BIFURCATIONS OF CUSPIDAL LOOPS PRESERVING NILPOTENT SINGULARITIES

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Abstract. A cuspidal loop $L$ of a smooth planar vector field $X_0$ is a singular cycle formed by the union of a cuspidal singularity with 2-jet equivalent to $y \frac{\partial}{\partial x} + (x^2 + b_0 xy) \frac{\partial}{\partial y}$ and a connection between its two local separatrices. We consider smooth unfoldings $X_\lambda$ along cuspidal loop $L$ of $X_0$ parameterized by $\lambda \in (\mathbb{R}^p, 0)$. We assume that the cuspidal point exists at all parameter values. Let $P_0$ be the Poincaré map of $X_0$ along $L$. If this map is not formally equal to the identity, then it has the asymptotic expansion $P_0 : u \to u + a_\pm |u| \tau + \cdots$, where $\pm$ is the sign of $u$, $a_\pm \neq 0$, and $\tau$ is a coefficient belonging to the sequence $S = \{1, 7/6, 11/6, 2, \ldots \} = \{n \in \mathbb{N}\} \cup \{m + 1/6, m \in \mathbb{N}, m \geq 1\} \cup \{p - 1/6, p \in \mathbb{N}, p \geq 2\}$. In this case we say that $(X_0, L)$ has finite codimension equal to the order of $\tau$ in the sequence $S$. The main result of this paper is that the cyclicity of the unfolding $X_\lambda$ has an explicit bound of $e.o. H_0(s)$, where $s$ is the codimension of $(X_0, L)$ ($e.o. H_0(s) \sim \frac{5}{7} s^3$ when $s \to \infty$). This bound is sharp for generic unfoldings. For analytic unfoldings, the cyclicity is always finite and is given by the codimension of the related Abelian integral in the case of a non-conservative perturbation of a Hamiltonian vector field.

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1. Introduction

The singular cycles or graphics are the simplest invariant sets of dynamical systems generated by 2-dimensional vector fields involving regular orbits and singular points and exhibiting recurrent behaviour. The study of these objects and their relation to isolated periodic orbits (also called limit cycles) goes back to Dulac [D].

The understanding of the unfoldings of graphics is a key ingredient in the study of Hilbert’s sixteenth problem [H1], [R3], [I] and of some dynamical models arising in applications.

In the works of Roussarie [R1], Joyal [J], Mourtada [Mo], Il’yashenko and Yakovenko [IY], Kaloshin [K], and many others, essential steps were made towards
the proof of the existence of a finite bound for the number of limit cycles (finite cyclicity) and towards the understanding of the bifurcation diagrams of elementary graphics, i.e., those with only hyperbolic and semi-hyperbolic singularities.

The study of unfoldings of non-elementary graphics, with nilpotent singularities say, involves analytical difficulties of a different nature, which were not resolved in the works mentioned above. They are related mainly to the fact that non-elementary singularities bifurcate in a non-trivial way (for instance, they may disappear at some parameter values). A general method for desingularizing parametrized families of 2-dimensional vector fields, called global desingularization, was introduced in [DR]. This method was used in [P] to desingularize families whose non-elementary singularities are nilpotent singular points. At such points, the linear part of the field is non-zero and has zero eigenvalues. In [DRS], this desingularization method was used to study the bifurcation diagrams of the generic 3-parameter unfoldings of a cuspidal loop (see Fig. 1), which is a singular loop $L$ formed by the union of a cuspidal singularity and a connection between its two local separatrices. A cuspidal singularity is a generic nilpotent singularity. Its 2-jet is equivalent to the polynomial vector field

$$y \frac{\partial}{\partial x} + (x^2 + b_0xy) \frac{\partial}{\partial y}$$

for some $b_0 \in \mathbb{R}$.

In this article, we analyze the unfoldings of a cuspidal loop which do not change the nature of the singularity. This means that the singular point remains nilpotent for each value of the parameter. As the cuspidal singularity is stable in the set of nilpotent singularities, the point remains in fact a cuspidal singularity. We shall say that such an unfolding is cusp-preserving. This study is intended as the first step towards the construction of a theory of general unfoldings of cuspidal loops (not necessarily cusp-preserving). These general unfoldings require applying global desingularization; this gives rise to new asymptotic problems (see, e.g., [DRS]). In the case of unfoldings which preserve the cuspidal singularity, no desingularization is needed, because the type of the singularity does not change. Moreover, the theory of expansion of functions on some simple asymptotic scale applies.

In Section 2, we analyze the Dulac map $D$ for a cuspidal singularity, i.e., the transition between the transversals to the local stable separatrix and to the local unstable separatrix. A consequence of this analysis is an expression for the principal part of the function $D - \text{Id}$ in terms of the Loray normal form of the
cuspoidal singularity. This result is used in Section 3 to define the codimension of a cuspoidal loop \((X, L)\). In the same Section 3, some properties of asymptotic scales are also recalled. In particular, we introduce the Chebyshev asymptotic scales. It can be proved, again as a consequence of the study made in Section 2, that the difference map \(\Delta(u, \lambda)\), which controls the position of the bifurcating limit cycles, can be represented as a collection of functions whose principal parts belong to an asymptotic scale \(\mathcal{S}_0\). It is a Chebyshev asymptotic scale on the side of \(u > 0\), which is formed by the monomials \(u^i\) for \(i \geq 0\), \(u^{j+1/6}\) for \(j \geq 1\), and \(u^{k-1/6}\) for \(k \geq 2\). To simultaneously study both sides of the cuspoidal loop, we use generalized Chebyshev asymptotic scales. The asymptotic scale \(\mathcal{S}_0\) considered on the two sides of 0 does have this generalized type. We define an effective order \(\text{e.o.}_{\mathcal{S}_0}(s)\) for this scale, which is a sequence of integers. Its explicit expression and the estimates \(\text{e.o.}_{\mathcal{S}_0}(s) < \frac{5}{3}s\) and \(\text{e.o.}_{\mathcal{S}_0}(s) \sim \frac{5}{3}s\) as \(s \to \infty\) are given in Lemma 4.9. Then, based on the properties of asymptotic expansions which are proved in Section 3, we obtain the following results, which are enunciated and proved in Section 4.5:

Let \((X, L)\) be a cuspoidal loop of finite codimension \(s\). Then the cyclicity of any cusp-preserving unfolding of this cuspoidal loop is bounded by \(\text{e.o.}_{\mathcal{S}_0}(s)\). Moreover, if the above unfolding is generic, then its cyclicity is equal to \(\text{e.o.}_{\mathcal{S}_0}(s)\).

The generic one-parameter cusp-preserving unfolding of a codimension one loop is included in the generic three-parameter unfolding studied in [DRS]. We find only one of the four possible limit cycles which appear when we allow the cuspoidal singularity to bifurcate. Note that, since \(\text{e.o.}_{\mathcal{S}_0}(2) = 3\), the number of limit cycles bifurcating generically from a cuspoidal loop of codimension 2 is three rather than two, as one might expect.

