k, x_{-\ell})$, and each $i \in N$, we have $f^i_\ell(R) = U^i_\ell(R)$.*

Proof. The proof proceeds by induction on k. Let $k \in \{1, 2, \ldots, n-2\}$. Assume that

(A) *For each $h \in \{0, 1, \ldots, k-1\}$, each $\hat{x}_{-\ell} \in Z_{-\ell}$, each $\bar{R} \in \bar{\mathcal{R}}^N(h, \hat{x}_{-\ell})$, and each $i \in N$, we have $f^i_\ell(\bar{R}) = U^i_\ell(\bar{R})$.*

By Step 1, we have already proven that (A) holds when $k = 1$. We will prove that

(B) *For each $x_{-\ell} \in Z_{-\ell}$, each $R \in \bar{\mathcal{R}}^N(k, x_{-\ell})$, and each $i \in N$, we have $f^i_\ell(R) = U^i_\ell(R)$.*

Let $x_{-\ell} \in Z_{-\ell}$ and $R \in \bar{\mathcal{R}}^N(k, x_{-\ell})$. Without loss of generality, we may assume that agents are indexed so that $p_\ell(R^1) \geq \cdots \geq p_\ell(R^n)$. Let $K \equiv \{1, 2, \ldots, k\}$.

Then, by the definition of U, for each $i \in K$, $U_\ell^i(R) = p_\ell(R^i)$, and for each $j \in N \setminus K$, $U_\ell^j(R) = \frac{W_\ell - \sum_{i=1}^k p_\ell(R^i)}{n-k} = \lambda_\ell(R)$. Note that, since $R \in \bar{\mathcal{R}}^N(k, x_{-\ell})$ implies $\sum_{i\in N} p_\ell(R^i) < W_\ell$, we have $\lambda_\ell(R) > 0$. We show that for each $i \in N$, $f_\ell^i(R) = U_\ell^i(R)$.

Let $\hat{R}^{-K} \in \mathcal{R}^{|-K|}$ be such that for each $i \in N \setminus K$, $p_\ell(\hat{R}^i) = 0$ and $p_{-\ell}(\hat{R}^i) = p_{-\ell}(R^i)$. Then, by the definition of U, for each $i \in N$, $U_\ell^i(R^K, \hat{R}^{-K}) = U_\ell^i(R)$. If for each $i \in N$, $f_\ell^i(R^K, \hat{R}^{-K}) = U_\ell^i(R^K, \hat{R}^{-K})$, then, by *group uncompromisingness* (Lemma 3-ii), $f(R) = f(R^K, \hat{R}^{-K})$. Thus, we only have to show for each $i \in N$, $f_\ell^i(R^K, \hat{R}^{-K}) = U_\ell^i(R^K, \hat{R}^{-K})$.

By (A) and Step 2, we have already proven that for each $i \in K$, $f_\ell^i(R^K, \hat{R}^{-K}) = p_\ell(R^i)$. Thus, we only show that for each $i \in N \setminus K$, $f_\ell^i(R^K, \hat{R}^{-K}) = \lambda_\ell(R)$.

By contradiction, suppose that there is $j \in N \setminus K$ such that $f_\ell^j(R^K, \hat{R}^{-K}) > \lambda_\ell(R)$. Without loss of generality, assume that $j = k+1$. Note that, since for each $i \in K$, $f_\ell^i(R^K, \hat{R}^{-K}) = p_\ell(R^i)$, and for each $i \in \{k+2, \dots, n\}$, $f_\ell^i(R^K, \hat{R}^{-K}) \geq 0$, by feasibility, $f_\ell^{k+1}(R^K, \hat{R}^{-K}) \leq W_\ell - \sum_{i=1}^k p_\ell(R^i)$.

Let $e_1 \equiv f_\ell^{k+1}(R^K, \hat{R}^{-K}) - \lambda_\ell(R)$. For each $h \in \{2, 3, \dots, n-k\}$, let $e_h \equiv \frac{e_{h-1}}{n-k-h+2}$. Then, (i) for each $h \in \{2, 3, \dots, n-k\}$, $e_h > 0$ and $e_h < e_{h-1}$, (ii) for each $h \in \{2, 3, \dots, n-k\}$,

$$e_h = \frac{e_1}{\prod_{t=0}^{h-2}(n-k-t)},$$

and (iii) for each $h \in \{2, 3, \dots, n-k\}$,

$$\frac{e_{h-1}}{n-k-h+1} - e_h = \frac{e_1}{\prod_{t=0}^{h-1}(n-k-t)} \geq \frac{e_1}{(n-k)!}. \tag{12}$$

Let $d \equiv \frac{e_1}{2\cdot(n-k)!}$. For each $i \in N \setminus K$, let $\bar{R}^i \in \mathcal{R}^V(0, p_{-\ell}(\hat{R}^i), d)$, and $\bar{R}^{-K} \equiv (\bar{R}^{k+1}, \dots, \bar{R}^n)$. Then, by *peak-onlyness* (Lemma 2), $f(R^K, \bar{R}^{-K}) = f(R^K, \hat{R}^{-K})$.

Let $\tilde{x}_{-\ell} \in Z_{-\ell}$, $\tilde{R}_0 \in \mathcal{R}_0^V(0, \tilde{x}_{-\ell}, d)$, and $\tilde{R}_0^{k+1} \equiv \tilde{R}_0$.

Step 3-1. $f_\ell^{k+1}(R^K, \tilde{R}_0^{k+1}, \bar{R}^{-K\cup\{k+1\}}) \geq \lambda_\ell(R) + e_1 = f_\ell^{k+1}(R^K, \hat{R}^{-K})$.

Proof. The proof is similar to Step 1-1. By contradiction, suppose not. Since $f(R^K, \bar{R}^{-K}) = f(R^K, \hat{R}^{-K})$, we have $f_\ell^{k+1}(R^K, \bar{R}^{-K}) = \lambda_\ell(R) + e_1$. Let $\check{R}^{k+1} \in \mathcal{R}$ be such that $p(\check{R}^{k+1}) = p(\bar{R}^{k+1})$ and $f^{k+1}(R^K, \tilde{R}_0^{k+1}, \bar{R}^{-K\cup\{k+1\}})\ \check{P}^{k+1}\ f^{k+1}(R^K, \bar{R}^{-K})$. Then, by *peak-onlyness* (Lemma 2), $f^{k+1}(R^K, \check{R}^{k+1}, \bar{R}^{-K\cup\{k+1\}}) = f^{k+1}(R^K, \bar{R}^{-K})$. Thus, $f^{k+1}(R^K, \tilde{R}_0^{k+1}, \bar{R}^{-K\cup\{k+1\}})\ \check{P}^{k+1}\ f^{k+1}(R^K, \check{R}^{k+1}, \bar{R}^{-K\cup\{k+1\}})$, contradicting *strategy-proofness*. □

Given $\hat{N} \subset N \setminus K$, let $\tilde{R}_0^{\hat{N}}$ be such that for each $i \in \hat{N}$, $\tilde{R}_0^i = \tilde{R}_0$.

Step 3-2. *For each $h \in \{1, 2, \dots, n-k\}$,*
(a) if h is even, then there is $\hat{N} \subset N \setminus K$ such that $|\hat{N}| = h$ and for each $i \in \hat{N}$, we have $f_\ell^i(R^K, \tilde{R}_0^{\hat{N}}, \bar{R}^{-K\cup\hat{N}}) \leq \lambda_\ell(R) - e_h$, and
(b) if h is odd, then there is $\hat{N} \subset N \setminus K$ such that $|\hat{N}| = h$ and for each $i \in \hat{N}$, we have $f_\ell^i(R^K, \tilde{R}_0^{\hat{N}}, \bar{R}^{-K\cup\hat{N}}) \geq \lambda_\ell(R) + e_h$.

Proof. The proof is similar to Step 1-2, and proceeds by induction on h. Let $h \in \{1, 2, \ldots, n-k\}$. When $h = 1$, by Step 3-1, we have already proven that (b) holds. Assume that $h \geq 2$.

Case (a): h is even.

