Two-time scale control and observer design for trajectory tracking of two cooperating robot manipulators moving a flexible beam

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ABSTRACT
In this paper, we present two-time scale control design for trajectory tracking of two cooperating planar rigid robots moving a flexible beam, which does not require vibration measurement for the beam. First, the kinematics and dynamics of the robots and the object are derived. Then, using the relations between different forces acting on the object by the manipulators’ end-effectors, dynamics equations of the robots and the object are combined. The resulting equations show that the coupled dynamics including beam vibration and the rigid motion take place in two different time domains. By applying two-time scale control theory on the combined dynamics, a composite control scheme is elaborated which makes the beam orientation and its center of mass position track a desired trajectory while suppressing the beam vibration. For the controller algorithm, first a slow controller is utilized for the slow (rigid) subsystem and then a fast stabilizing controller is considered for the fast (flexible) subsystem. To avoid requiring measurement of beam vibration for the fast control law, a linear observer is also designed. The simulation results show the efficiency of the proposed control scheme.

1. Introduction
Many manufacturing processes exploit deformable objects such as rubber tubes, beams, sheet metals, cords, leather products, and paper sheets which are now being automatically handled by special equipment with high costs and low efficiency or still done by human workers.

In space, large, lightweight structures, satellite solar arrays, and structural members all exhibit noticeable flexibility. In the aerospace industry, non-rigid composite materials are used to replace metals in many products. Underwater, large bundles of cabling, for power or for signals (e.g., sonar) are necessarily flexible. Many vehicles, including automobiles and airplanes, are manufactured by joining flexible plates of metal or composite materials together. In shipbuilding industries, flexible frames and plates are used in the assembly of various kinds of ships. In industry, spring-loaded parts are becoming more common, particularly as design for assembly becomes more prevalent.

In many of these applications, vibration-free movement of flexible objects is required. To handle flexible objects by automatic machines, other than using some special tools and devices, robot manipulators can also accomplish the task. Robotic manipulation of flexible objects is a complex and challenging problem and has recently attracted a lot of attention due to its current and potential applications in industry and space.

There are two problems that need to be addressed in flexible object manipulation; the object dynamics modeling and the control design. Mathematical modeling of handling flexible objects has been extensively studied, [1–4]. Kita et al., [5], used stereo cameras and 3D shape estimation for observation, modeling and handling of clothes. Wakamatsu et al. used differential geometry for static modeling, [6], and dynamic modeling, [7], of linear object deformation. The motion of a flexible object, for example a beam, consists of its rigid body motion and its vibration. The rigid body motion is regarded as the position and orientation of the beam as if the beam were a rigid body. The vibration of the beam is then taken into account with respect to the rigid body motion. This vibration can be modeled by assuming the flexible object as a distributed parameter system or assuming that it consists of lumped masses and springs (infinite or finite dimensional modeling). Usually mode summation procedure, finding the mode shapes and natural frequencies of the beam, is used. Tanner and Kyrriakopoulos viewed a manipulated deformable object as an underactuated mechanical system [3]. They discussed controllability issues and the results on the nature of the constraints and the controllability properties of an important class of deformable objects being modeled by finite element method were stated.

Zheng and Chen [8] and Arai et al., [9] examined position control of flexible objects by one robot manipulator. Their purpose was to insert flexible object’s one end into a hole in concrete while moving the other end by a robot manipulator. Nakagaki et al., studied...
the same problem using static shape functions of the object, [10]. Yukawa and Uchiyama, unlike the above studies, dealt with the problem of handling one end of the flexible object by a robot while the other end was fixed in the wall, [11].

The importance of multi-arm robotic systems has been realized by the robotics community. Having a wider set of functional capabilities, robots with several manipulators can be successfully used not only in industry but in unstructured environments as well. The advantages of multi-arm robot utilization have been especially realized in the space applications where, as it is expected, remote manipulation systems with cooperating arms may be required to perform future in-orbit construction, servicing and repair tasks with the minimum involvement of astronauts. In these applications the object to be manipulated may be too heavy, too large or too flexible. If the load is heavier than the carrying capacity of a single robot, multiple robots can distribute the load among them and better move the object. Also, if the object shows some flexibility, multiple robots can move the object much better and more accurately than one robot, just the same as a human with two arms can move a flexible object better than moving it by only one arm.

Chen and Zheng studied the coordinating of two grippers to handle a deformable object, [12]. They investigated passive approaches for vibration-free handling of deformable beams, [13]. Soni and Kohno dealt with the geometrical analysis to perform the position control and suppression of the flexible object vibration, [14]. Sun, Liu and Mills studied a more general case: handling a flexible object with an arbitrary shape, [15]. They showed that under a simple PD position feedback, the position/orientation of a flexible object handled by two manipulators is able to approach the desired ones and at the same time the vibration of each contact point can be suppressed. Kosuge et al., used static bending functions of the flexible object for the same purpose, [16]. Dologluer and Peitekis considered a rectangular object grasped by two robot fingers with spherical end-effectors that were allowed to roll along the object surface, [4]. Yukawa et al., assumed the handled beam has two free ends while the contact points with the robots’ end-effectors were not located at the ends of the beam, [17]. They also assumed that the beam dynamic parameters are not negligible and proposed a controller which was robust while the vibration could occur in transverse and rotational directions, [18]. Sun and Liu proposed a coupled position/force control law and also an impedance control for handling a flexible beam by two robots, [19]. Also, Wada et al., [20], used simple PID for indirect positioning of deformable objects. They considered the stretch of a flexible object by using a springs-lumped masses system.

The above works only contribute to the regulation problem of handling a flexible object to a desired position/orientation using robot manipulators. The regulation problem is less complicated when compared to the tracking problem since stability analysis of vibration suppression when the object arrives at the desired position/orientation and ends its rigid body motion is much simpler than when it continues its motion to track a desired trajectory where the object motion may induce additional vibration.

Svinin and Uchiyama dealt with controlling the transfer motion of a flexible object using a combination of feedforward and feedback coordinations, [21]. AlYahmadi and Hsia presented a simple and computationally efficient scheme for handling a flexible object by two coordinated manipulators, [22]. They also used sliding mode control for the same problem, [23,24]. Jiang and Kohno dealt with the issues of vibration measurement and control design in order to establish a flexible objects manipulating system using industrial robot arms, [25]. They presented a method for vibration measurement of the flexible object using a force/torque sensor equipped at the wrist of the robot and proposed a linear feedback control for the flexible object. Sun and Liu discussed the issue of hybrid position and force control of a two-manipulator system moving a flexible beam using saturation control approach, [26].