In the last section, we study the cusp-preserving unfoldings \((X_{\bar{\lambda}, \varepsilon}, L)\) of Hamiltonian cuspoidal loops. That is, we assume that, for \(\varepsilon = 0\), the dual 1-form \(\omega_{(\bar{\lambda}, 0)}\) is equal to \(dH\), where \(H(x, y)\) is a smooth function defined on a neighbourhood of \(L\). To such an unfolding, we associate an Abelian integral unfolding \(I(h, \bar{\lambda})\). This unfolding \(I(h, \bar{\lambda})\) is expandable in the asymptotic scale \(\mathcal{S}_0\). We define codimension and genericity for Abelian integrals and prove the following assertions:

Let \((X_{\bar{\lambda}, \varepsilon}, L)\) be a cusp-preserving unfolding of a Hamiltonian cuspoidal loop with Abelian integral unfolding \(I(h, \bar{\lambda})\). Suppose that \(I(h, 0)\) has codimension \(s\). Then the cyclicity of the unfolding \((X_{\bar{\lambda}, \varepsilon}, L)\) is bounded by \(\text{e.o.}_{\mathcal{S}_0}(s)\). Moreover, if we suppose that the unfolding \(I(h, \bar{\lambda})\) is generic, then the cyclicity of \((X_{\bar{\lambda}, \varepsilon}, L)\) is equal to \(\text{e.o.}_{\mathcal{S}_0}(s)\).

Finally, we give an explicit polynomial example of a generic cusp-preserving unfolding of a Hamiltonian cuspoidal loop. We also prove that any analytic cusp-preserving unfolding of a Hamiltonian cuspoidal loop has a finite cyclicity.

2. The Dulac Map for a Cuspidal Singularity

The goal of this section is to obtain asymptotic expansions for the Dulac maps \(D_+\) and \(D_-\), which are illustrated in Fig. 2.
Let us briefly outline the main steps of our analysis. We start by considering only analytic cuspidal 1-forms (see Definition 2.1 below). In Section 2.1, we study the desingularization of the complex foliation associated to an analytic cuspidal 1-form and describe the elementary singularities which appear on the exceptional divisor after such a procedure. In Section 2.2, we recall the definition of the projective holonomy group associated to cuspidal 1-forms (see, e.g., [CM]) and relate the Dulac maps $D_+$ and $D_-$ to certain elements of this holonomy group. In Section 2.3, we use the normal form theorem of Loray [L] and the study of solvable groups made in [LM] to obtain a classification of cuspidal 1-forms in terms of finite jet truncations of the projective holonomy group. This classification is used in Section 2.4 to prove the main results of our analysis, Theorem 2.17 and its corollary.

We start by introducing some basic notions.

**Definition 2.1.** Let $\omega = F(x, y)dx + G(x, y)dy$ be a real smooth 1-form. We say that $\omega$ is **cuspidal** if its 1-jet at the origin is a nonzero nilpotent differential form and

$$\dim_{\mathbb{C}} \mathbb{C}[[x, y]](F, G) = 2.$$ 

Up to a local smooth change of coordinates, a cuspidal 1-form $\omega$ can always be written as

$$\omega = d(y^2 - x^3) + (xR_2 + yR_1)dx,$$

where $R_i$ is an analytic germ of multiplicity larger than $i$ (for $i = 1, 2$). If $\omega$ has real coefficients, then the real leaves of the Pfaffian system $\omega = 0$ near the origin are always topologically equivalent to level sets of the cusp $d(y^2 - x^3) = 0$.

![Figure 2. The Dulac map](image)

Our goal is to compute the asymptotic expansion of the **Dulac map**

\[ D: \Gamma^1 \to \Gamma^2, \]

which is shown in Fig. 2. Up to rescaling, we can assume that the transversal sections $\Gamma^1$ and $\Gamma^2$ are segments of the line $\{x = 1\}$. We shall always orient these transversals as shown in the figure. Their parametrizations will be defined later on, in terms of normal form coordinates given on a certain saddle point which will result from the desingularization of $\omega$. 
From now on, we shall suppose that $\omega$ is an analytic 1-form. The reason for this assumption is that we compute $D$ by complexifying the Pfaffian equation $\omega = 0$. At the end of the section, we shall see how to transpose this result to the $C^\infty$ setting.

According to [CM], there exists an analytic change of coordinates such that the cuspidal form $\omega$ can be written as

$$\omega = d(y^2 - x^3) + g(x, y)(3y \, dx - 2x \, dy)$$

for some analytic germ $g$ vanishing at the origin. We say that this is the prenormal form of $\omega$.

In these coordinates, the separatrix of $\omega$ is precisely the cusp $\Sigma = \{y^2 - x^3 = 0\}$.

To compute the Dulac map $D: \Gamma^1 \to \Gamma^2$, it is convenient to decompose each section $\Gamma^i$ into two components,

$$\Gamma^i_+ = \Gamma^i \cap \{y^2 - x^3 \leq 0\} \quad \text{and} \quad \Gamma^i_- = \Gamma^i \cap \{y^2 - x^3 \geq 0\}, \quad i = 1, 2,$$

and consider separately the positive and negative Dulac maps

$$D_+: \Gamma^i_+ \to \Gamma^2_+ \quad \text{and} \quad D_-: \Gamma^i_- \to \Gamma^2_-$$

(see Fig. 2). For what follows, it is important to distinguish between the analytic and the transcendental components of such Dulac maps. For this purpose, we fix some small $\varepsilon > 0$ and consider the curves

$$\Sigma^\varepsilon_+ = \{y^2 - (1 - \varepsilon)x^3 = 0\} \quad \text{and} \quad \Sigma^\varepsilon_- = \{y^2 - (1 + \varepsilon)x^3 = 0\},$$

which are transversals to $\omega$ outside the origin.

As shown in Fig. 3, the Dulac maps $D_+$ and $D_-$ can be written as

$$D_+ = D^{-1}_{-2} \circ T_+ \circ D_{+, 1} \quad \text{and} \quad D_- = D^{-1}_{-2} \circ T_- \circ D_{-, 1}.$$

Our plan is as follows. First of all, we shall compute the transition maps $T_{\pm}: \Sigma^\varepsilon_{\pm} \to \Sigma^\varepsilon_{\pm}$. We shall prove that their germs are members of the projective holonomy group of $\omega$. Then, we shall compute the local Dulac maps $D_{+, i}: \Gamma^i \to \Sigma^\varepsilon_{\pm}$ by standard normal form techniques. Our principal result is Theorem 2.17: we prove that the first order of the maps $D_\pm(Y)$ is equivalent to $|Y|^n + 5/6$ or $|Y|^n + 7/6$ depending
2.1. The desingularization of $\omega$. The minimal desingularization $\pi: M \to (\mathbb{C}^2, 0)$ of a cuspidal differential form $\omega$ in prenormal form is given by a sequence of three quadratic blowing-ups

$$
\begin{align*}
\pi_1: & M_1 \to (\mathbb{C}^2, 0), \\
\pi_2: & M_2 \to M_1, \\
\pi_3: & M_3 \to M_2
\end{align*}
$$

with $M_3 = M$. Let us denote the usual affine charts in a neighbourhood of the exceptional divisor $D_i$ by $(x_i, y_i)$ and $(x_i', y_i')$. For each $i = 1, 2, 3$, $D_i$ is given either by $\{x_i = 0\}$ or by $\{y_i = 0\}$, and we have $y_i x_i' = 1$.