Our induction hypothesis is that there is $\bar{N} \subset N \setminus K$ such that $|\bar{N}| = h-1$ and for each $i \in \bar{N}$, $f^i_\ell(R^K, \tilde{R}^{\bar{N}}_0, \bar{R}^{-K\cup\bar{N}}) \geq \lambda_\ell(R) + e_{h-1}$.

Suppose on the contrary that for each $j \in N \setminus (K \cup \bar{N})$, $f^j_\ell(R^K, \tilde{R}^{\bar{N}}_0, \bar{R}^{-K\cup\bar{N}}) > \lambda_\ell(R) - \frac{e_{h-1}}{n-k-h+1}$. By (A) and Step 2, for each $i \in K$, $f^i_\ell(R^K, \tilde{R}^{\bar{N}}_0, \bar{R}^{-K\cup\bar{N}}) = p_\ell(R^i)$. Thus,

$$\begin{aligned}
W_\ell &= \sum_{i\in N} f^i_\ell(R^K, \tilde{R}^{\bar{N}}_0, \bar{R}^{-K\cup\bar{N}}) \qquad \text{(by feasibility)}\\
&> \sum_{i\in K} p_\ell(R^i) + (h-1)\cdot(\lambda_\ell(R) + e_{h-1}) + (n-k-h+1)\cdot\left(\lambda_\ell(R) - \frac{e_{h-1}}{n-k-h+1}\right)\\
&= W_\ell + (h-2)\cdot e_{h-1} \qquad \left(\text{ by } \sum_{i\in K} p_\ell(R^i) + (n-k)\cdot\lambda_\ell(R) = W_\ell\right)\\
&\geq W_\ell, \qquad \text{(by } h \geq 2 \text{ and } e_{h-1} > 0)
\end{aligned}$$

which is a contradiction. Thus, there is $j \in N\setminus(K\cup\bar{N})$ such that $f^j_\ell(R^K, \tilde{R}^{\bar{N}}_0, \bar{R}^{-K\cup\bar{N}}) \leq \lambda_\ell(R) - \frac{e_{h-1}}{n-k-h+1}$.

Let $\hat{N} \equiv \bar{N} \cup \{j\}$ and $\tilde{R}^j_0 \equiv \tilde{R}_0$. Let $i \in \hat{N}$. Then,

$$\begin{aligned}
f^i_\ell(R^K, \tilde{R}^{\hat{N}}_0, \bar{R}^{-K\cup\hat{N}}) &< f^j_\ell(R^K, \tilde{R}^{\bar{N}}_0, \bar{R}^{-K\cup\bar{N}}) + 2\cdot d \qquad \text{(by Lemma 6)}\\
&\leq \lambda_\ell(R) - \frac{e_{h-1}}{n-k-h+1} + 2\cdot d\\
&= \lambda_\ell(R) - \frac{e_{h-1}}{n-k-h+1} + \frac{e_1}{(n-k)!} \qquad \left(\text{by } d = \frac{e_1}{2\cdot(n-k)!}\right)\\
&\leq \lambda_\ell(R) - e_h. \qquad \text{(by } h \geq 2 \text{ and (12))}
\end{aligned}$$

Case (b): h is odd.

Our induction hypothesis is that there is $\bar{N} \subset N \setminus K$ such that $|\bar{N}| = h-1$ and for each $i \in \bar{N}$, $f^i_\ell(R^K, \tilde{R}^{\bar{N}}_0, \bar{R}^{-K\cup\bar{N}}) \leq \lambda_\ell(R) - e_{h-1}$.

Suppose on the contrary that for each $j \in N \setminus (K \cup \bar{N})$, $f^j_\ell(R^K, \tilde{R}^{\bar{N}}_0, \bar{R}^{-K\cup\bar{N}}) < \lambda_\ell(R) + \frac{e_{h-1}}{n-k-h+1}$. By (A) and Step 2, for each $i \in K$, $f^i_\ell(R^K, \tilde{R}^{\bar{N}}_0, \bar{R}^{-K\cup\bar{N}}) = p_\ell(R^i)$. Thus,

$$\begin{aligned}
W_\ell &= \sum_{i\in N} f^i_\ell(R^K, \tilde{R}^{\bar{N}}_0, \bar{R}^{-K\cup\bar{N}}) \qquad \text{(by feasibility)}\\
&< \sum_{i\in K} p_\ell(R^j) + (h-1)\cdot(\lambda_\ell(R) - e_{h-1}) + (n-k-h+1)\cdot\left(\lambda_\ell(R) + \frac{e_{h-1}}{n-k-h+1}\right)\\
&= W_\ell - (h-2)\cdot e_{h-1} \qquad \left(\text{ by } \sum_{i\in K} p_\ell(R^i) + (n-k)\cdot\lambda_\ell(R) = W_\ell\right)\\
&\leq W_\ell, \qquad \text{(by } h \geq 2 \text{ and } e_{h-1} > 0)
\end{aligned}$$

which is a contradiction. Thus, there is $j \in N \backslash (K \cup \bar{N})$ such that $f_\ell^j(R^K, \tilde{R}_0^{\bar{N}}, \bar{R}^{-K\cup\bar{N}}) \geq \lambda_\ell(R) + \frac{e_{h-1}}{n-k-h+1}$.

Let $\hat{N} \equiv \bar{N} \cup \{j\}$ and $\tilde{R}_0^j \equiv \tilde{R}_0$. Let $i \in \hat{N}$. Then,

$$\begin{aligned}
f_\ell^i(R^K, \tilde{R}_0^{\hat{N}}, \bar{R}^{-K\cup\hat{N}}) &> f_\ell^j(R^K, \tilde{R}_0^{\bar{N}}, \bar{R}^{-K\cup\bar{N}}) - 2 \cdot d \qquad \text{(by Lemma 6)} \\
&\geq \lambda_\ell(R) + \frac{e_{h-1}}{n-k-h+1} - 2 \cdot d \\
&= \lambda_\ell(R) + \frac{e_{h-1}}{n-k-h+1} - \frac{e_1}{(n-k)!} \qquad \left(\text{by } d = \frac{e_1}{2 \cdot (n-k)!}\right) \\
&\geq \lambda_\ell(R) + e_h. \qquad (\text{by } h \geq 2 \text{ and } (12))
\end{aligned}$$

Thus, Step 3-2 holds. □

Step 3-3. *We derive a contradiction to conclude that for each* $i \in N \backslash K$, $f_\ell^i(R^K, \hat{R}^{-K}) = \lambda_\ell(R)$.

By (A) and Step 2, for each $i \in K$, $f_\ell^i(R^K, \tilde{R}_0^{-K}) = p_\ell(R^i)$.

There are two cases.

Case 1: $n - k$ is even.

Then, by Case (a) of Step 3-1, for each $i \in N \setminus K$, we have $f_\ell^i(R^K, \tilde{R}_0^{-K}) \leq \lambda_\ell(R) - e_{n-k}$. Thus,

$$\begin{aligned}
W_\ell &= \sum_{i \in N} f_\ell^i(R^K, \tilde{R}_0^{-K}) \qquad \text{(by feasibility)} \\
&\leq \sum_{i \in K} p_\ell(R^i) + (n-k) \cdot \lambda_\ell(R) - (n-k) \cdot e_{n-k} \\
&= W_\ell - (n-k) \cdot e_{n-k} \qquad \left(\text{by } \sum_{i \in K} p_\ell(R^i) + (n-k) \cdot \lambda_\ell(R) = W_\ell\right) \\
&< W_\ell, \qquad (\text{by } n-k \geq 2 \text{ and } e_{n-k} > 0)
\end{aligned}$$

which is a contradiction.

Case 2: $n - k$ is odd.

Then, by Case (b) of Step 3-1, for each $i \in N \setminus K$, we have $f_\ell^i(R^K, \tilde{R}_0^{-K}) \geq \lambda_\ell(R) + e_{n-k}$. Thus,

$$\begin{aligned}
W_\ell &= \sum_{i \in N} f_\ell^i(R^K, \tilde{R}_0^{-K}) \qquad \text{(by feasibility)} \\
&\geq \sum_{i \in K} p_\ell(R^i) + (n-k) \cdot \lambda_\ell(R) + (n-k) \cdot e_{n-k} \\
&= W_\ell + (n-k) \cdot e_{n-k} \qquad \left(\text{by } \sum_{i \in K} p_\ell(R^i) + (n-k) \cdot \lambda_\ell(R) = W_\ell\right) \\
&> W_\ell, \qquad (\text{by } n-k \geq 2 \text{ and } e_{n-k} > 0)
\end{aligned}$$

which is a contradiction.