By a thorough look at the literature, the following points can be observed:

- Often, researchers use non-model-based and vision-based approaches.
- In the model-based works, regularly, very simple models are utilized; they model deformable objects as simple lumped-mass systems or for more complicated cases, they mostly use rods or strings.
- A few researches for beams and plates are noticed; however, commonly the static effects are considered in those works.
- The most recent researches focus generally on regulation and only a few works for tracking of flexible objects and simultaneously their vibration suppression are seen.
- Control of two rigid robots moving a flexible object, in all of the above mentioned works for tracking, has required vibration measurement of the flexible object, which resulted in the use of some devices such as strain gauges and piezoelectric materials at different points of the object or force/torque sensors at its contact points with the robots.

The motion of a flexible beam consists of its rigid body motion and its vibration. These two motions happen in two different time scales where the rigid motion can be regarded as the slow motion while the vibration appears as the fast motion. Therefore, for moving the mass center of the rigid body on a trajectory of desired positions/orientations, while suppressing the vibration of the beam, two-time scale control theory can be applied. The underlying idea of two-time scale control theory is to couple the system dynamics into the slow and fast subsystems with separate time scales which is already the case for our system. Control design may then proceed for each lower-order subsystem, and the results are combined to yield a composite controller for the original system. The design is sequential in general, since the fast control design depends on the slow control design, [27]. For our system, the slow subsystem represents the rigid system where the object has no flexibility and the fast subsystem is a linear time variant system. To control the slow subsystem we can use any control scheme applied for the two cooperating rigid robots moving a rigid object, [28]. For the fast subsystem we may use the linear control theory where the poles of the fast subsystem can be placed on their desired values. To avoid vibration measurement difficulty, in this paper, for our proposed control scheme, we design a linear time variant observer that estimates the flexible coordinates for the feedback control of the fast subsystem.

In this paper, two planar robots, each with three revolute joints, grasping and moving a flexible beam are considered, see Fig. 1. The kinematics and dynamics of rigid robot manipulators and the flexible beam are derived. Replacing the contact forces/moments, between two rigid robots and flexible beam, from the robots dynamics into the beam dynamics, the combined system dynamics is obtained. To carry out trajectory tracking of the mass center of the beam, while suppressing the vibration of the beam, we use two-time scale control theory where the system is decoupled into slow (rigid) and fast (flexible) subsystems is applied. To control the slow subsystem, two rigid robots moving a rigid object, a PD controller is used. Linear control theory and observer design are considered to place the poles of the linear fast subsystem into their desired values without vibration measurement of the beam. At the end, the simulation results show that the proposed control and observer scheme is a convenient and effective choice.
The kinematics of each robot can be easily developed by using rotation matrices and translation vectors of the links. We consider \( r_1 = [p_1 \quad \theta]^T \) and \( r_2 = [p_2 \quad \theta]^T \), see Fig. 2, as the position and orientation vectors from reference frame origin to the robot grippers, see Fig. 2. Each vector time derivative can be mapped into the corresponding robot joint space by the Jacobian matrix. Considering \( J_1 \) and \( J_2 \) as the Jacobian matrices and \( q_1 \) and \( q_2 \) as the vectors of joint variables of robot 1 and robot 2, respectively, one may write

\[
\dot{r}_1 = J_1 \dot{q}_1 \quad \text{and} \quad \dot{r}_2 = J_2 \dot{q}_2. \tag{1}
\]

2. Kinematics of two cooperating robot manipulators and a flexible beam

2.1. Coordinate frames

To analyze motion of two rigid robots moving an object, five principal coordinate frames are considered: \( F_1 \) and \( F_2 \), the inertial coordinate frames of robot manipulator bases, \( F_{r1} \) and \( F_{r2} \), the frames of grippers at the contact points with the object, and \( F_0 \), the mobile frame that is attached to rigid object and its origin is at the mass center of the beam. All kinematic equations will be written with respect to the reference frame, \( F_1 \), see Fig. 2.

2.2. Robots kinematics

The kinematics of each robot can be easily developed by using rotation matrices and translation vectors of the links. We consider \( r_1 = [p_1 \quad \theta]^T \) and \( r_2 = [p_2 \quad \theta]^T \), see Fig. 2, as the position and orientation vectors from reference frame origin to the robots grippers, see Fig. 2. Each vector time derivative can be mapped into the corresponding robot joint space by the Jacobian matrix. Considering \( J_1 \) and \( J_2 \) as the Jacobian matrices and \( q_1 \) and \( q_2 \) as the vectors of joint variables of robot 1 and robot 2, respectively, one may write

\[
\dot{r}_1 = J_1 \dot{q}_1 \quad \text{and} \quad \dot{r}_2 = J_2 \dot{q}_2. \tag{1}
\]

Eqs. (1) can be assembled as

\[
\dot{r} = J \dot{q} \tag{2}
\]

where

\[
\dot{r} = \begin{bmatrix} \dot{r}_1 \\ \dot{r}_2 \end{bmatrix}, \quad J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \quad \text{and} \quad \dot{q} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}. \tag{3}
\]

2.3. Beam kinematics

Consider a flexible Euler–Bernoulli beam with length \( l \), rigidly grasped by two robot manipulators. For simplicity assume that longitudinal deformation of the beam is negligible, and thus only the transverse vibration takes place. For any point on the beam, the transverse deflection \( w(s, t) \) and the slope \( \psi(s, t) \) can be approximated by finite series of assumed modes, [29,30],

\[
w(s, t) = \Phi(s)\xi(t) = \sum_{k=1}^{m} \Phi_k(s)\xi_k(t) \quad \text{and} \quad \psi(s, t) = \sum_{k=1}^{m} \Phi_k'(s)\xi_k(t) \tag{4}\]

where \( s \) denotes the normalized curvilinear coordinate of the rigid beam and is in the range of 0–1, \( \Phi_k(s) \) is the \( k \)th mode shape of the flexible beam corresponding to its boundary conditions and \( \Phi_k'(s) \) is its derivative with respect to \( s \). \( \xi_k(t) \) is the \( k \)th generalized coordinate representing the contribution of the \( k \)th mode shape in the flexible motion. We consider \( w_1 \) and \( w_2 \), see Fig. 3, as the deformation of the beam at the two ends \( s = 0, l \) and \( w' \) and \( w'' \) the corresponding slopes.