In these coordinates, the blowing-ups are given by

$$
\begin{align*}
\pi_1: & \begin{cases}
x = x_1 = x_1', y_1, \\
y = x_1 y_1 = y_1',
\end{cases} \\
\pi_2: & \begin{cases}
x_1' = x_2 = x_2', y_2, \\
y_1' = x_2 y_2 = y_2',
\end{cases} \\
\pi_3: & \begin{cases}
x = x_3 = x_3', y_3, \\
y_2 = x_3 y_3 = y_3'.
\end{cases}
\end{align*}
$$

Consider the chart $(x_3, y_3)$. Using the abbreviated notation $\bar{x} = x_3$, $\bar{y} = y_3$, we write the blowing-up map $\pi$ in these coordinates as

$$
\begin{align*}
x = \bar{x}^2 \bar{y}, \\
y = \bar{x}^3 \bar{y}^2.
\end{align*}
$$

Therefore, the total transforms of the differential forms $d(y^2 - x^3)$ and $d(y^2/x^3)$ are given, respectively, by

$$
d(\bar{x}^6 \bar{y}^3 (\bar{y} - 1)) \quad \text{and} \quad d\bar{y} = 0. \tag{1}
$$

Since $\omega = d(y^2 - x^3) + g(x, y) \frac{\bar{x}^2}{\bar{y}} d(y^2/x^3)$, this implies that the total transform of $\omega$ is given by

$$
d(\bar{x}^6 \bar{y}^3 (\bar{y} - 1)) + g(\bar{x}^2 \bar{y}, \bar{x}^3 \bar{y}^2) \bar{x}^5 \bar{y}^2 d\bar{y}.
$$

Dividing by $\bar{x}^5 \bar{y}^2$, we obtain the strict transform of $\omega$, which is

$$
\bar{\omega} = 6\bar{y}(\bar{y} - 1) d\bar{x} + [\bar{x}(4\bar{y} - 3) + g(\bar{x}^2 \bar{y}, \bar{x}^3 \bar{y}^2)] d\bar{y}.
$$

Similarly, the strict transforms of the sections $\Sigma_+^\varepsilon$ and $\Sigma_-^\varepsilon$ are given by the straight lines

$$
\Sigma_+^\varepsilon = \{\bar{y} = 1 - \varepsilon\} \quad \text{and} \quad \Sigma_-^\varepsilon = \{\bar{y} = 1 + \varepsilon\}.
$$

The above 1-form $\bar{\omega}$ is a local generator of the strict transform $\tilde{F}_\omega$ of the foliation $\mathcal{F}_\omega = \{\omega = 0\}$. On a sufficiently small neighbourhood of $\pi^{-1}(0)$, the singularities of $\tilde{F}_\omega$ are the points

$$
s_2 = D_2 \cap D_3, \quad s_3 = D_3 \cap D_1, \quad \text{and} \quad s = \{\bar{x} = \bar{y} - 1 = 0\},
$$

and the sets $L_i = D_i \setminus \{s_2, s_3, s\}$ are the regular leaves of $\tilde{F}_\omega$ ($i = 1, 2, 3$). These points $s_2$, $s_3$, and $s$ are singularities of saddle type for $\tilde{F}_\omega$. Moreover, the following assertions are valid:

- In a neighbourhood of $s_3 = \{x_3' = y_3' = 0\}$, there exists a local (near identity) analytic change of coordinates

$$
(x_3', y_3') = (X, Y(1 + P(X, Y)), \quad \text{where} \quad P(0, 0) = 0,
$$

such that $\tilde{F}_\omega = \{Y dX + 3X dY = 0\}$.
In a neighbourhood of \( s_2 = \{ \bar{x} = \bar{y} = 0 \} \), there exists a local (near identity) analytic change of coordinates
\[
(\bar{x}, \bar{y}) = (X(1 + Q(X, Y)), Y), \quad \text{where } Q(0, 0) = 0,
\]
such that \( \tilde{\mathcal{F}}_\omega = \{ 2YdX + XdY = 0 \} \).

We refer to [CM] for a more detailed description of such foliations.

2.2. Projective holonomy. Let \( \text{Diff}(\mathbb{C}, 0) \) denote the group of germs of analytic diffeomorphisms in \((\mathbb{C}, 0)\) under the composition operation. According to [CM], the projective holonomy of \( \omega \) is the subgroup \( H_\omega \subset \text{Diff}(\mathbb{C}, 0) \) obtained as follows: we fix a local parametrized transversal \( \tilde{\Sigma} \subset M \) to the foliation at some point \( p \in \tilde{\Sigma} \cap L_3 \) and let the projective holonomy (based on the transversal \( \tilde{\Sigma} \) ) \( H_\omega \) be the set of germs of the diffeomorphism \( f: (\tilde{\Sigma}, p) \to (\tilde{\Sigma}, p) \) obtained by the suspension of closed paths \( \gamma \ni p \in L_3 \) to neighbour leaves.

Remark 2.2. The suspension can always be made to an open neighbourhood of \( \gamma \) where some transversal fibration to \( \tilde{\mathcal{F}}_\omega \) is defined.

Note that homotopic paths determine the same germ \( f \), and a different choice of the transversal gives a conjugate group in \( \text{Diff}(\mathbb{C}, 0) \). Thus, the projective holonomy can be regarded as a map from the fundamental group \( \pi_1(L_3, p) \) to the set of germs of diffeomorphisms \( \text{Diff}(\tilde{\Sigma}, p) \).

We are interested in the holonomy based on the transversals \( \Sigma_+^\varepsilon = \{ \bar{y} = 1 - \varepsilon \} \) and \( \Sigma_-^\varepsilon = \{ \bar{y} = 1 + \varepsilon \} \). To perform explicit computations, we can fix the following generators for \( \pi_1(L_3, 1 - \varepsilon) \):
\[
\delta: [0, 1] \ni \theta \mapsto \bar{y} = (1 - \varepsilon)e^{-2\pi i \theta}, \quad \gamma: [0, 1] \ni \theta \mapsto \bar{y} = 1 + \varepsilon e^{2\pi i \theta}.
\]
Thus, $\delta$ is a closed path of index $-1$ around $s_2$ and $\gamma$ is a closed path of index 1 around $s$. Of course, $\eta = \delta^{-1} \cdot \gamma$ is a closed path of index $-1$ around $s_3$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{paths.png}
\caption{The paths $\delta$, $\gamma$, and $\eta = \delta^{-1} \cdot \gamma$ on the $\bar{y}$ coordinate}
\end{figure}

**Remark 2.3.** Considering the path $\xi : [0, 1] \ni \theta \mapsto \bar{y} = 1 + \varepsilon e^{\pi i \theta}$, we see that $\tilde{\delta} = \xi \cdot \delta \cdot \xi^{-1}$ and $\tilde{\gamma} = \xi \cdot \delta \cdot \xi^{-1}$ are generators of $\pi_1(L_3, 1 + \varepsilon)$. We set $\tilde{\eta} = \tilde{\delta}^{-1} \cdot \tilde{\gamma}$.

Let $\Phi : \pi_1(L_3, 1 - \varepsilon) \to \text{Diff}(\Sigma_\pm \varepsilon, 0)$ be the representation of the holonomy group computed on the transversal $\Sigma_\pm \varepsilon$. Consider the germs of diffeomorphisms $f_+ = \Phi([\delta])$ and $g_+ = \Phi([\eta])$. We define $f_-$, $g_-$ in a similar way by using the transversal $\Sigma^- \varepsilon$ and the paths $\tilde{\delta}$ and $\tilde{\eta}$.

Note that the properties of $\tilde{F}_\nu$ near the saddles $s_2$ and $s_3$ immediately imply $f_+^3 = g_+^3 = \text{id}$. Moreover, the holonomy of $s$ with respect to the leaf $L_3$ is given by $h_\pm = f_+ \circ g_\pm$.

The following theorem relates these germs with the transition maps $T_+$ and $T_-$.  

**Theorem 2.4.** The transition maps $T_+$ and $T_-$ are given by $T_+ = f_+$ and $T_- = g_- \circ f_+ \circ g_-^{-1}$.

**Proof.** We shall give only the main idea of the proof. The details can be easily restored from the very similar proofs in [B] and [BM].