Therefore, for each $i \in N \setminus K$, we have $f_\ell^i(R^K, \hat{R}^{-K}) = \lambda_\ell(R)$. □

Step 4. *For each $x_{-\ell} \in Z_{-\ell}$, each $R \in \bar{\mathcal{R}}^N(n-1, x_{-\ell})$, and each $i \in N$, we have $f^i_\ell(R) = U^i_\ell(R)$.*

Proof. Let $x_{-\ell} \in Z_{-\ell}$ and $R \in \bar{\mathcal{R}}^N(n-1, x_{-\ell})$. Without loss of generality, we may assume that $p_\ell(R^1) \geq \cdots \geq p_\ell(R^n)$. Then, by the definition of U, for each $i \in N \setminus \{n\}$, $U^i_\ell(R) = p_\ell(R^i)$, and $U^n_\ell(R) = W_\ell - \sum_{i=1}^{n-1} p_\ell(R^i)$. Let $\hat{R}^n \in \mathcal{R}$ be such that $p_\ell(\hat{R}^n) = 0$ and $p_{-\ell}(\hat{R}^n) = p_{-\ell}(R^n)$. By Steps 1 and 3, when $k = n-2$, we have already proven that the assumption of Step 2 holds. Thus Step 2 implies that for each $i \in N \setminus \{n\}$, $f^i_\ell(\hat{R}^n, R^{-n}) = p_\ell(R^i)$. By feasibility, $f^n_\ell(\hat{R}^n, R^{-n}) = W_\ell - \sum_{i=1}^{n-1} p_\ell(R^i)$. Since $\sum_{i \in N} p_\ell(R^i) < W_\ell$, we have $W_\ell - \sum_{i=1}^{n-1} p_\ell(R^i) > 0$. Thus $p_\ell(\hat{R}^n) < f_\ell(\hat{R}^n, R^{-n})$. By *own uncompromisingness* (Lemma 3-i), $f(R) = f(\hat{R}^n, R^{-n})$. Thus, for each $i \in N$, we have $f^i_\ell(R) = U^i_\ell(R)$. □

In the same way, we can also show that for each $x_{-\ell} \in Z_{-\ell}$, each $R \in \bar{\mathcal{R}}^N(x_{-\ell})$ such that $\sum_{i \in N} p_\ell(R^i) > W_\ell$, and each $i \in N$, we have $f^i_\ell(R) = U^i_\ell(R)$.

We have completed the proof of Lemma 7. □

Proof of the Theorem. Let f be a *strategy-proof*, *unanimous*, *symmetric*, and *nonbossy* rule. We will show that for each $R \in \mathcal{R}^n$ and each $\ell \in M$, $f_\ell(R) = U_\ell(R)$. Let $R \in \mathcal{R}^n$ and $\ell \in M$. Let $\bar{R} \in \mathcal{R}^n$ be such that for each $i \in N$, (i) $p_\ell(\bar{R}^i) = p_\ell(R^i)$, (ii) $p_{-\ell}(\bar{R}^i) = f^i_{-\ell}(R)$, and (iii) $UC(\bar{R}^i, f^i(R)) \subset UC(R^i, f^i(R))$ and $UC(\bar{R}^i, f^i(R)) \cap LC(R^i, f^i(R)) = \{f^i(R)\}$ (Figure 10). By *strategy-proofness*, $f^1(\bar{R}^1, R^{-1}) = f^1(R)$. By *nonbossiness*, $f(\bar{R}^1, R^{-1}) = f(R)$. Repeating the same argument for $i = 2, \ldots, n$, we have $f(\bar{R}) = f(R)$.

By feasibility, $f_{-\ell}(R) \in Z_{-\ell}$. Since for each $i \in N$, $p_{-\ell}(\bar{R}^i) = f^i_{-\ell}(R)$, we have $\bar{R} \in \bar{\mathcal{R}}^N(f_{-\ell}(R))$. Thus Lemma 7 implies $f_\ell(\bar{R}) = U_\ell(\bar{R})$. By the definition of U, $U_\ell(\bar{R}) = U_\ell(R)$. Thus, $f_\ell(\bar{R}) = U_\ell(R)$. Since $f(\bar{R}) = f(R)$, we obtain $f_\ell(R) = U_\ell(R)$. □

Proof of the Corollary. Let f be a *strategy-proof*, *unanimous*, *symmetric*, and *nonbossy* rule defined on the domain $\mathcal{R}^n_M$ of continuous, strictly convex, and multidimensional single-peaked preferences. Let $R \in \mathcal{R}^n_M$. Let $\hat{R} \in \mathcal{R}^n$ be such that for each $i \in N$, (i) $p(\hat{R}^i) = p(R^i)$, and (ii) $UC(\hat{R}^i, f^i(R)) \subset UC(R^i, f^i(R))$ and $UC(\hat{R}^i, f^i(R)) \cap LC(R^i, f^i(R)) = \{f^i(R)\}$. Then, by *strategy-proofness*, $f^1(\hat{R}^1, R^{-1}) = f^1(R)$. By *nonbossiness*, $f(\hat{R}^1, R^{-1}) = f(R)$. Repeating the same argument for $i = 2, \ldots, n$, we have $f(\hat{R}) = f(R)$. By the Theorem, $f(\hat{R}) = U(\hat{R})$. Since the uniform rule is *peak-only*, $U(R) = U(\hat{R})$. Hence, $f(R) = U(R)$. □

4 Concluding Remarks

We considered the problem of allocating several infinitely divisible commodities among agents with continuous, strictly convex, and separable preferences. We established that a rule on this class of preferences satisfies *strategy-proofness*, *unanimity*, *symmetry*, and *nonbossiness* if and only if it is the uniform rule.

We conclude by commenting on further research. The *only if* part of our Theorem does not hold when we drop any of the three axioms of *strategy-proofness*, *unanimity*, and *symmetry*. The proportional rule[19] satisfies *unanimity*, *symmetry*, and *nonbossiness*, but not *strategy-proofness*. The queuing rules[20] satisfy *strategy-proofness*, *unanimity*, and *nonbossiness*, but not *symmetry*. The equal distribution rule[21] satisfies *strategy-proofness*, *symmetry*, and *nonbossiness*, but not *unanimity*. Thus, it is an open question whether the uniqueness part of our theorem holds without *nonbossiness*.

In the one-commodity case, since the uniform rule is *nonbossy*, *strategy-proofness*, *Pareto-efficiency*, and *symmetry* imply *nonbossiness* (Sprumont, 1991; Ching, 1994). Moreover, *effectively pairwise strategy-proofness* and *unanimity* imply *nonbossiness* (Serizawa, 2006).[22] Therefore, it is an interesting question whether their characterizations of the uniform rule for the one-commodity case extend to the multiple-commodity case.

Appendix

Fact. *For each $\ell \in M$, each $x^i_\ell \in \{0, W_\ell\}$, each $x^i_{-\ell} \in X_{-\ell}$, and each $d \in (0, \frac{W_\ell}{2n})$, $\mathcal{R}^V(x^i_\ell, x^i_{-\ell}, d)$ is nonempty.*

Proof. Let $\ell \in M$, $x^i_\ell \in \{0, W_\ell\}$, $x^i_{-\ell} \in X_{-\ell}$, and $d \in (0, \frac{W_\ell}{2n})$.
Case 1: $x^i_\ell = 0$.