Various types of boundary conditions can be considered for the rigid body motion of the beam, but boundary conditions that allow us to determine beam’s rigid body motion from end-effectors’ motions should be considered. Therefore, a clamped-free model in which the contact points have no deformation relative to the end-effectors can be utilized, see Fig. 1. The first six assumed modes for these boundary conditions are given in Appendix A.1.

The position vector from reference frame origin, \( F_1 \), to a point on the beam is considered as \( p = [x \quad y]^T \) which can be written as, see Figs. 2 and 3,

\[
x = x_0 + v_0 - w_0 \quad \text{and} \quad y = y_0 + v_0 + w_0 \tag{5}\]

where \( p_0 = [x_0 \quad y_0]^T \), see Fig. 2, shows the position of rigid body motion of the mass center of the beam, \( \theta \) is the beam orientation, \( \cos(\theta) \) and \( \sin(\theta) \) and \( v = s - l/2 \). Considering
$r_o = [p_0 \; \theta]^T$ as the position and orientation of the rigid body motion of mass center of the object, under clamped-free boundary conditions, we have

\begin{align}
  r_1 = r_o - \left[ \frac{1}{2} c_0 \; \frac{1}{2} s_0 \; 0 \right]^T \quad \text{and} \\
  r_2 = r_o + \left[ \frac{1}{2} c_0 \; \frac{1}{2} s_0 \; \pi \right]^T + \left[ -w_2 s_0 \; w_2 c_0 \; w_2 \right]^T \tag{6}
\end{align}

Differentiating Eqs. (6), we can write

\begin{align}
  i_1 = R_1(\theta) \dot{r}_o \quad \text{and} \quad i_2 = R_2(\theta, w_2) \dot{r}_o + R_\xi \dot{\xi}
\end{align}

where

\begin{align}
  R_1(\theta) &= \begin{bmatrix}
    1 & 0 & \frac{1}{2} c_0 \\
    0 & 1 & -\frac{1}{2} c_0 \\
    0 & 0 & 1
  \end{bmatrix}, \\
  R_2(\theta, w_2) &= \begin{bmatrix}
    1 & 0 & -\frac{1}{2} s_0 - w_2 c_0 \\
    0 & 1 & \frac{1}{2} c_0 - w_2 s_0 \\
    0 & 0 & 1
  \end{bmatrix} \\
  \text{and} \quad R_\xi(\theta) &= \begin{bmatrix}
    -\phi^T(l)s_0 & \phi^T(l)c_0 & \phi^T(l)\theta
  \end{bmatrix}^T.
\end{align}

Defining

\begin{align}
  r = \begin{bmatrix}
    r_1^T \\
    r_2^T
  \end{bmatrix}, \quad X = \begin{bmatrix}
    r_o^T \\
    \xi^T
  \end{bmatrix}
\end{align}

and

\begin{align}
  R = \begin{bmatrix}
    R_1(\theta) & 0 \\
    R_2(\theta, w_2) & R_\xi(\theta)
  \end{bmatrix}
\end{align}

we can be concluded that

\begin{align}
  \dot{r} = R X.
\end{align}

Differentiating Eq. (11), we get the following relation for the accelerations,

\begin{align}
  \ddot{r} = \dot{R} X + R \ddot{X}.
\end{align}

### 3. Dynamics of two cooperating robot manipulators moving a flexible beam

#### 3.1. Robot dynamics

The dynamic equation of each robot can be written as, \[31, 32,\]

\begin{align}
  M_i(q_i) \ddot{q}_i + C_i(q_i, \dot{q}_i) \dot{q}_i + G_i(q_i) &= \tau_i + f_i, \quad i = 1, 2 \tag{13}
\end{align}

where $M_i(q_i)$, $C_i(q_i, \dot{q}_i)$ and $G_i(q_i)$ denote the inertia matrix, matrix of Coriolis and centrifugal effects and vector of gravitational terms of robot $i$, $\tau_i$ and $f_i$ are the vectors of applied generalized torques at joints of robot $i$ and interaction force between robot $i$ and the beam, respectively.

Using Eq. (1), we have

\begin{align}
  \dot{q}_i &= J_i^{-1} \dot{r}_i \quad \text{and} \quad \ddot{q}_i = J_i^{-1} \ddot{r}_i + J_i^{-1} \dot{r}_i \tag{14}
\end{align}

Therefore, the following form for each robot is held true

\begin{align}
  M_i(r_i) \ddot{r}_i + C_i(r_i, \dot{r}_i) \dot{r}_i + G_i(r_i) &= u_i + f_i \tag{15}
\end{align}

where

\begin{align}
  M_i = J_i^{-T} M_i(q_i) J_i^{-1}, \quad C_i = (J_i^{-T} C_i(q_i, \dot{q}_i) J_i^{-1} + J_i^{-T} M_i(q_i) J_i^{-1}) \tag{16}
\end{align}

Assembling dynamic equations of the two robots, one may write

\begin{align}
  M(r) \ddot{r} + C(r, \dot{r}) \dot{r} + G(r) &= u + f \tag{17}
\end{align}

in which

\begin{align}
  M(r) &= \begin{bmatrix}
    M_1(r_1) & 0 \\
    0 & M_2(r_2)
  \end{bmatrix}, \\
  C(r, \dot{r}) &= \begin{bmatrix}
    C_1(r_1, \dot{r}_1) & 0 \\
    0 & C_2(\dot{r}_2, r_2)
  \end{bmatrix}, \\
  G(r) &= \begin{bmatrix}
    G_1(r_1) \\
    G_2(r_2)
  \end{bmatrix}, \quad f = \begin{bmatrix}
    f_1 \\
    f_2
  \end{bmatrix} \quad \text{and} \quad u = \begin{bmatrix}
    u_1 \\
    u_2
  \end{bmatrix}.
\end{align}

#### 3.2. Beam dynamics

An Euler–Bernoulli beam with length $l$, moment inertia $I_0$ and mass $m_0 = \rho A l$ where $\rho$ and $A$ denote the mass density and area of cross section of the beam, respectively, is considered as the flexible object example. To obtain beam dynamic equations of motion, one may use Lagrange’s method. For this, first, kinetic and potential energies of the beam should be derived. The kinetic energy, $T$, of the beam is

\begin{align}
  T = \frac{1}{2} \int_{-l/2}^{l/2} \rho A (\dot{x}^2 + \dot{y}^2) \; dv. \tag{19}
\end{align}

Differentiating Eq. (5), we have

\begin{align}
  \dot{x} = x_0 - \dot{w}_c s_0 - w_c \dot{s}_0 - \dot{w}_s c_0 - \dot{w}_w c_0 \\
  \dot{y} = y_0 + \dot{w}_c c_0 - \dot{w}_s c_0 - \dot{w}_w s_0 \tag{20}
\end{align}