Let us fix a point $x \in \Sigma_\pm \varepsilon \setminus L_3$ (for $* = +, -$). The real integral curve $c_x$ of $\tilde{\omega}$ through $x$ can be presented on the real blow-up space $M \cap \pi^{-1}(\mathbb{R}^2)$ as a curve which wanders around the exceptional divisor $(D_1 \cup D_2 \cup D_3) \cap \pi^{-1}(\mathbb{R}^2)$ as shown in Fig. 6.

We need to interpret these curves as suspensions of closed paths in the leaf $L_3$. For this purpose, we use the pull-back of the fibration $l_\lambda = \{y^2 = \lambda x^3\}$, where $\lambda \in \mathbb{P}^1$, 

\[
\]
under the blow-up $\pi$ and use this fibration to project the paths on $L_3$. We denote such a pull-back by the same symbol $l_\lambda$; it is easy to see from (1) that each fiber $l_\lambda$, except for the fibers $l_0$ and $l_\infty$, cuts $L_3$ transversally.

Therefore, in order to project the curves $c_x$ on $L_3$, we need to modify them so as to keep them away from the fibers $l_0$ and $l_\infty$. This is possible by making small detours to the complex leaf $L_x \subset M$, whose real trace is $c_x$. Such small detours on the leaf $L_3$ are shown in Fig. 7 (the computations are exactly the same as in [BM]).
2.3. Transition maps in normal form coordinates. Following [L], we define a quasi-homogeneous valuation on \( \mathbb{C}[[x, y]] \) by

\[
\deg_{(2,3)}(x^ky^l) = 2k + 3l.
\]

A germ \( g \in \mathbb{C}[[x, y]] \) can be written uniquely as \( g = \sum_{r \geq 1} g_r \), where \( g_r \) is the \( r \)-quasi-homogeneous term of \( g \), i.e., \( \deg_{(2,3)}(g_r) = r \).

For each \( r \geq 1 \), we define the \( r \)-\( \text{quasi-homogeneous jet} \) of a cuspidal differential form \( \omega \) in prenormal form to be the polynomial form

\[
j^r \omega = d(y^2 - x^3) + \sum_{2 \leq i \leq r} g_i(x, y) (3y \, dx - 2x \, dy).
\]

Remark 2.5. It turns out that the equivalence between two differential forms \( \omega_1 \) and \( \omega_2 \) in prenormal form is naturally filtered by such quasi-homogeneous degrees. More precisely, it was proved in [BM] that if \( \omega_1 \) and \( \omega_2 \) are equivalent, then this equivalence can be realized by a change of coordinates \( \phi \) of the form

\[
(X, Y) = (x U(x, y)^2, y U(x, y)^3)
\]

for some germ \( U(x, y) \) such that \( U(0, 0) \neq 0 \). In particular, a simple computation shows that the \( r \)-\( \text{quasi-homogeneous jet} \) of \( \phi \cdot \omega_1 \) depends only on the \( r \)-\( \text{quasi-homogeneous jet} \) of \( \omega_1 \) and the \((r-1)\)-\( \text{quasi-homogeneous jet} \) of \( U(x, y) \).

We say that the \( r \)-jets of two cuspidal differential forms \( \omega_1 \) and \( \omega_2 \) are equivalent if there exists some change of coordinates \( \phi \) of the form special in Remark 2.5 such that \( j^r(\phi \cdot \omega_1) \) is equivalent to \( j^r \omega_2 \).

Proposition 2.6 (see [L]). Let \( \omega = d(y^2 - x^3) + g(x, y) (3y \, dx - 2x \, dy) \), where \( g \) is an analytic germ vanishing at the origin. Then either \( \omega \) is analytically equivalent to \( d(y^2 - x^3) = 0 \) or there exists a unique integer \( n \) such that exactly one of the following cases occur:

(i) \( n \geq 1 \) and the \((6n+1)\)-\( \text{quasi-homogeneous jet} \) of \( \omega \) is equivalent to

\[
\alpha_n = d(y^2 - x^3) + (y^2 - x^3)^n (3y \, dx - 2x \, dy);
\]

(ii) \( n \geq 0 \) and the \((6n+5)\)-\( \text{quasi-homogeneous jet} \) of \( \omega \) is equivalent to

\[
\beta_n = d(y^2 - x^3) + x(y^2 - x^3)^n (3y \, dx - 2x \, dy).
\]

Proof. This is a direct consequence of the normal form theorem of Loray [L] (see also Proposition 4.5 below).

Remark 2.7. In case (ii), note that the term \( x(y^2 - x^3)^n \) has quasi-homogeneous degree \( 6n + 2 \). In this case, the proposition also says that all terms of quasi-homogeneous degrees \( 6n + 3, 6n + 4 \), and \( 6n + 5 \) can be eliminated by a polynomial change of coordinates.

For brevity, we shall denote the cases (i) and (ii) by \( \omega \sim \alpha_n \) and \( \omega \sim \beta_n \), respectively.

Definition 2.8. The \text{codimension} of a cuspidal 1-form \( \omega \) is defined as follows.

1. If \( \omega \sim \alpha_n \), then \( \text{cod}(\omega) = 2n - 1 \), where \( n \geq 1 \);
2. If \( \omega \sim \beta_n \), then \( \text{cod}(\omega) = 2n \), where \( n \geq 0 \);
(3) If \( \omega \) is formally equivalent to \( d(y^2 - x^3) \), then \( \text{cod}(\omega) = \infty \).

We say that the germ is of \( \alpha \)-type in case (1), of \( \beta \)-type in case (2), and of Hamiltonian type in case (3).

The uniqueness statement in Proposition 2.6 implies that the codimension of a cuspidal 1-form is well-defined.

**Remark 2.9.** The above notion of codimension is relative to unfoldings which pre-
serve the cuspidal singularity. That is, the germ \( \omega \) appears generically in unfoldings with \( \text{cod}(\omega) \) parameters. If we consider arbitrary unfoldings, the germ \( \omega \) appears generically in unfoldings with \( \text{cod}(\omega) + 2 \) parameters; see [R2].

The following result is a finite jet version of Theorem III-2 in [LM]. To enumerate it, we need the following definition: Given a natural number \( n \geq 1 \), we denote the subgroup of germs tangent to the identity up to order \( n \) by \( I_n \subset \text{Diff}(\mathbb{C}, 0)/G \); thus,

\[ I_n = \{ f \in G: f'(0) = 1, f^{(i)}(0) = 0 \text{ for } i = 2, \ldots, n \} \]

Clearly, \( I_n \) is a normal subgroup of \( G \), and the quotient \( j^n G = G/I_n \) (whose elements are the right cosets \( I_n f \)) can be identified with the group of \( n \)-jets of elements of \( G \) under the natural operation

\[ j^n f \circ j^n g = j^n (f \circ g) \quad \text{for } f, g \in G. \]

Given a subgroup \( H \subset G \), we denote the subgroup of elements \( j^n h \) for all \( h \in H \) by \( j^n H \subset j^n G \).

**Proposition 2.10.** Let \( \omega \) be the cuspidal differential form \( \omega = d(y^2 - x^3) + g(x, y) (3y \, dx - 2x \, dy) \), and let \( H_\omega \) be the corresponding projective holonomy group. Then

(i) \( \omega \sim \alpha_n \) if and only if the \((6n - 1)\)-jet of the projective holonomy group \( j^{6n-1} H_\omega \) is conjugate to the cyclic group \( (e^{2\pi i / 6}) \) but \( j^{6n} H_\omega \) is not cyclic;

(ii) \( \omega \sim \beta_n \) if and only if the \((6n + 1)\)-jet of the projective holonomy group \( j^{6n+1} H_\omega \) is conjugate to the cyclic group \( (e^{2\pi i / 6}) \) but \( j^{6n+2} H_\omega \) is not cyclic.