---

[19]**Proportional rule, *Pro*:** For each $R \in \mathcal{R}^n$, each $\ell \in M$, and each $i \in N$,

$$Pro^i_\ell(R) = \begin{cases} \frac{p_\ell(R^i)\cdot W_\ell}{\sum_{j\in N} p_\ell(R^j)} & \text{if } \sum_{j\in N} p_\ell(R^j) > 0 \\ \frac{W_\ell}{n} & \text{otherwise.} \end{cases}$$

[20]**Queuing rule, *Q*:** There is a permutation π of N, and for each $R \in \mathcal{R}^n$ and each $\ell \in M$,

$$\begin{aligned} Q^{\pi(1)}_\ell(R) &= p_\ell(R^{\pi(1)}) \\ Q^{\pi(2)}_\ell(R) &= \min\{p_\ell(R^{\pi(2)}),\ W_\ell - Q^{\pi(1)}_\ell(R)\} \\ Q^{\pi(3)}_\ell(R) &= \min\{p_\ell(R^{\pi(3)}),\ W_\ell - Q^{\pi(1)}_\ell(R) - Q^{\pi(2)}_\ell(R)\} \\ &\vdots \\ Q^{\pi(n)}_\ell(R) &= W_\ell - \sum_{j=1}^{n-1} Q^{\pi(j)}_\ell(R). \end{aligned}$$

[21]**Equal distribution rule, *E*:** For each $R \in \mathcal{R}^n$, each $\ell \in M$, and each $i \in N$,

$$E^i_\ell(R) = \frac{W_\ell}{n}.$$

[22]*Effective pairwise strategy-proofness* requires that rules are strategy-proof and that no pair of agents can increase the welfare of any agent of the pair without decreasing the welfare of the other member of the pair, and neither member of the pair has an incentive to betray his partner. Serizawa (2006) characterized the uniform rule by *effectively pairwise strategy-proofness*, *unanimity*, and *symmetry*.

Define

$$L_\ell^i \equiv \max_{\hat{y}\in X} \left\{ \frac{\sum_{\ell'\neq\ell} (\hat{y}_{\ell'} - x_{\ell'}^i)^2}{d\cdot(2\cdot\hat{y}_\ell + d)} \right\}. \tag{13}$$

Since the set X is compact, L_ℓ^i exists. By $d>0$, $0<L_\ell^i<\infty$. Let $a_\ell^i > L_\ell^i$. Let $u^i : X \to \mathbb{R}$ be a utility function such that for each $y \in X$,

$$u^i(y) \equiv -a_\ell^i \cdot (y_\ell)^2 - \sum_{\ell'\neq\ell} \left(y_{\ell'} - x_{\ell'}^i\right)^2. \tag{14}$$

Let $R^i \subset X\times X$ be the preference relation such that for each $y,z\in X$, yR^iz if and only if $u^i(y) \geq u^i(z)$. Then, $p(R^i) = (0, x_{-\ell}^i)$, and R^i is continuous, strictly convex, and separable. We show that for each $y_\ell \in [0, W_\ell - d]$ and each $y_{-\ell}\in X_{-\ell}$, $(y_\ell, y_{-\ell})\ P^i\ (y_\ell + d, x_{-\ell}^i)$. Let $y_\ell\in[0,W_\ell-d]$ and $y_{-\ell}\in X_{-\ell}$. Then,

$$\begin{aligned}
& u^i(y_\ell, y_{-\ell}) - u^i\left(y_\ell + d, x_{-\ell}^i\right) \\
=& -a_\ell^i\cdot(y_\ell)^2 - \sum_{\ell'\neq\ell}\left(y_{\ell'} - x_{\ell'}^i\right)^2 + a_\ell^i\cdot(y_\ell+d)^2 \qquad \text{(by (14))} \\
=& \, a_\ell^i \cdot d\cdot(2\cdot y_\ell + d) - \sum_{\ell'\neq\ell}\left(y_{\ell'} - x_{\ell'}^i\right)^2 \\
>& \, L_\ell^i \cdot d\cdot(2\cdot y_\ell + d) - \sum_{\ell'\neq\ell}\left(y_{\ell'} - x_{\ell'}^i\right)^2 \qquad \text{(by } a_\ell^i > L_\ell^i,\ d>0, \text{ and } y_\ell\geq 0) \\
=& \, d\cdot(2\cdot y_\ell + d)\left\{L_\ell^i - \frac{\sum_{\ell'\neq\ell}(y_{\ell'} - x_{\ell'}^i)^2}{d\cdot(2\cdot y_\ell + d)}\right\} \\
\geq& 0. \quad \text{(by (13), } d>0, \text{ and } y_\ell \geq 0)
\end{aligned}$$

Thus, $R^i \in \mathcal{R}^V(0, x_{-\ell}^i, d)$.

Case 2: $x_\ell^i = W_\ell$.

Define

$$\bar{L}_\ell^i \equiv \max_{\hat{y}\in[0,W_\ell-d]\times X_{-\ell}} \left\{ \frac{\sum_{\ell'\neq\ell} (\hat{y}_{\ell'} - x_{\ell'}^i)^2}{d\cdot(2\cdot(W_\ell - \hat{y}_\ell) - d)} \right\}.$$

Note that since the set $[0, W_\ell - d]\times X_{-\ell}$ is compact, $\bar{L}_\ell^i$ exists. Since $d>0$ and for each $\hat{y}_\ell \in [0, W_\ell - d]$, $2\cdot(W_\ell - \hat{y}_\ell) - d > 0$, we have $0 < \bar{L}_\ell^i < \infty$. Let $\bar{a}_\ell^i > \bar{L}_\ell^i$. Let $\bar{u}^i : X\to\mathbb{R}$ be a utility function such that for each $y\in X$,

$$\bar{u}^i(y) \equiv -\bar{a}_\ell^i\cdot(y_\ell - W_\ell)^2 - \sum_{\ell'\neq\ell}\left(y_{\ell'} - x_{\ell'}^i\right)^2.$$

Let $\bar{R}^i \subset X\times X$ be the preference relation represented by $\bar{u}^i$. Similarly to Case 1, we can show that $\bar{R}^i \in \mathcal{R}^V(W_\ell, x_{-\ell}^i, d)$. □

References

Amorós, P. (2002) Single-peaked preferences with several commodities. *Social Choice and Welfare* 19: 57-67

Barberà, S., Jackson, M.O. and Neme, A. (1997) Strategy-proof allotment rules. *Games and Economic Behavior* 18: 1-21

Benassy, J-P. (1982) *The Economics of Market Disequilibrium.* Academic Press, New York

Border, K.C. and Jordan, J.S. (1983) Straightforward elections, unanimity and phantom voters. *Review of Economic Studies* 50: 153-170

Ching, S. (1994) An alternative characterization of the uniform rule. *Social Choice and Welfare* 11: 131-136

Ching, S. and Serizawa, S. (1998) A maximal domain for the existence of strategy-proof rules. *Journal of Economic Theory* 78: 157-166

Foley, D. (1967) Resource allocation and the public sector. *Yale Economic Essays* 7: 45-98

Maskin, E. (1999) Nash equilibrium and welfare optimality. *Review of Economic Studies* 66: 23-38

Massó, J. and Neme, A. (2001) Maximal domain of preferences in the division problem. *Games and Economic Behavior* 37: 367-387

Massó, J. and Neme, A. (2007) Bribe-proof rules in the division problem. *Games and Economic Behavior* 61: 331-343

Moulin, H. (1980) On strategy-proofness and single-peakedness. *Public Choice* 35: 437-456

Otten, G., Peters, H. and Volij, O. (1996) Two characterizations of the uniform rule for division problems with single-peaked preferences. *Economic Theory* 7: 291-306

Sasaki, H. (2003) Limitation of efficiency: strategy-proofness and single-peaked preferences with many commodities. Working Paper, Rice University

Satterthwaite, M. A. and Sonnenschein, H. (1981) Strategy-proof allocation mechanisms at differentiable points. *Review of Economic Studies* 48: 587-597

Sprumont, Y. (1991) The division problem with single-peaked preferences: a characterization of the uniform allocation rule. *Econometrica* 59: 509-519

Serizawa, S. (2006) Pairwise strategy-proofness and self-enforcing manipulation. *Social Choice and Welfare* 26: 305-331

Thomson, W. (1994a) Consistent solutions to the problem of fair division when preferences are single-peaked. *Journal of Economic Theory* 63: 219-245