To calculate the kinetic energy of the beam we should derive $\dot{x}^2 + \dot{y}^2$

\begin{align}
  \dot{x}^2 &= \dot{x}_0^2 + \dot{v}_c^2 \dot{s}_0^2 + \dot{w}_c^2 \dot{c}_0^2 + \dot{w}_s^2 \dot{c}_0^2 - 2 \dot{w}_c \dot{s}_0 \dot{w}_c \dot{c}_0 - 2 \dot{w}_s \dot{c}_0 \dot{w}_c \dot{c}_0 \\
  \dot{y}^2 &= \dot{y}_0^2 + \dot{v}_c^2 \dot{c}_0^2 + \dot{w}_c^2 \dot{s}_0^2 + \dot{w}_s^2 \dot{c}_0^2 + 2 \dot{v}_c \dot{c}_0 \dot{w}_c \dot{c}_0 - 2 \dot{v}_c \dot{s}_0 \dot{w}_c \dot{c}_0 + 2 \dot{v}_s \dot{c}_0 \dot{w}_c \dot{c}_0 + 2 \dot{v}_w \dot{c}_0 \dot{w}_c \dot{c}_0 \tag{21}
\end{align}

Using these equations, we get

\begin{align}
  \dot{x}^2 + \dot{y}^2 &= \dot{x}_0^2 + (\dot{v}_c^2 + \dot{w}_c^2)^2 - 2 \dot{w}_c \dot{s}_0 \dot{w}_c \dot{c}_0 + 2 \dot{v}_c \dot{c}_0 \dot{w}_c \dot{c}_0 + 2 \dot{v}_c \dot{s}_0 \dot{w}_c \dot{c}_0 + 2 \dot{v}_w \dot{c}_0 \dot{w}_c \dot{c}_0 + 2 \dot{v}_w \dot{s}_0 \dot{w}_c \dot{c}_0 \tag{22}
\end{align}

After taking the integral of Eq. (19), as shown in Appendix A.2, the kinetic energy can be written as

\begin{align}
  T = \frac{1}{2} \int_{-l/2}^{l/2} \rho A (\dot{x}^2 + \dot{y}^2) \; dv, \tag{23}
\end{align}

where

\begin{align}
  M_0 &= \begin{bmatrix}
    m_0 & 0 & -c_0 \alpha \xi \\
    0 & m_0 & -s_0 \alpha \xi \\
    -c_0 \alpha \xi & -s_0 \alpha \xi & I_0 + \xi^2 M_0 \xi
  \end{bmatrix}, \\
  M_0 &= \text{diag} \left( \int_0^l \rho A \phi(s) \; ds \right) \tag{24}
\end{align}

and

\begin{align}
  W &= \begin{bmatrix}
    -s_0 \alpha^T \\
    c_0 \alpha^T \\
    \beta^T
  \end{bmatrix}^T
\end{align}

in which

\begin{align}
  \alpha &= \int_0^l \rho A \phi(s) \; ds \quad \text{and} \quad \beta = \int_0^l \rho A (s - l/2) \phi(s) \; ds. \tag{25}
\end{align}
The elastic potential energy of the beam is defined as

\[ U = \frac{1}{2} \int_{-l/2}^{l/2} \left( \frac{\partial^2 W}{\partial x^2} \right)^2 \, ds = \frac{1}{2} k \xi^2 \]  

where

\[ k = \text{dig} \left[ \int_0^l E I \left( \frac{\partial^2 \phi}{\partial x^2} \right)^2 \, ds \right] \]  

and \( E I \) is the bending stiffness of the beam.

Applying Lagrange's equation and considering \( X = \begin{bmatrix} r_0^T & \xi^T \end{bmatrix}^T \), the equation of motion of the beam can be written as

\[ A \ddot{X} + B\dot{X} + KX = -R^2 \theta f \]  

where

\[ A = \begin{bmatrix} M_0 & W \cr W^T & M_0 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & 0 \\ 0 & k \end{bmatrix}, \quad \dot{B} = \begin{bmatrix} \dot{M}_0 - \frac{1}{2} \frac{\partial \dot{M}_0}{\partial r} W - \frac{1}{2} \frac{\partial \dot{W}}{\partial r} \\ W^T - \frac{1}{2} \frac{\partial \dot{W}}{\partial \xi} \end{bmatrix} \]  

and \( B = \begin{bmatrix} \frac{\partial \dot{M}_0}{\partial r} & \frac{\partial \dot{W}}{\partial r} \\ \frac{\partial \dot{W}}{\partial \xi} & 0 \end{bmatrix} \).

3.3. Combined dynamics of the system

Substituting \( \dot{r} \) and \( \dot{\xi} \) from Eqs. (11) and (12) into Eq. (17), the interaction forces/moments vector between two robots and the flexible beam, \( f \), can be expressed in the beam coordinates, \( X, \dot{X} \), and applied torques vector, \( u \), as

\[ f = M_0(\dot{r}) \ddot{X} + (M_0(\dot{\xi}) \dot{R} + C_0(\dot{r}, \dot{\xi}) \dot{X} + G_0(\dot{r}) - u. \]  

Replacing \( r \) and \( \xi \) from Eqs. (6) and (11) in terms of \( X \) and \( \dot{X} \) into \( M_0(\dot{r}), C_0(\dot{r}, \dot{\xi}) \) and \( G_0(\dot{r}) \) and renaming the corresponding terms \( M_0(X), C_0(X, \dot{X}) \) and \( G_0(X) \), Eq. (30) can be written as

\[ f = M_0(X) \ddot{X} + (M_0(X) \dot{R} + C_0(X, \dot{X}) \dot{X} + G_0(X) - u. \]  

Replacing the above equation for \( f \) into Eq. (28), the combined system dynamics is obtained as

\[ M(X) \ddot{X} + C(X, \dot{X}) \dot{X} + KX + R^2 C_0(X) = R^2 u \]  

where

\[ M(X) = A(X) + R^2 M_0(X) \]  

and

\[ C(X, \dot{X}) = B(X, \dot{X}) + R^2 C_0(X, \dot{X}) \dot{R} + R^2 M_0(X) \dot{R}. \]

Remark. Eq. (32) has the two principal properties, [31,32]:

1. The mass matrix \( M(X) \) is positive definite.
2. The matrix \( \dot{M}(X) - 2C(X, \dot{X}) \) is skew symmetric.

The first property can be inferred from positive definiteness property of matrix \( M_0(\dot{r}) \) for each robot and matrix \( A(X) \) in the beam dynamics. Since the matrix \( \dot{M}_0(\dot{r}) - 2C_0(\dot{r}, \dot{\xi}) \) in Eq. (15), robot dynamics, and also the matrix \( \dot{A}(X) = 2B(X, \dot{X}) \) in Eq. (28), beam dynamics, are skew symmetric, the second property is concluded.