**Proof.** In [LM], it was proved that \( H_{\alpha_n} \) and \( H_{\beta_n} \) are always given (up to analytic conjugation) by

\[ H_{\alpha_n} = \left\langle -x, \frac{j x}{(1 + x^{6n-1})^{1/(6n-1)}} \right\rangle \quad \text{and} \quad H_{\beta_n} = \left\langle -x, \frac{j x}{(1 + x^{6n+1})^{1/(6n+1)}} \right\rangle, \]

where \( j = e^{2\pi i / 3} \). Therefore,

- \( j^{6n-1} H_{\alpha_n} = (e^{2\pi i / 6}) \) but \( j^{6n} H_{\alpha_n} \) in not cyclic;
- \( j^{6n+1} H_{\beta_n} = (e^{2\pi i / 6}) \) but \( j^{6n+2} H_{\beta_n} \) in not cyclic.

Thus, it suffices to prove that if \( \omega \sim \alpha_n \) (or \( \sim \beta_n \)), then, up to a polynomial change of coordinates, \( j^{6n} H_\omega = j^{6n} H_{\alpha_n} \) (respectively, \( j^{6n+2} H_\omega = j^{6n+2} H_{\alpha_n} \)).

Suppose that \( \omega \sim \alpha_n \). Then, up to a polynomial change of coordinates, we can assume that \( \omega \) is given by

\[ \omega = \alpha_n + G(x, y) (3y \, dx - 2x \, dy), \]
where \( G(x, y) \) contains only quasi-homogeneous terms of degree strictly larger than \( 6n \). In the blow-up coordinates \( x = \bar{x}^2 \bar{y}, \ y = \bar{x}^3 \bar{y}^2 \), the strict transform of \( \omega \) is given by

\[
\tilde{\omega} = \tilde{\alpha}_n + G(\bar{x}^2 \bar{y}, \bar{x}^3 \bar{y}^2) \, d\bar{y},
\]

where \( \tilde{\alpha}_n \) is the strict transform of \( \alpha_n \). For the quasi-homogeneous expansion \( G = \sum G_r \), where \( \deg_{(2,3)}(G_r) = r \), we have

\[
G_r(\bar{x}^2 \bar{y}, \bar{x}^3 \bar{y}^2) = \bar{x}^r G_r(\bar{y}, \bar{y}^2) \quad \text{for each} \ r \geq 2.
\]

Therefore, for an arbitrary function \( O(\bar{x}^k) \) divisible by \( \bar{x}^k \), we can write

\[
\tilde{\omega} = \tilde{\alpha}_n + O(\bar{x}^{6n+2}) \, d\bar{y}.
\]

Now, take any transversal \( \tilde{\Sigma} = \{ \tilde{y} = \text{const} \} \) to the leaf \( L_3 \) parameterized by \( \tilde{x} \) and let \( \Phi_{\alpha_n}(\tilde{\delta}) \) and \( \Phi_{\alpha_n}(\tilde{\gamma}) \) (\( \Phi_{\omega}(\tilde{\delta}) \) and \( \Phi_{\omega}(\tilde{\gamma}) \)) be generators of the holonomy group of \( H_{\alpha_n} \) (respectively, \( H_{\omega} \)). An elementary computation shows that

\[
\Phi_{\omega}(\tilde{\delta}) = \Phi_{\alpha_n}(\tilde{\delta}) + O(\bar{x}^{6n+2}), \quad \Phi_{\omega}(\tilde{\gamma}) = \Phi_{\alpha_n}(\tilde{\gamma}) + O(\bar{x}^{6n+2}),
\]

and therefore \( j^{6n} H_{\omega} = j^{6n} H_{\alpha_n} \). The proof in the case \( \omega \sim \beta_n \) is similar. \( \square \)

### 2.4. Normal form coordinates.

Our next goal is to compute the transition maps \( T_+ \) and \( T_- \) in normal form coordinates at the saddle \( s \). We define normal form coordinates as follows. For each \( \nu \in \mathbb{N} \), a normal form coordinate (of order \( \nu \)) for \( \tilde{\omega} \) at \( s \) is a change of coordinates of the form

\[
(X, Y) = (\bar{x}(1 + P(\bar{x}, 1 - \tilde{y})), 1 - \tilde{y})
\]

centered at \( s \) (where \( P \) is an analytic germ vanishing at the origin), and such that \( \tilde{\omega} \) is equivalent to

\[
6Y \, dX + X \left[ 1 + \sum_{i=1}^{\nu} \rho_i u^i + u^{\nu+1} R(X, Y) \right] \, dY, \quad \text{where} \ u = X^6 Y,
\]

for some real numbers \( \rho_1, \ldots, \rho_\nu \) and an analytic germ \( R(X, Y) \). A theorem of Du- lac [D] assures that for each \( \nu \) there always exists an analytic change of coordinates which satisfies the above requirements.

**Remark 2.11.** In general, the normal form coordinates are not uniquely determined. However, the order of the first non-vanishing nonlinear term \( \rho_i u^i \) in the above expansion is a formal invariant of the differential form \( \omega \) (see, e.g., [MR]).

Using this remark, we say that the saddle point \( s \) is linear up to order \( k \in \mathbb{N} \) if \( \rho_1 = \cdots = \rho_k = 0 \).

**Proposition 2.12.** Suppose that a cuspidal 1-form \( \omega \) is such that either \( \omega \sim \alpha_n \) or \( \omega \sim \beta_n \). Let \( (X, Y) \) be arbitrary normal form coordinates of sufficiently high order at \( s \). Then, in these coordinates, \( \tilde{\omega} \) is linear up to order \( n \).

**Proof.** By Remark 2.11, it suffices to prove the existence of one normal form coordinate in which \( \tilde{\omega} \) is linear up to order \( n \).

Suppose that \( \omega \sim \alpha_n \). Then the strict transform of \( \omega \) is given by

\[
\tilde{\omega} = \tilde{\alpha}_n + O(\bar{x}^{6n+2}) \, d\bar{y}
\]
(see the proof of Proposition 2.10). According to [LM], the holonomy $h_{+,\bar{\alpha}} : \Sigma^+_\varepsilon \rightarrow \Sigma^+_\varepsilon$ of $\bar{\alpha}$ at $s$ is periodic (in fact, its 6th power is the identity, because the projective holonomy $H_{\bar{\alpha}}$ is solvable). Hence $s$ is a linearizable singular point of $\alpha$, and there exists a local analytic change of coordinates of the form
\[
(\bar{X}, \bar{Y}) = (\bar{x}(1 + P(\bar{x}, 1 - \bar{y})), 1 - \bar{y}), \quad \text{with } P(0, 0) = 0,
\]
such that $\alpha$ is equivalent to $6\bar{Y}d\bar{X} + \bar{X}d\bar{Y}$. In these coordinates, $\tilde{\omega}$ is equivalent to $6\bar{Y}d\bar{X} + \bar{X}d\bar{Y} + O(\bar{X}^{6n+2})d\bar{Y}$.

It follows from Dulac’s construction [D] that we can obtain normal form coordinates $(X, Y)$ for $\tilde{\omega}$ such that $X = \bar{X}(1 + Q_1)$ and $Y = \bar{Y}$, where $Q_1$ is an analytic function of multiplicity larger than $6n + 2$. We conclude that $\rho_1 = \cdots = \rho_n = 0$.