Thomson, W. (1994b) Resource-monotonic solutions to the problem of fair division when preferences are single-peaked. *Social Choice and Welfare* 11: 205-223

Thomson, W. (1995) Population-monotonic solutions to the problem of fair division when preferences are single-peaked. *Economic Theory* 5: 229-246

Thomson, W. (1997) The replacement principle in economies with single-peaked preferences. *Journal of Economic Theory* 76: 145-168

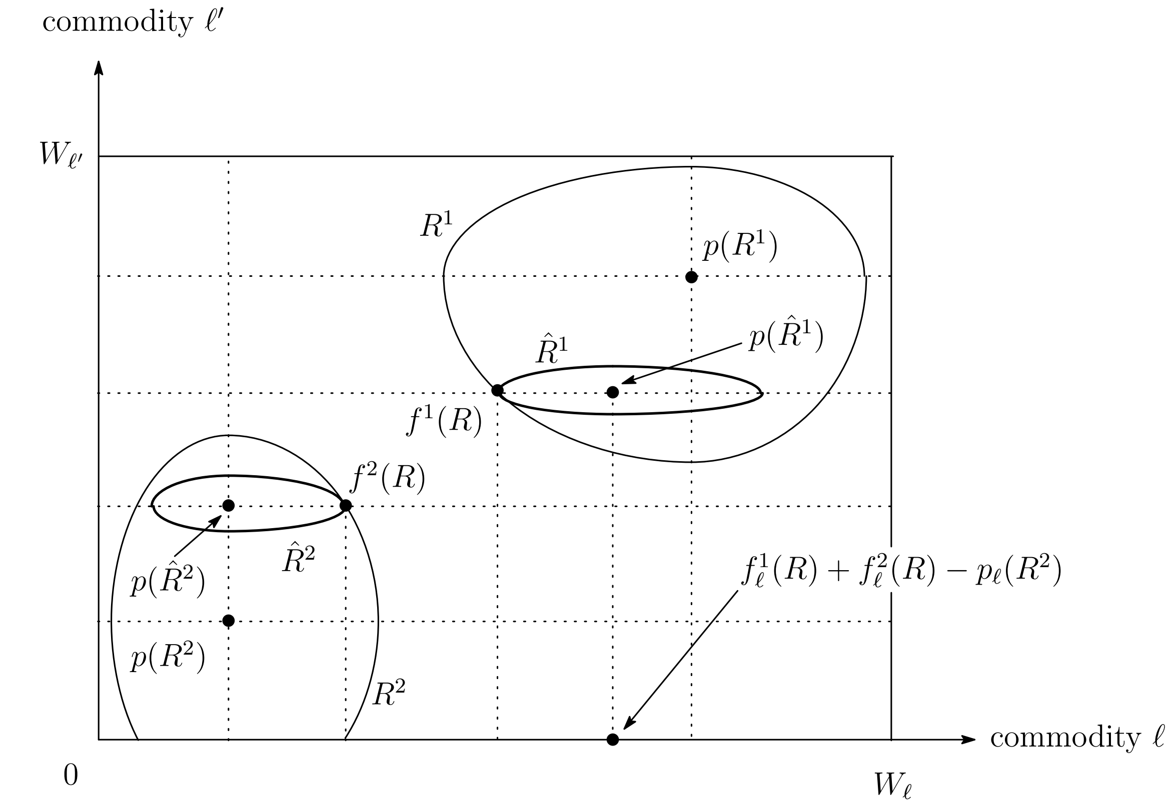

Figure 1. Illustration of $(\hat{R}^1, \hat{R}^2)$ in the proof of Lemma 1.

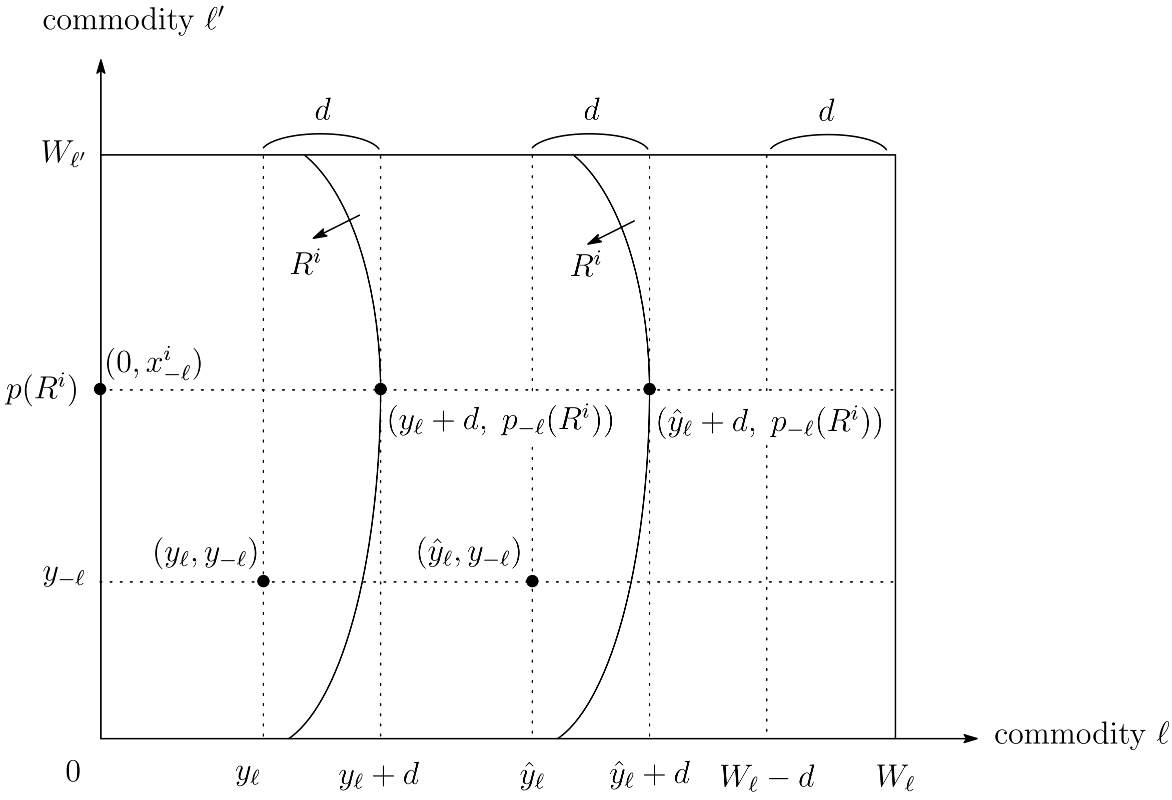

Figure 2. Illustration of $R^i \in \mathcal{R}^V(0, x^i_{-\ell}, d)$.