4. Two-time scale control

An alternative nonlinear approach to the problem of trajectory tracking of two rigid manipulators moving a flexible object can be devised using the different time scales between the flexible system dynamics and the rigid dynamics. This can be made possible by singular perturbation approach, [27]. Specifically, the smallest stiffness constant of \( k \) in Eq. (27), say \( k_{11} \), can be regarded as the inverse of a perturbation parameter, i.e. \( \mu = 1/k_{11} \). For a given manipulator geometry, the limit \( \mu \to 0 \) corresponds to the case of an equivalent system with rigid object, say rigid system. It should be stressed that the approach is a candidate to work only if \( \mu \) is actually small (typically \( \mu < 1 \)). For example, for a slender steel beam with \( E \approx 200 \text{ GPa} \) and diameter \( d \approx 50 \text{ mm} \), \( E \cdot I \approx 6 \times 10^4 \text{ kgm}^2/s^2 \) and \( E \cdot A \approx 4 \times 10^4 \text{ kgm}/s^2 \) and therefore, \( \varepsilon \), the perturbation parameter, is very small. In general, however, the entries of \( k \) depend on the type of expansion functions used when generating the mathematical model and then the two-time scale approach is effective only when the above mentioned condition is satisfied.

It is worth mentioning from a practical point of view that it may not always be possible to separate rigid body dynamics from flexible dynamics. This depends on the location of the first flexible mode, and the bandwidth required in the slow subsystem design, [33].

In this section we will modify the dynamics obtained in the previous section for the two-time scale control theory. For this purpose, we should consider the following relations

\[ C(X, \dot{X})\dot{X} = \begin{bmatrix} C_1(r_0, \dot{r}_0, \dot{\xi}, \dot{\xi}) \\ C_2(r_0, \dot{r}_0, \dot{\xi}, \dot{\xi}) \end{bmatrix}, \quad KX = \begin{bmatrix} 0 \\ k \dot{\xi} \end{bmatrix}, \]  

\[ R^2 \dot{G}_0(X) = \begin{bmatrix} G_1(r_0, \dot{\xi}) \\ G_2(r_0, \dot{\xi}) \end{bmatrix}, \quad R^2 \dot{u} = \begin{bmatrix} R^2 u_1 + R^2 u_2 \\ R^2 u_1 \end{bmatrix}, \quad \dot{\xi} = \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} \]  

\[ R^{-1}(X) = \begin{bmatrix} [M_{11}(r_0, \dot{\xi}) & M_{12}(r_0, \dot{\xi})]^{-1} \\ [M_{21}(r_0, \dot{\xi}) & M_{22}(r_0, \dot{\xi})]^{-1} \end{bmatrix}, \quad \dot{r}_0 = \begin{bmatrix} \dot{r}_{11} \\ \dot{r}_{12} \end{bmatrix}, \quad \dot{r}_1 = \begin{bmatrix} \dot{r}_{21} \\ \dot{r}_{22} \end{bmatrix}. \]  

Then, Eq. (32) becomes – drops the dependencies for notation compactness –

\[ \dot{r}_0 = H_{11}(\dot{r}_0) + H_{12}(\dot{r}_2) - H_{11}(r_1) - H_{12}(r_2) - H_{12}(\dot{r}_2) - H_{12}(\dot{r}_2) \]  

\[ \dot{\xi} = H_{21}(\dot{r}_1) + H_{22}(\dot{r}_2) - H_{21}(r_1) - H_{22}(r_2) - H_{22}(\dot{r}_2) - H_{22}(\dot{r}_2) \]  

At this point, factoring \( k \) as \( k = k_1 k' \) and defining the new variable \( z = k_2 \dot{\xi} \), yield the equations of the system as

\[ \dot{r}_0 = H_{11}(\dot{r}_0) + H_{12}(\dot{r}_2) - H_{11}(r_1) - H_{12}(r_2) - H_{12}(\dot{r}_2) - H_{12}(\dot{r}_2) \]  

\[ \dot{z} = H_{21}(\dot{r}_1) + H_{22}(\dot{r}_2) - H_{21}(r_1) - H_{22}(r_2) - H_{22}(\dot{r}_2) - H_{22}(\dot{r}_2) \]  

(39)

where the superscript indicates that the corresponding quantities have been multiplied by \( k \).

Now, the typical steps of a singular perturbation formulation can be taken, [27,31]. Because of the presence of \( \mu \), the system described by Eq. (41) exhibits a boundary layer phenomenon in the fast variable \( z \).

For two-time scale control, we consider the overall control as

\[ u = u_1(r_0, \dot{r}_0) + u_2(z, \dot{r}_0, \dot{r}_2) \]  

from which we can conclude that \( u'_i \) in Eq. (35) can be written as

\[ u'_i = u'_i(r_0, \dot{r}_0) + u'_i(z, \dot{r}_0, \dot{r}_2) \]  

(43)

In Eq. (42), \( u_i = u_i|_{z=0} \), i.e. \( u'_i \) and then \( u_i \) is designed for the rigid system using only slow variables, and \( u'_i \) or \( u'_i \) counters the effects of beam elasticity. It should be noticed that \( u'_i|_{z=0} = 0 \).