In the case $\omega \sim \beta$, we have $\tilde{\omega} = \tilde{\beta} + O(\bar{x}^{5n+6})d\bar{y}$. Considering linearizing coordinates $(\bar{X}, \bar{Y})$ for $\tilde{\beta}$ at $s$, we see that $\tilde{\omega}$ is equivalent to $6\bar{Y}d\bar{X} + \bar{X}d\bar{Y} + O(\bar{X}^{6n+6})d\bar{Y}$. The same arguments as above implies the existence of normal form coordinates for $\tilde{\omega}$ such that $\rho_1 = \cdots = \rho_n = 0$.

It is immediate to deduce the following result.

Corollary 2.13. Let $(X, Y)$ be normal form coordinates (of sufficiently high order) at $s$. Then the holonomy map $h_{+,\bar{\alpha}} : \Sigma^+_\varepsilon \rightarrow \Sigma^+_\varepsilon$ of $\omega$ computed in these coordinates is given by
\[
h_{+}(X) = e^{-2\pi i/6}X + O(X^m),
\]
where $m = 6n + 2$ if $\omega \sim \alpha$ and $m = 6n + 6$ if $\omega \sim \beta$.

Now, we are ready to compute the asymptotic expansion of the regular transition maps $T_+$ and $T_-$. 

Theorem 2.14. For the expression of the transition map $T_+$ in normal form coordinates $(X, Y)$ (of sufficiently high order) at $s$, the following cases can occur.

(i) If $\omega \sim \alpha$, then $T_+$ has the asymptotic expansion
\[
T_+ (X) = -X + A_n(\varepsilon)X^{6n} + O(X^{6n+1}),
\]
where $A_n(\varepsilon) = A_n e^{\varepsilon - 1/6} + O(\varepsilon)$ for some constant $A_n \neq 0$;

(ii) If $\omega \sim \beta$, then $T_+$ has the asymptotic expansion
\[
T_+(X) = -X + B_n(\varepsilon)X^{6n+2} + O(X^{6n+3}),
\]
where $B_n(\varepsilon) = B_n e^{\varepsilon 1/6} + O(\varepsilon)$ for some constant $B_n \neq 0$.

Proof. Suppose that $\omega \sim \alpha$ (the proof in the case $\omega \sim \beta$ is similar). Then, according to Proposition 2.10, $j^{6n-1}H_{\omega}$ is cyclic but $j^{6n}H_{\omega}$ is not cyclic.

Since $H_{\omega}$ is generated by the germs $f$ and $h$, $j^kH_{\omega}$ is generated by $j^k f$ and $j^k h$ for each $k \in \mathbb{N}$. On the other hand, if $(X, Y)$ are normal form coordinates, we have
\[
j^{6n}h_+(X) = e^{-2\pi i/6}X \text{ by Corollary 2.13. Therefore, } j^{6n}T_+ = j^{6n}f_+ \text{ is given by}
\]
\[
j^{6n}f_+(X) = -X + A_n(\varepsilon)X^{6n}
\]
for some function $A_n(\varepsilon) \in \mathbb{R}$, which is non-zero for all small $\varepsilon > 0$ (because otherwise $j^{6n}H_{\omega}$ would be cyclic).
It remains to prove that $A_n(\varepsilon) = A_n \varepsilon^{n-1/6} + O(\varepsilon)$. Take two sufficiently small reals $\varepsilon, \tilde{\varepsilon} > 0$. The expression for $f_+$ computed in the section $\Sigma_\varepsilon^+$ can be obtained from the expression of $f_+$ in the section $\Sigma^{\tilde{\varepsilon}}_+$ by conjugation by the mapping

$$r_{\varepsilon, \tilde{\varepsilon}}: \Sigma^{\tilde{\varepsilon}}_+ \to \Sigma^{\varepsilon}_+,$$

which is the suspension of the path $\zeta_{\varepsilon, \tilde{\varepsilon}}: [1 - \varepsilon, 1 - \tilde{\varepsilon}] \ni t \mapsto \bar{y} = t$ contained in $L_3$. For the special case of the normal form coordinates $(X, Y)$ used in the proof of Proposition 2.12, we, obviously, have

$$r_{\varepsilon, \tilde{\varepsilon}}(X) = (\varepsilon / \tilde{\varepsilon})^{1/6} X + O(X^{6n+1}).$$

For a general normal form coordinate, a similar (but cumbersome) computation gives the desired result. □

Now, let us compute the asymptotic expansion of $T_-$. Since $g = f^{-1} \circ h$, we can write

$$T_- = f^{-1} \circ h_- \circ f_- \circ h_-^{-1} \circ f_-.$$  \hspace{1cm} (2)

Using the coordinates of the normal form $(X, Y)$, we obtain

$$h_-(X) = e^{-2\pi i / 6} X + O(X^m),$$

where $m = 6n + 2$ if $\omega \sim \alpha_n$ and $m = 6n + 6$ if $\omega \sim \beta_n$. Moreover, the map $f_-(X)$ can be obtained from the map $f_+(X)$ by conjugation by the half-holonomy map $R: \Sigma^{\tilde{\varepsilon}}_+ \to \Sigma^{\varepsilon}_+$ given by

$$R: X \mapsto e^{-\pi i / 6} X + O(X^m),$$

where $m$ is the same as above.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{half-holonomy.png}
\caption{The half-holonomy map}
\end{figure}

Therefore, using the expression for $f_+(X) = T_+(X)$ given by Theorem 2.14 and formula (2) and performing simple computations, we obtain the following result:

- If $\omega \sim \alpha_n$, then $T_-$ has the asymptotic expansion
  $$T_-(X) = -X + (-1)^n \sqrt{3} \alpha_n \varepsilon X^{6n} + O(X^{6n+1});$$

- If $\omega \sim \beta_n$, then $T_-$ has the asymptotic expansion
  $$T_-(X) = -X + (-1)^n \sqrt{3} \beta_n \varepsilon X^{6n+2} + O(X^{6n+3}).$$
Remark 2.15. It is easy to see that for the generic Bogdanov–Takens singularity

$$\omega = d(y^2 - x^3) + (x + O(y, x^2)) (3y dx - 2x dy),$$

we have \( \omega \sim \beta_0 \). Using MAPLE, we can explicitly compute the first coefficient of \( B_0(\varepsilon) \):

$$B_0 = \frac{2(\sqrt{\pi} \Gamma\left[\frac{5}{6}\right])}{3\Gamma\left[\frac{2}{3}\right]} = 1.493668400444373626 \ldots ,$$

where \( \Gamma[\cdot] \) is the Gamma function. From this, we conclude that the transition maps \( T_+ \) and \( T_- \) are given by

$$T_+(X) = -X + (B_0 \varepsilon^{-1/6} + O(\varepsilon)) X^2 + O(X^3),$$

$$T_-(X) = -X + \sqrt{3}(B_0 \varepsilon^{-1/6} + O(\varepsilon)) X^2 + O(X^3).$$

Now, consider the local Dulac maps \( D_{\pm,i} : \Gamma_i \to \Sigma_{\pm,i} \) (see Figs. 3 and 9).

Lemma 2.16. Let \( (X, Y) \) be normal form coordinates of sufficiently high order at \( s \). Then, in these coordinates, the local Dulac maps \( D_{\pm,i} \) have the asymptotic expansion

$$D_{\pm,i}(Y) = (-1)^{i+1} \varepsilon^{-1} |Y| + O(|Y|^{n+2} \ln |Y|))^{1/6},$$

where \( O(|Y|^{n+2} \ln |Y|) \) denotes some function which approaches zero faster than \( |Y|^{n+2} \ln |Y| \), uniformly in \( \varepsilon \).