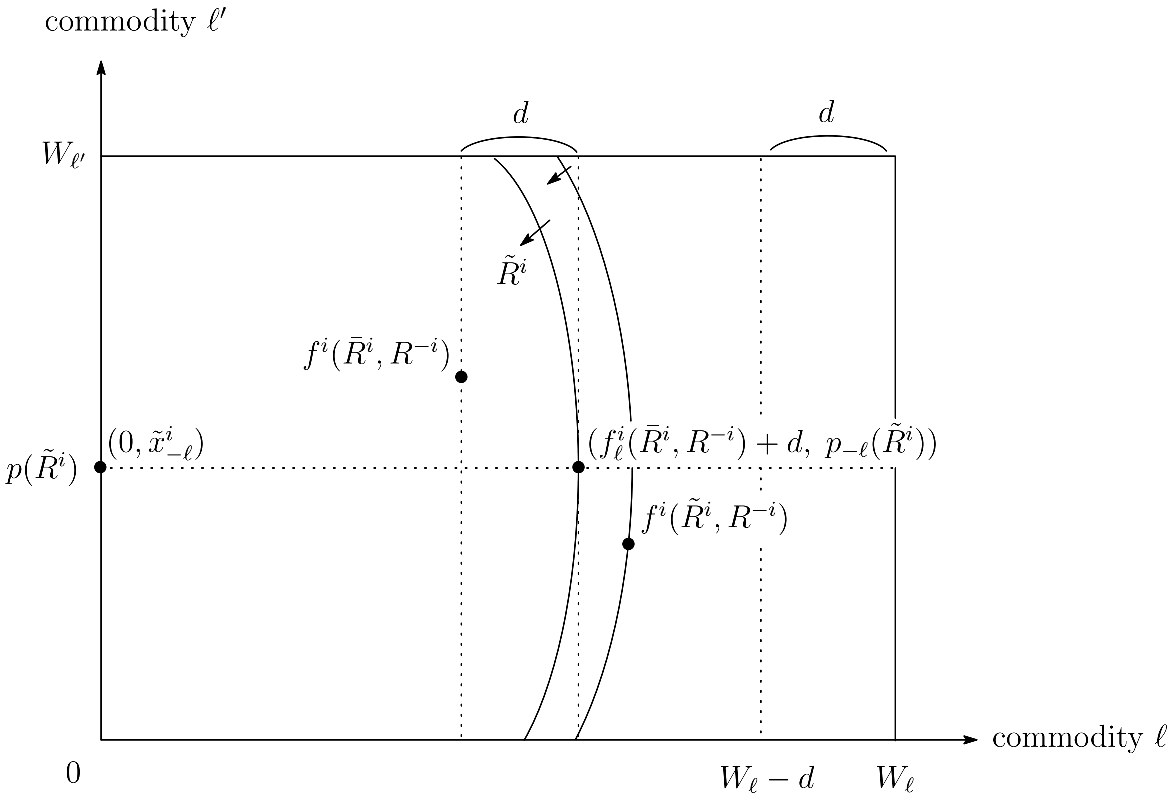

Figure 3. Illustration of Case 1-2 of Step 1 in the proof of Lemma 4.

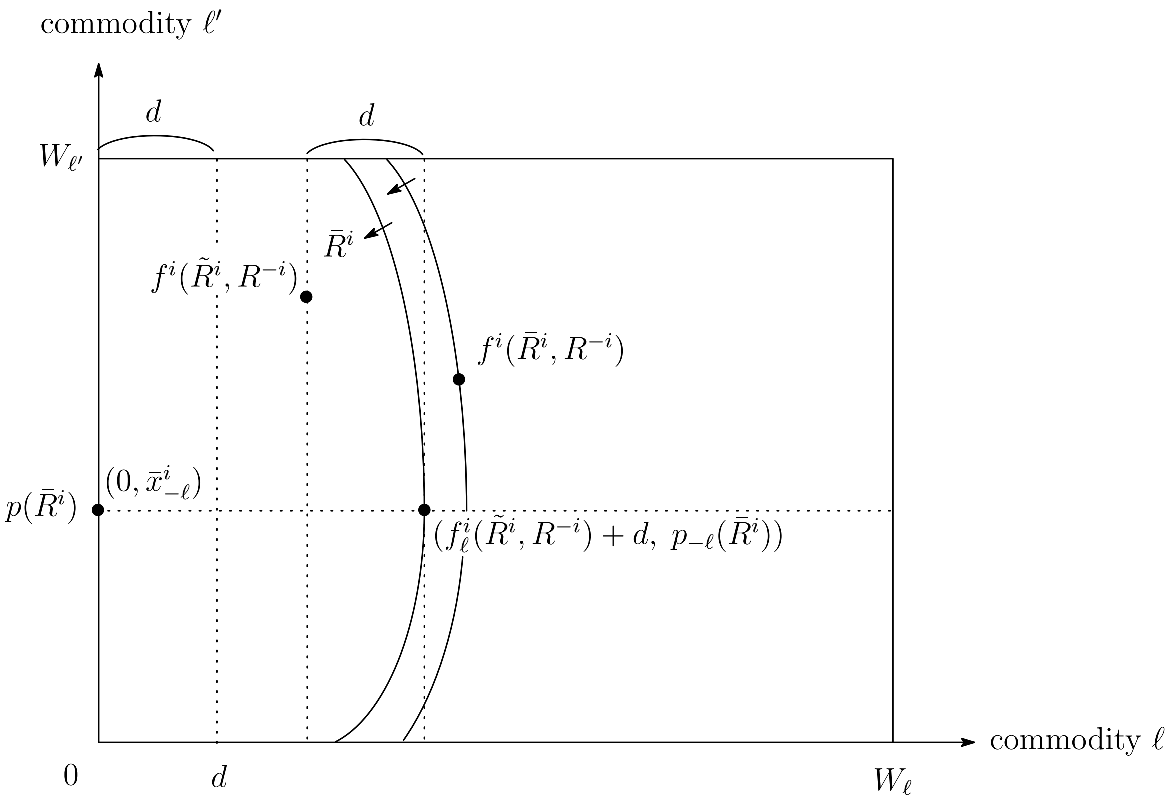

Figure 4. Illustration of Case 2-2 of Step 2 in the proof of Lemma 4.

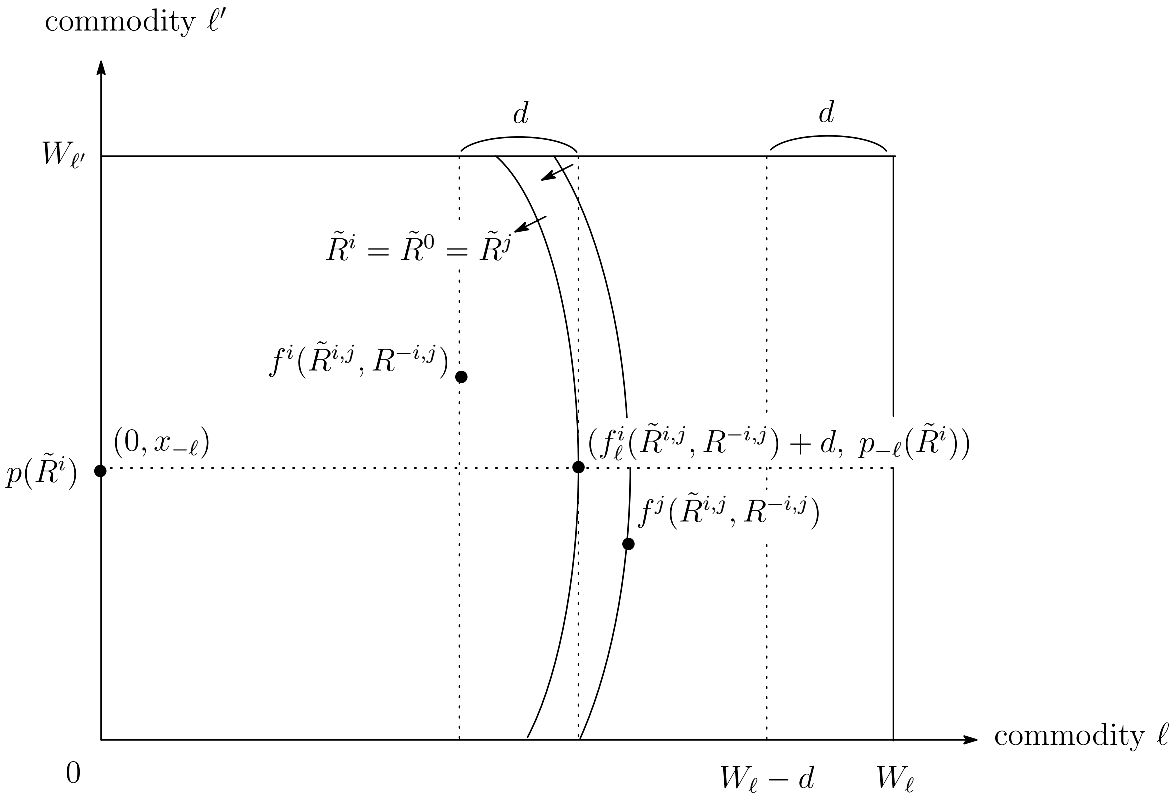

Figure 5. Illustration of Case 1-2 of Step 1 in the proof of Lemma 5.