At the first step, setting \( \mu = 0 \) we design \( u_i \) or \( u'_i \) for the slow subsystem and then for the second step using \( u_i \) we design \( u_i \) or \( u'_i \).
4.1. Slow control design for the rigid subsystem

In fact, setting \( \mu = 0 \) and solving for \( z \) in (35) gives
\[
\dot{z}_r = -H_{22}^{1}H_{21}(u'_s - C_{11} - G_{11}) - u_{21} + C_{22} + G_{22},
\]
where subscript \( s \) indicates that the system is considered in the slow time scale, i.e. when \( \mu = 0 \). Plugging (43) into (40) with \( \mu = 0 \) yields
\[
\dot{\tilde{z}}_r = (H_{11} - H_{12}H_{22}^{-1}H_{21})(u'_s - C_{11} - G_{11}).
\]
It is not difficult to show that
\[
(H_{11} - H_{12}H_{22}^{-1}H_{21}) = M_{11}^{-1}
\]
where \( M_{11} \) is the inertia matrix of the equivalent rigid system, so that Eq. (45) becomes
\[
M_{11}\ddot{r}_{12} + C_{11} + G_{11} = u'_s = R_{o1}u_{1s} + R_{o2}u_{2s}.
\]
This equation represents the dynamics of two cooperating robots handling a rigid object; therefore, the slow controller for the rigid nonlinear subsystem can be designed according to the control schemes used for the rigid robots, [28]. For this purpose we may consider, for example, the well-known PD control concept. Then, the slow control law can be designed as
\[
u_1 = K_F(r_{o1} - r_0) + K_P(r_{o2} - \dot{r}_0)
\]
where \( r_0 \) is the desired trajectory of the mass center of the rigid object and \( \dot{r}_0 \) is its derivative. \( K_F \) and \( K_P \) are constant, positive definite matrices, [31,32].

Since for the clamped-free model, see Fig. 2, \( r_0 \) can be expressed in terms of robots joint angels, it is seen that for the slow subsystem we require to measure only joint variables of the robots.

4.2. Control design for the fast subsystem

Rewriting Eq. (41) and considering Eq. (43) we have
\[
\mu \ddot{z} = -H_{22}^{1}z + H_{2}^{1}u_{1s} + H_{22}^{1}u_{2s} + H_{2}^{1}(u'_s - C_{11} - G_{11})
\]
\[
+ H_{22}^{1}(u_{2s} - C_{22} - G_{22}).
\]
Due to the time scale separation, we can assume that slow variables, and also \( z_r \), are at steady state with respect to variations of the fast variable \( z \). Consequently, considering Eqs. (44) and (35), Eq. (49) can be written as
\[
\ddot{z} = -H_{22}^{1}z + H_{2}^{1}(R_{o1}u_{1s} + R_{o2}u_{2s}) + H_{22}^{1}R_{o1}u_{2s} + H_{22}^{1}z_r.
\]
Defining the fast variable \( z_r = z - z_s \) and the new time scale \( \tau = t/\sqrt{\mu} \) and rearranging Eq. (50), the fast subsystem dynamics becomes
\[
\frac{d^2z_r}{d\tau^2} = -H_{22}z_r + H_{2}^{1}R_{o1}u_{1s} + (H_{2}^{1}R_{o2}^2 + H_{22}^{1}R_{o1}^2)u_{2s}.
\]
At this point, it is required that the fast subsystem (51) be uniformly stable along the trajectories \( z_r \), given by (44). For this purpose, let’s use a state space representation of (51) as
\[
\dot{\eta} = A_f \eta + B_f u_f,
\]
where
\[
\eta = \begin{bmatrix} z_r \\ \frac{dz_r}{d\tau} \end{bmatrix}, \quad A_f = \begin{bmatrix} 0 & 1 \\ -H_{22} & 0 \end{bmatrix}, \quad u_f = \begin{bmatrix} u_{1s} \\ u_{2s} \end{bmatrix}
\]
and
\[
B_f = \begin{bmatrix} 0 \\ H_{2}^{1}R_{o1} \\ H_{2}^{1}R_{o2}^2 + H_{22}^{1}R_{o1}^2 \end{bmatrix}.
\]
The system described by Eq. (53) is a linear system whose poles are on the imaginary axis in the \( s \)-plane. Since the pair \((A_f, B_f)\) can be shown to be completely controllable, a fast feedback control of the type
\[
u_f = -K_f \eta
\]
can be devised to arbitrarily place the poles of the closed-loop system. It is important to note that matrix \( K_f \) is not unique for a given system, but depends on the desired closed-loop pole locations (which determine the speed and damping of the fast subsystem response). Also, note that the selection of the desired closed-loop poles is a compromise between the rapidity of the response of the error vector and the sensitivity to disturbances and measurement noises, [34], in our system in measurement of the joint variables of the two rigid robots. That is, for our system, we should select the desired poles of the fast subsystem to increase the speed of error response of the fast subsystem, while the adverse effects of disturbances and measurement noises do not increase in such a way that the combined system becomes unstable.

4.3. Observer design for the fast subsystem control

It is seen that to control the fast subsystem we require information about the vibration of the flexible object and hence it is necessary to use measurement instruments in different points of the beam. To avoid this problem, measuring flexible coordinates and their time derivatives, we can design an observer for linear fast subsystem.

First, we should notice that for the clamped-free model, the deflection and slope \((w_2, w_3)\) at the free end can be written in terms of the two rigid robot joint angels.

Defining \( \phi_1 = \phi(s)_{\|o1} \) and \( \phi_2' = \phi'(s)_{\|o}, \) and considering Eqs. (4.3) and (53) and \( z_r = z - z_s \), we can write
\[
\begin{bmatrix} w_2 \\ w_3 \\ \frac{dz_r}{d\tau} \end{bmatrix} = \begin{bmatrix} \phi_k^{-1}z_r \\ \phi_k^{-1}z_r \\ 0 \end{bmatrix}
\]
and
\[
\begin{bmatrix} w_2 \\ w_3 \\ \frac{dz_r}{d\tau} \end{bmatrix} = \begin{bmatrix} \phi_k^{-1}(k\beta - z_r) \\ \phi'_k^{-1}(k\beta - z_r) \\ 0 \end{bmatrix}.
\]
Because \( w_2 \) and \( w_3 \) can be computed from the two robots joint angels and also \( z_r \) is calculated in terms of the slow variables in Eq. (44), the right side of the Eq. (55) can be obtained from the joint variables of two robots and, therefore, we can consider this term as the output of the fast subsystem (52).

Defining
\[
C_f = \begin{bmatrix} \phi_k^{-1} 0 \\ \phi'_k^{-1} 0 \end{bmatrix}
\]
and considering the first equation of (52), the output of the fast subsystem can be written as
\[
y = C_f \eta.
\]
It can be shown that the pair \((A_f, C_f)\) is full state observable; therefore, linear observer can be designed to estimate the fast variables \( \eta \). If we consider \( \hat{\eta} \) as the estimate of the \( \eta \) in any time, \( \hat{\eta} \) can be obtained as, [34]
\[
\dot{\hat{\eta}} = A_f \hat{\eta} + B_f u_f + K_f (y - C_f \hat{\eta})
\]
where \( K_f \) should be designed to force \( \hat{\eta} \) to converge to \( \eta \) very fast. Notice that the feedback signal through the observer gain matrix \( K_f \) serves as a correction signal to the fast subsystem to account for the unknowns in the fast subsystem. If significant unknowns are involved, the feedback signal through the matrix \( K_f \) should
be relatively large. However, if the output signal is contaminated significantly by disturbances and measurement noises, then the output $y$ is not reliable and the feedback signal through the matrix $K_r$ should be relatively small. In determining the matrix $K_r$, we should carefully examine the effects of disturbances and noises involved in the output $y$, [34].