Proof. The lemma is proved by a straightforward computation using Proposition 2.12 (see [R3]). \( \square \)

![Figure 9. The local Dulac maps](image)

Finally, combining Lemma 2.16 with the asymptotic expansion for the maps \( T_+ \) and \( T_- \), we obtain the following theorem.

Theorem 2.17. Let \( \omega \) be an analytic cuspidal 1-form. Then the Dulac maps \( D_+ : \Gamma^+_1 \to \Gamma^+_2 \) and \( D_- : \Gamma^-_1 \to \Gamma^-_2 \) have the following asymptotic expansion

(i) If \( \omega \sim \alpha_n \), then \( D_+(Y) = Y - 6A_n Y^{n+5/6} + o(Y^{n+5/6}) \) and

$$D_-(Y) = Y + (-1)^n 6\sqrt{3} A_n |Y|^{n+5/6} + o(|Y|^{n+5/6});$$
(ii) If \( \omega \sim \beta_n \), then
\[
D_+(Y) = Y - 6B_n Y^{n+7/6} + o(Y^{n+7/6}) \quad \text{and} \quad
D_-(Y) = Y + (-1)^n 6\sqrt{3} B_n |Y|^{n+7/6} + o(|Y|^{n+7/6}),
\]
where \( A_n \) and \( B_n \) are the nonzero constants introduced in Theorem 2.14.

As mentioned at the beginning of this section, the same result for smooth cuspidal 1-forms follows immediately.

**Corollary 2.18.** The Theorem 2.17 is also valid for smooth cuspidal 1-forms.

**Proof.** We can find local smooth coordinates such that \( \omega \) is equivalent to
\[
d(y^2 - x^3) + g(x, y)(3y \, dx - 2x \, dy),
\]
where \( g \) is a smooth function. Now, it suffices to note that each initial segment of the asymptotic expansion of the Dulac maps \( D_\pm \) depends only on the quasi-homogeneous jet of \( g \) of finite order. \( \Box \)

### 3. Asymptotic Scales of Functions and Cyclicity

In Sections 3.1–3.3, we study functions \( f(u) \) and unfoldings of functions \( f(u, \lambda) \) for \( u \in [0, U] \) (with some fixed \( U > 0 \)) and \( \lambda \in (\mathbb{R}^p, 0) \). We consider the expansion of such unfoldings on Chebyshev asymptotic scales and study their cyclicity and bifurcation diagrams.

In the last subsection, we consider unfoldings, Chebyshev asymptotic scales, and generalized Chebyshev asymptotic scales defined on intervals of the form \((-U, U)\). We show how to reduce their study to the above situation.

We assume that the functions under consideration are \( C^\infty \) for \( u \neq 0 \).

#### 3.1. Expansion in asymptotic scales.

**Definition 3.1.** An asymptotic scale of functions is a collection \( \mathcal{F} = \{f_i\}_{i \in \mathbb{N}} \) of continuous functions \( f_i: [0, U) \to \mathbb{R} \), where \( U > 0 \), such that
\[
f_0 \equiv 1, \quad f_1 \text{ is strictly monotonic on } (0, U), \quad f_{i+1}(u)/f_i(u) \to 0 \text{ as } u \to 0^+ \text{ (we denote this by } f_{i+1} \succ f_i).\]

**Definition 3.2.** A Chebyshev asymptotic scale of functions is an asymptotic scale of functions \( \mathcal{F} = \{f_i\}_{i \in \mathbb{N}} \) satisfying the following conditions.

1. Each \( \mathcal{F}_j = \{f_{i,j}^0, f_{i,j}^1, \ldots \} \) is defined by the recursive formula
   \[
f_{i+1}^j(u) = \frac{(f_i^j)'(u)}{(f_i^j)'} \quad \text{for } u \in (0, u_{j+1}), \quad i, j \geq 1,
   \]

   where \( h' = \frac{dh}{du} \) (we suppose that \( (f_i^j)'(u) \neq 0 \) for \( u \in (0, u_{j+1}) \));

2. Each sequence \( \mathcal{F}_j \) is an asymptotic scale of functions.

**Examples.** (1) The Taylor scale of functions \( \mathcal{T} = \{1, u, u^2, \ldots \} \).

(2) \( \mathcal{L}_S = \{u^r \log^p u\} \), where \( S = \{(r_i, p_i)\}_{i \in \mathbb{N}} \) is a sequence in \( \mathbb{R}^2 \) such that \( r_0 = p_0 = 0 \) and either (a) \( r_i < r_{i+1} \) or (b) \( r_i = r_{i+1} \) and \( p_i > p_{i+1} \) for all \( i \geq 0 \).

The next definition extends the usual notion of expansion in a given asymptotic scale.
Definition 3.3. We say that a function \( f: [0, U) \to \mathbb{R} \) is expandable at order \( N \) in a Chebyshev asymptotic scale \( \mathcal{F} \) if there exists a sequence of real numbers \( \{\alpha_i\}_{i \in \mathbb{N}} \) and a \( U_N > 0 \) such that
\[
f(u) = \sum_{i=0}^{N} \alpha_i f_i(u) + \phi_N(u) \quad \text{for} \ u \in [0, U_N),
\]
where \( \phi_N(u) \) is a remainder such that
\[
\phi_0^N = \phi_N, \quad \phi_{j+1}^N = \frac{(\phi_j^N)'}{(f_j^N)'} \quad \text{for} \ j = 0, \ldots, N - 1,
\]
and the following condition \( (R_N) \) holds: \( \phi_j^N > f_{N-j}^j \) for \( j = 0, \ldots, N \). We say that \( f(u) \) is expandable in \( \mathcal{F} \) if it is expandable at all orders \( N \in \mathbb{N} \).

Definition 3.4. If \( f(u) \) is expandable in some asymptotic scale \( \mathcal{F} \), we say that \( f \) has codimension \( n \) in the scale \( \mathcal{F} \) if \( \alpha_i = 0 \) for all \( 0 \leq i \leq n - 1 \) and \( \alpha_n \neq 0 \).

The following definition introduces the more general notion of expandable unfoldings.

Definition 3.5. We say that an unfolding \( f(u, \lambda) \) is expandable at order \( N \) if it is expandable for each fixed \( \lambda \in \Lambda_N \), where \( \Lambda_N \) is a neighbourhood of the origin. Moreover, we suppose that the coefficients \( \alpha_i(\lambda) \) are \( \mathcal{C}^\infty \) and the remainder \( \phi_N(u, \lambda) \) satisfies condition \( (R_N) \) uniformly in \( \lambda \).

Remark 3.6. If a function (or unfolding) is expandable in some Chebyshev asymptotic scale, then the coefficients (or the germ of coefficients) \( \alpha_i \) are determined uniquely and do not depend on \( N \).