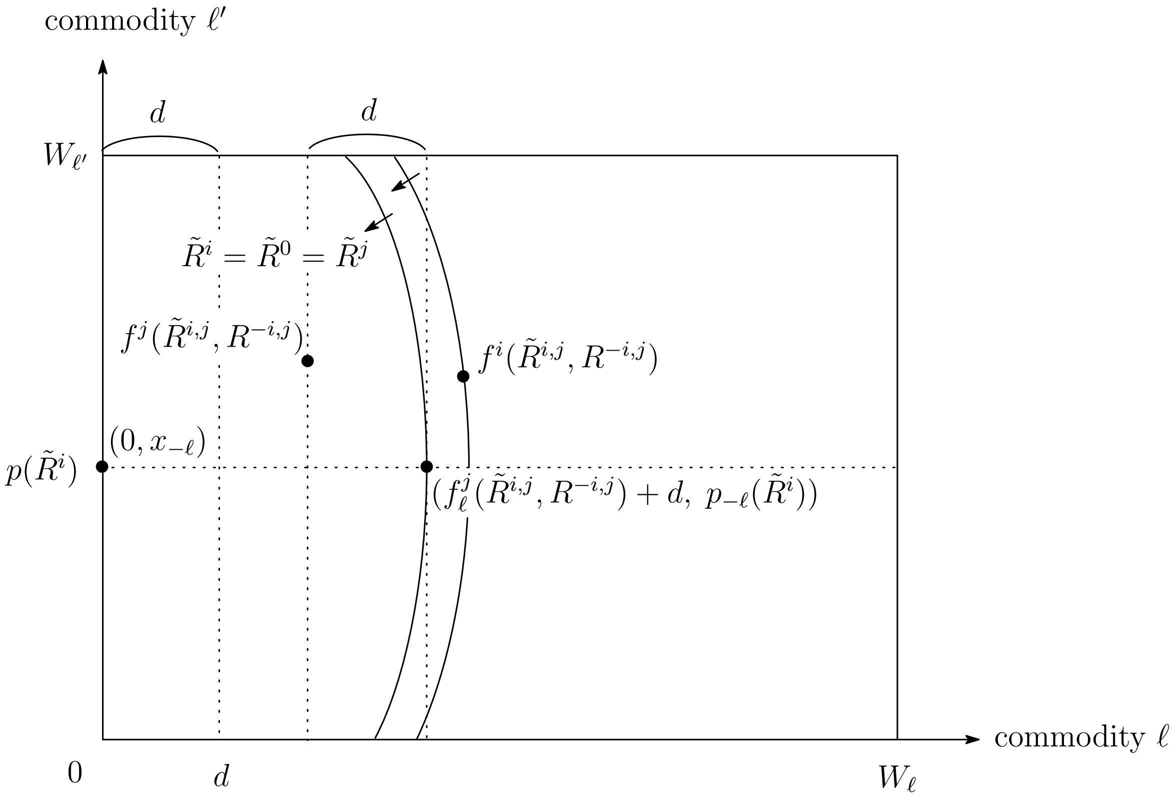

Figure 6. Illustration of Case 2-2 of Step 2 in the proof of Lemma 5.

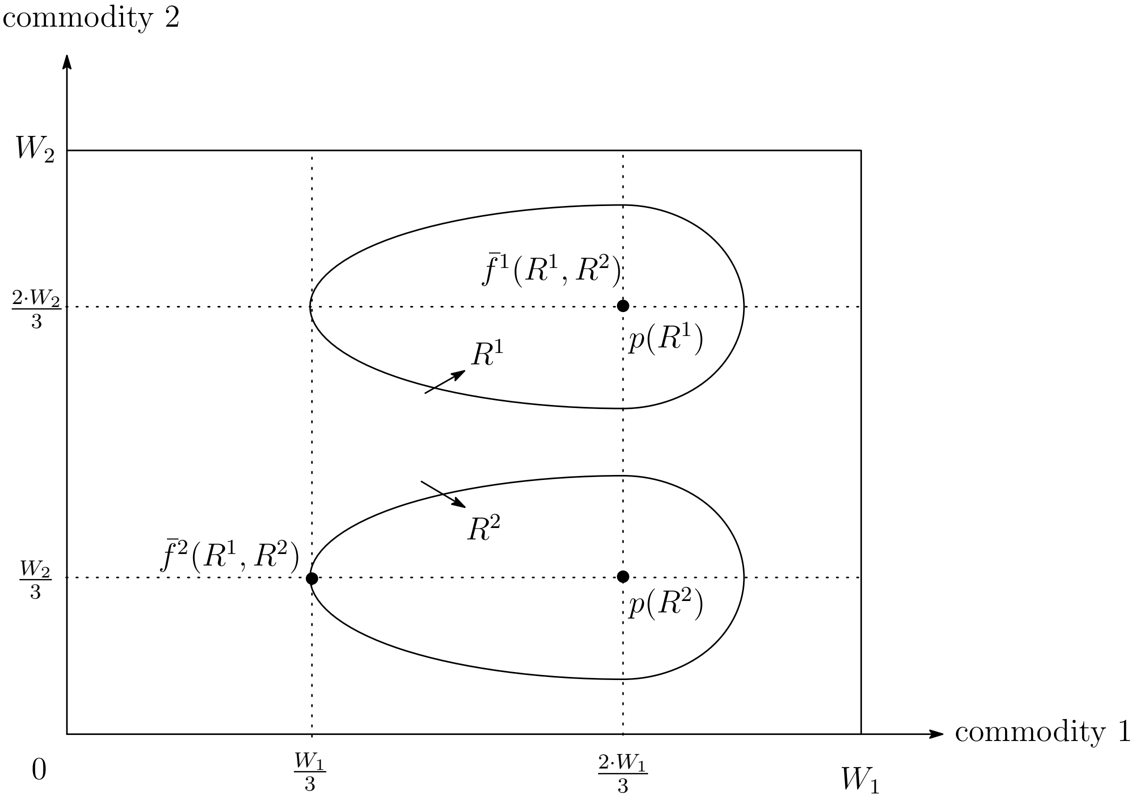

Figure 7. Illustration of $\bar{f}$ in Example 3 for $p(R^1) = (\frac{2\cdot W_1}{3}, \frac{2\cdot W_2}{3})$ and $p(R^2) = (\frac{2\cdot W_1}{3}, \frac{W_2}{3})$. Note that $\bar{R}(R^1) = \bar{R}(R^2)$.