By Eq. (58) the fast control input in Eq. (54) can be modified by

$$u_r = -K_f \bar{y}.$$  \hspace{1cm} (59)

5. Simulation results

In the simulation, we have considered two planar robots, each with three revolute joints, see Figs. 1-3. Jacobian matrix $J$, inertia matrix $M$, matrix of Coriolis and centrifugal effects, $C_i$, and vector of gravitational terms, $G_i$, of each robot are given in Appendix A.3. These two robots are handling a flexible beam that is moved from the initial position and orientation of $r_0 = [2.2 \ 1.7 \ 0]$ to the final position and orientation of $r_f = [2.4 \ 1.5 \ .3]$ through the desired designed trajectories for orientation and position of the center of the beam. These trajectories consist of some polynomials, see Figs. 4-6. The area cross section and length of the beam are $(0.0001 \pi \text{ m}^2)$ and $(1 \text{ m})$, respectively. The beam flexural rigidity and mass of the beam are $(EI = 1 \text{ Nm}^2)$ and $(1 \text{ kg})$, respectively. Each link of the robots has weight of $(1 \text{ kg})$ and length of $(1 \text{ m})$. The initial conditions of the deflection of the beam and its time derivative have been considered to be zero. The slow controller gains in (48) are considered as $K_0 = 5 I_{6 \times 6}$ and $K_s = 7 I_{6 \times 6}$. For the fast subsystem in any time $K_f$ and $K_e$ in (58) and (59) are designed so that the poles of the linear system and the observer can be placed on $P = [-2 + .005i -2 - .005i -2.25 -2.5 -2.75 -3 -3.5 -4 -4.5 -5 -5.5 -6] \text{ and } P_{\text{observer}} = 2 * [-2 + .005i -2 - .005i -2.25 -2.5 -2.75 -3 -3.5 -4 -4.5 -5 -5.5 -6]$, respectively. The number of the flexible coordinates, $\xi_i$, is selected to be 6. By the considered values for the beam parameters and 6 mode shapes of the flexible beam the perturbation parameter, $\mu$, becomes 0.0808 which is suitable for the two time scale control.

Figs. 4 to 6 show the trajectory tracking of the rigid body motion vs. time, where the first two curves represent the $X$ and $Y$ coordinates of the position and the last curve shows the orientation of the beam. Also, Figs. 7-12 show the flexible coordinates of the beam vs. time. It can be seen from Figs. 4 to 6 and Figs. 7 to 12 that the rigid body trajectory tracking errors converge to zero as well as the flexible coordinates. For the deflection along the beam length, for example, the deflection of the free end of the beam (in cm) throughout the trajectory tracking is shown in Fig. 13.

Comparison

The results presented in Refs. [22,23,26] can be compared with the corresponding results presented in this paper. For example, the last Fig. 13, in compare to the corresponding results of Reference [26], shows much better vibration suppression, [35].

6. Conclusion

In this paper, two planar robots, each with three revolute joints, grasping and moving a flexible beam were considered. The kinematics and combined dynamics of rigid robot manipulators and the flexible beam were derived. For trajectory tracking of the mass center of the beam, while suppressing the vibration of the beam, a sufficiently accurate control synthesis is needed. So, to deal with this problem, we proposed to develop the following procedure:

![Fig. 4. Trajectory tracking for the beam rigid motion, X-coordinate of center of mass, vs. time.](image)

![Fig. 5. Trajectory tracking for the beam rigid motion, Y-coordinate of center of mass, vs. time.](image)

![Fig. 6. Trajectory tracking for the beam rigid motion, orientation, vs. time.](image)

![Fig. 7. The behavior of flexible coordinate $\xi_1$, vs. time.](image)
• Decompose the system dynamics into robots dynamics and beam dynamics.
• Decompose beam dynamics into rigid and flexible motions.

Fig. 8. The behavior of flexible coordinate $\xi_2$, vs. time.

Fig. 9. The behavior of flexible coordinate $\xi_3$, vs. time.

Fig. 10. The behavior of flexible coordinate $\xi_4$, vs. time.

Fig. 11. The behavior of flexible coordinate $\xi_5$, vs. time.

Fig. 12. The behavior of flexible coordinate $\xi_6$, vs. time.

Fig. 13. Beam deflection at the free end, ($s = l$), in cm vs. time.

• Relate the two rigid dynamics and also the flexible dynamics by a cascade structure.
• Control the two parts of the system dynamics by two-time scale (TTS) control method.
• Control the rigid dynamics by some PD control scheme.
For control of linear flexible dynamics, use pole placement.
To avoid use of distributed measurement/actuation systems or use of a number of sensors/actuators, design a linear observer.
Combine these controllers and the observer as a composite controller for the ultimate aim of trajectory tracking of the rigid subsystem while stabilizing the flexible subsystem (vibration suppression) by applying Tikhonov’s theorem [27,36].

Simulation results showed the efficiency and usefulness of the proposed control and observer scheme.

Appendix

A.1. Assumed mode shapes

The first six assumed modes for clamped-free boundary conditions are given as, [29,30]:
\[ \Phi(s) = A_i \left( \cosh(\beta_i s) + B_i \sinh(\beta_i s) + C_i \cos(\beta_i s) + D_i \sin(\beta_i s) \right) \]
with
\[ \beta_1 = \sqrt{\frac{3.25}{l}}, \quad \beta_2 = \sqrt{\frac{22.03}{l}}, \quad \beta_3 = \sqrt{\frac{61.7}{l}} \]
\[ \beta_4 = \sqrt{\frac{120.9}{l}}, \quad \beta_5 = \sqrt{\frac{199.86}{l}}, \quad \beta_6 = \sqrt{\frac{298.56}{l}} \]
where
\[ A_i = 1, \quad B_i = A_i \frac{\sin(\beta_i l) - \sinh(\beta_i l)}{\cos(\beta_i l) + \cosh(\beta_i l)}, \quad C_i = -A_i, \]
\[ D_i = -B_i. \]

A.2. Taking integral of Eq. (19)