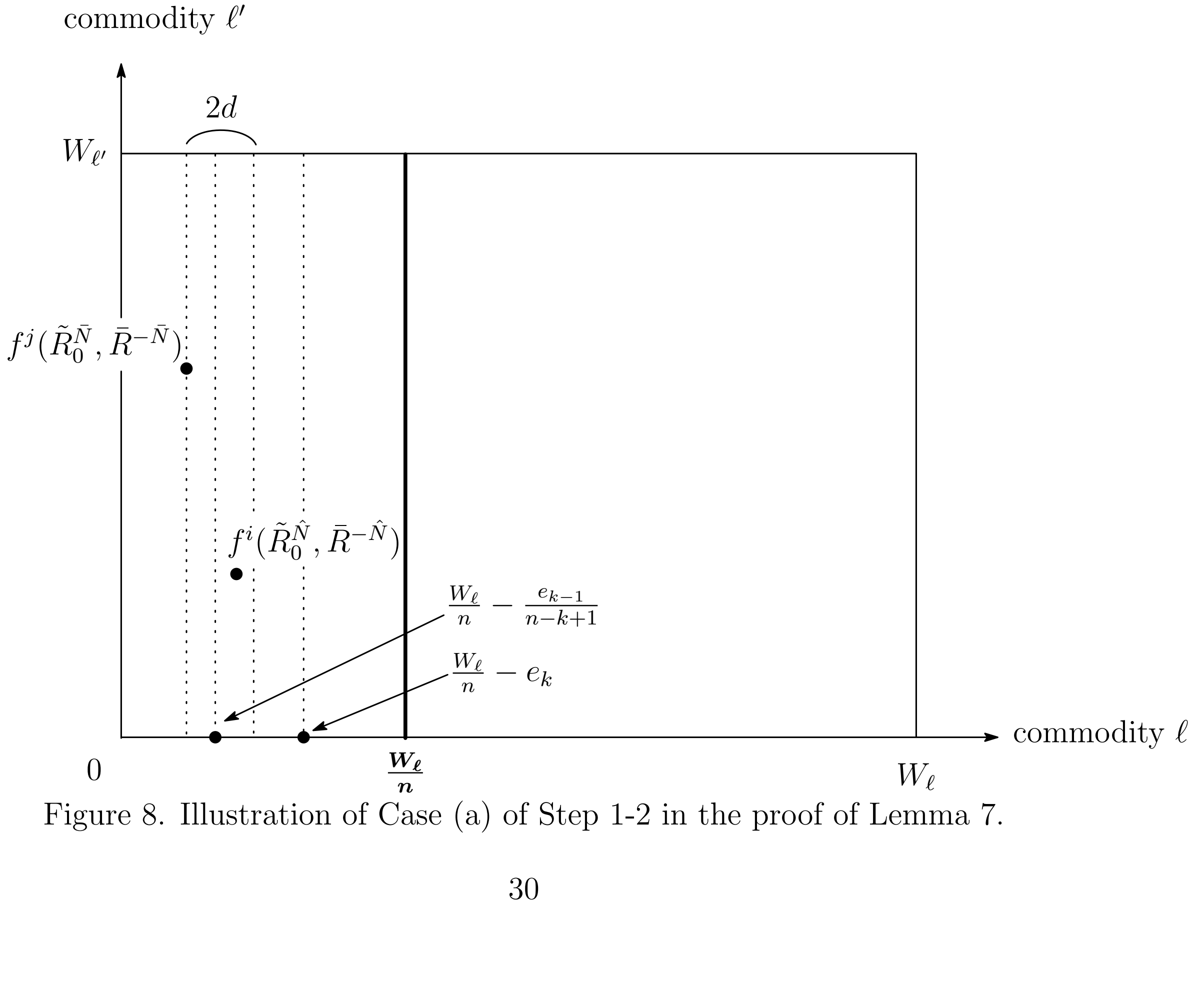

Figure 8. Illustration of Case (a) of Step 1-2 in the proof of Lemma 7.

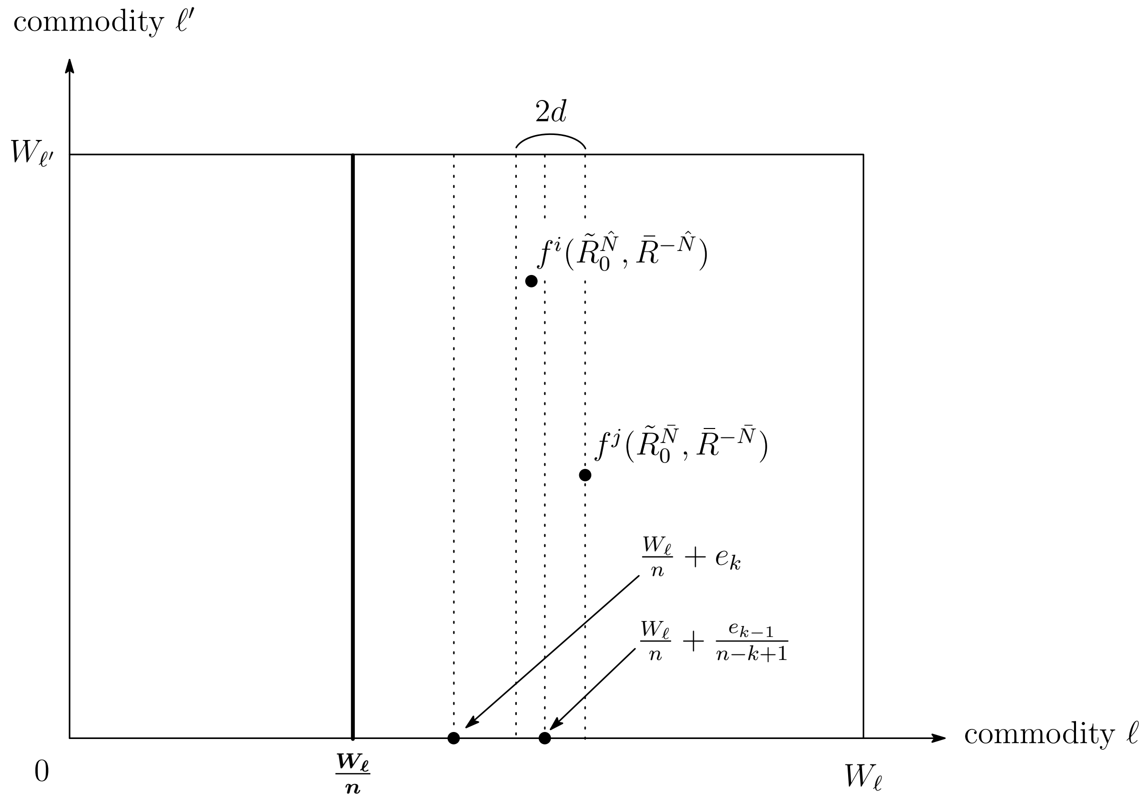

Figure 9. Illustration of Case (b) of Step 1-2 in the proof of Lemma 7.

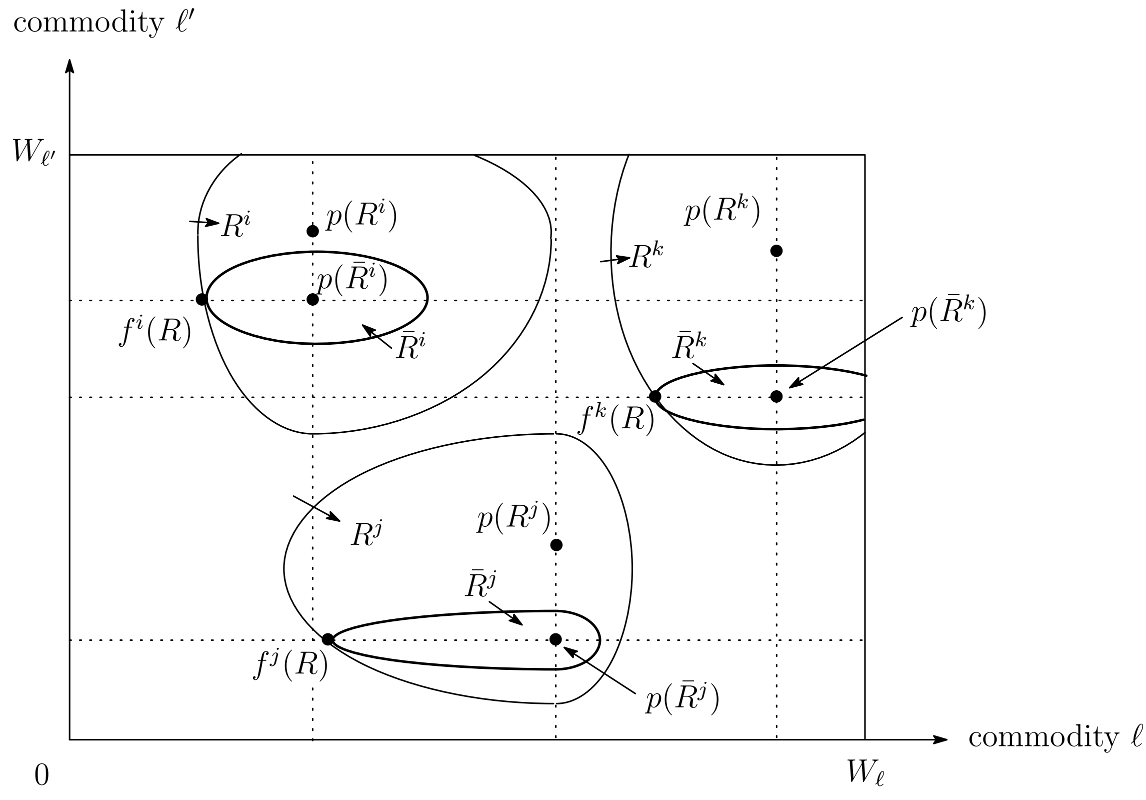

Figure 10. Illustration of $\bar{R}$ in the proof of Step 2 (Lemma 7) and Theorem.