By considering Eqs. (21), we derive the following equations below
\[ \int_{l/2}^{l/2} \rho \dot{A} \beta_i^2 dv = \ddot{p}_i^T H_0 \ddot{p}_0 \]
\[ \int_{l/2}^{l/2} \rho \dot{A} (\omega^2 + \omega^2) \beta_i^2 dv = \ddot{\theta}^T \left[ \int_{l/2}^{l/2} \rho \dot{A} \omega^T \phi ds \right] \dot{\xi} = \ddot{\theta}^T \left[ I_s + \xi^T H_0 \xi \right] \dot{\xi} \]
\[ \int_{l/2}^{l/2} \rho \dot{A} \omega^2 dv = \dot{\xi}^T \left[ \int_{l/2}^{l/2} \rho \dot{A} \omega^T \phi ds \right] \dot{\xi} = \dot{\xi}^T M_0 \dot{\xi} \]
\[ \int_{l/2}^{l/2} \rho \dot{A} \ddot{p}_i^T \left[-s_0 \quad c_0 \right] \dot{v} dv = 0 \]
\[ \int_{l/2}^{l/2} \rho \dot{A} \ddot{p}_i^T \left[-s_0 \quad -c_0 \quad c_0 \right] \dot{v} dv = \dot{p}_i^T \left[-s_0 \quad -c_0 \quad c_0 \right] \left[ \int_{l/2}^{l/2} \rho \dot{A} \phi dv \right] \dot{\xi} = \dot{p}_i^T w_1(\theta) \dot{\xi} \]
\[ \int_{l/2}^{l/2} \rho \dot{A} \ddot{p}_i^T \left[-c_0 \quad -s_0 \quad -c_0 \right] \dot{v} dv = \dot{p}_i^T \left[-c_0 \quad -s_0 \quad -c_0 \right] \left[ \int_{l/2}^{l/2} \rho \dot{A} \phi dv \right] \dot{\theta} = \dot{p}_i^T w_2(\theta, \xi) \dot{\theta} \]
\[ \int_{l/2}^{l/2} \rho \dot{A} \dot{\phi} dv = \ddot{\theta} \int_{l/2}^{l/2} \rho \dot{A} \phi dv \ddot{\xi} = \ddot{\theta} \beta_i \dot{\xi} \]
where
\[ H_0 = \begin{bmatrix} m_0 & 0 \\ 0 & m_i \end{bmatrix} \]
\[ w_1(\theta) = \left[ -s_0 \omega^T \begin{bmatrix} c_0 \\ s_0 \end{bmatrix} \right]^T \]
\[ w_2(\theta, \xi) = \left[ -c_0 \omega q \quad -s_0 \omega q \right]^T. \]

Then, by summing the above equations and substituting Eq. (19), Eq. (23) will be obtained.

A.3. Robots’ Jacobian and dynamics matrices and vectors

Jacobian matrix of each robot is, [35]:
\[ J_i = \begin{bmatrix} -a_{i1} s_{i1} - a_{i2} s_{i2} - a_{i3} s_{12} & -a_{i1} s_{i1} - a_{i3} s_{12} - a_{i3} s_{23} & -a_{i4} s_{12} \\ a_{i1} c_{i1} + a_{i2} c_{i2} + a_{i3} c_{12} & a_{i2} c_{i1} + a_{i3} c_{12} + a_{i4} c_{23} & a_{i3} c_{12} \end{bmatrix} \]

Elements of inertia matrix and vector of gravitational terms of each robot are as follows, [35]:
\[ M_{11} = l_{i22} + l_{i23} + m_i (a_1 + r_{i1})^2 \]
\[ + m_i (a_2 + r_{i2}) (2c_i a_1 + a_2 + r_{i3}) \]
\[ + m_i a_1^2 (m_i + m_2 (2c_i a_1 + a_2 + r_{i3})^2 + m_3 (a_1 + r_{i1}) (a_1 + r_{i3} + 2c_i a_2 + 2c_3 a_1) \]
\[ M_{12} = l_{i22} + l_{i23} + m_i (a_2 + r_{i2})^2 + m_3 a_2^2 \]
\[ + m_3 (a_1 + r_{i1}) (a_1 + r_{i3} + 2c_i a_2 + 2c_3 a_1) \]
\[ M_{13} = l_{i21} + m_i (a_3 + r_{i3}) (a_3 + r_{i3} + c_3 a_2 + c_2 a_3) \]
\[ M_{22} = l_{i22} + l_{i23} + m_i (a_2 + r_{i2})^2 + m_3 a_2^2 \]
\[ + m_3 (a_1 + r_{i1}) (a_1 + r_{i3} + 2c_i a_2 + 2c_3 a_1) \]
\[ M_{23} = l_{i22} + m_i (a_3 + r_{i3}) (a_3 + r_{i3} + c_3 a_2 + c_2 a_3) \]
\[ M_{33} = l_{i23} + m_i (a_3 + r_{i3})^2 \]

and
\[ G_1 = g \left[ m_i c_1 a_1 + r_{i1} + m_i c_1 a_1 + m_i c_1 a_2 + m_i c_1 a_2 + m_i c_1 a_2 \right] \]
\[ G_2 = g \left[ m_i c_1 a_1 + m_i c_1 a_2 + m_i c_1 a_2 + m_i c_1 a_2 \right] \]
\[ G_3 = g \left( m_i c_1 a_1 + r_{i1} \right) \]

Matrix of Coriolis and centrifugal effects can be found by using the following formula, [31,32]:
\[ C_i(q, \dot{q}) \dot{q} = \ddot{M}_i(q, \dot{q}) \dot{q} = \frac{1}{2} \frac{\partial}{\partial \dot{q}} (\ddot{q}^T \dot{M}_i(q, \dot{q}) \dot{q}). \]

In the above equation the following definitions, see Fig. 1, have been used:
\[ c_{ij} = \cos(q_{ij}) \]
\[ s_{ij} = \sin(q_{ij}) \]
\[ c_{ij} = \cos(q_{ij} + q_{kl}) \]
\[ s_{ij} = \sin(q_{ij} + q_{kl}) \]
\[ a_{ij} = \text{length of } j \text{th link of robot } i \]
\[ m_{ij} = \text{mass of } j \text{th link of robot } i \]
\[ r_{ij} = \text{distance from center of mass of } j \text{th link of robot } i \text{ to the origin of the link frame} \]
\[ I_{ij} = \text{moment of inertia of } j \text{th link of robot } i \text{ about } z \text{ axis (perpendicular to the plane).} \]

References

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