3.2. The \( \mathcal{N} \)-division property.

Definition 3.7. Let \( f(u, \lambda) \) be an unfolding expandable in a Chebyshev asymptotic scale \( \mathcal{F} = \{f_0, f_1, \ldots\} \), and let \( \mathcal{N} = \{n_0 < n_1 < \cdots < n_k\} \) be a sequence of natural numbers. We say that \( f(u, \lambda) \) has the \( \mathcal{N} \)-division property in \( \mathcal{F} \) if there exists a sequence of unfoldings \( g_0(u, \lambda), \ldots, g_k(u, \lambda) \) such that each \( g_i \) is expandable in \( \mathcal{F} \) with expansion at order \( n_i \) given by
\[
g_i(u, \lambda) = f_{n_i}(u) + o(f_{n_i})
\]
and there are \( \mathcal{C}^\infty \) functions \( \beta_0(\lambda), \ldots, \beta_k(\lambda) \) defined on some neighbourhood \( \Lambda \) of the origin in the parameter space such that \( \beta_0(0) = \cdots = \beta_{k-1}(0) = 0 \) and
\[
f(u, \lambda) = \sum_{i=1}^{k} \beta_i(\lambda) g_i(u, \lambda)
\]
for \( (u, \lambda) \in [0, U) \times \Lambda \) and some \( U > 0 \). We say that the cardinality \( k \) of the sequence \( \mathcal{N} \) is the length of the division.

Observe that if \( f(u, \lambda) \) has \( \mathcal{N} \)-division property, then it is necessarily expandable in \( \mathcal{F} \).
Remark 3.8. Note that if \( f(u, \lambda) \) has an expansion of the form (3) with \( g_i \) as above but \( \beta_i(0) \neq 0 \) for some \( 0 \leq l \leq k - 1 \), then we can divide \( f(u, \lambda) \) by the functions \( g_0, \ldots, g_l \).

Definition 3.9. Suppose that an unfolding \( f(u, \lambda) \) has the \( \mathfrak{R} \)-division property (3). We say that the division is nondegenerate if \( \beta_k(0) \neq 0 \). We say that the division is generic if it is nondegenerate and the mapping

\[ \lambda \mapsto (\beta_0(\lambda), \ldots, \beta_{k-1}(\lambda)) \]

is a local diffeomorphism at \( \lambda = 0 \). In particular, \( k = p \).

Note that the division is degenerate if and only if \( f(u, 0) \equiv 0 \).

Definition 3.10. Let \( \beta = \{\beta_0(\lambda), \ldots, \beta_{k-1}(\lambda)\} \) be a collection of elements in the ring \( C^\infty(\lambda) \) of \( C^\infty \) germs at \( (\mathbb{R}^p, 0) \). We say that \( \beta \) has the radicality property if, for each \( 0 \leq i \leq k - 1 \), the ideal generated by \( \beta_0, \ldots, \beta_i \) is radical. This means the following. For each \( 0 \leq i \leq k - 1 \), let \( Z_i \) be the germ of \( Z_i = \{\beta_0(\lambda) = \cdots = \beta_i(\lambda) = 0\} \). If \( \alpha \in C^\infty(\lambda) \) is such that \( \alpha|_{Z_i} \equiv 0 \), then there exist \( \xi_0, \ldots, \xi_i \in C^\infty(\lambda) \) such that \( \alpha = \sum_{l=0}^{i} \xi_l \beta_l \).

Example. Suppose that \( \beta_0, \ldots, \beta_{k-1} \) are such that \( d\beta_0(0) \wedge \cdots \wedge d\beta_{k-1}(0) \neq 0 \). Then \( \beta = \{\beta_0, \ldots, \beta_{k-1}\} \) has the radicality property.

Proposition 3.11. Let \( \mathfrak{R} = \{n_0 < n_1 < \cdots < n_k\} \) be a sequence in \( \mathbb{N} \), and let \( \{\beta_0, \ldots, \beta_k\} \) be a collection of elements in \( C^\infty(\lambda) \) such that \( \{\beta_0, \ldots, \beta_{k-1}\} \) has the radicality property and \( \beta_k(0) \neq 0 \). Suppose that \( f(u, \lambda) \) is an unfolding expandable in an asymptotic scale \( \mathfrak{S} = \{f_0, f_1, \ldots\} \). Then the following two conditions are equivalent:

1. \( f(u, \lambda) = \beta_0(\lambda)f_{n_0}(u) + o(f_{n_0}(u)) \) for all \( \lambda \) and, for each \( 0 \leq i \leq k - 1 \),
   \[
   f(u, \lambda) = \beta_{i+1}(\lambda)f_{n_{i+1}}(u) + o(f_{n_{i+1}}(u))
   
   \text{if } \lambda \text{ is in } Z_i = \{\beta_0(\lambda) = \cdots = \beta_i(\lambda) = 0\};
   
2. The unfolding \( f(u, \lambda) \) has an \( \mathfrak{R} \)-division in \( \mathfrak{S} \) of the form (3) with the coefficients \( \beta_i(\lambda) \) specified above.

Remark 3.12. In Proposition 3.11 and in what follows, the notation \( g(u, \lambda) = o(f(u)) \) means that the limit \( \lim_{u \to 0} g(u, \lambda)/f(u) = 0 \) is uniform in \( \lambda \).

Proof. The implication (2) \( \Rightarrow \) (1) is trivial.

Let us prove (1) \( \Rightarrow \) (2). Let us write the expansion of \( f(u, \lambda) \) at order \( n_k \):

\[
 f(u, \lambda) = \alpha_0(\lambda)f_{n_0}(u) + \cdots + \alpha_{n_k}(\lambda)f_{n_k}(u) + \phi_{n_k}(u, \lambda).
\]

From now on, we do not indicate the dependence on \( u \) and \( \lambda \) to simplify the notation. The assumption of the proposition and uniqueness of the expansion imply \( \alpha_0 \equiv \cdots \equiv \alpha_{n_k-1} \equiv 0 \) and \( \alpha_{n_k} \equiv \beta_0 \). Therefore,

\[
 f(u, \lambda) = \beta_0 f_{n_0} + \alpha_{n_0+1} f_{n_0+1} + \cdots + \alpha_{n_k} f_{n_k} + \phi_{n_k}(u, \lambda).
\]  

(4)

We shall prove the division formula (3) by induction. Suppose that, for some \( 0 \leq i \leq k \),

\[
 f(u, \lambda) = \beta_0 g_0 + \beta_1 g_1 + \cdots + \beta_i g_i + \phi_i(u, \lambda),
\]

(5)
where $g^i_j = (f_{n_j} + o(f_{n_j}))$ for each $0 \leq j \leq i$ and $\phi_n(u, \lambda) = o(f_{n+1})$. Note that (4) gives this formula for $i = 0$, and if we can prove it for $i = k$, then we can easily obtain the division formula (3), because $\beta_k(0) \neq 0$ and $\beta_k f_{n_k} + o(f_{n_k}) = \beta_k(f_{n_k} + \cdots)$.

Suppose that (5) holds for some $0 < i < k - 1$. Since $f(u, \lambda)$ is expandable in $\mathcal{G}$, we can write the last term $\phi_n(u, \lambda)$ as
\[
\phi_n(u, \lambda) = \alpha_{n+1} f_{n+1} + \cdots + \alpha_{n+i+1} f_{n+i+1} + o(f_{n+1}).
\]

Take $\lambda \in \mathbb{Z}_i$. We have $f(u, \lambda) = \phi_n(u, \lambda)$ and, by assumption, $f(u, \lambda) = \beta_{i+1} f_{n+i+1} + o(f_{n+i+1})$. The uniqueness of the expansion implies $\alpha_{n+1} = \cdots = \alpha_{n+i+1}(\lambda) = 0$ and $\alpha_i(\lambda) = \beta_{i+1}(\lambda)$.

The radicality property of $\{\beta_0, \ldots, \beta_{i-1}\}$ implies that, for each $j \in \{n_i + 1, \ldots, n_{i+1} - 1\}$, $\alpha_j = \sum_{k=0}^i \xi_k^j \beta_k$ and $\alpha_{n+i+1} = \beta_{i+1} + \sum_{k=0}^i \xi_{n+i+1}^j \beta_k$ for some collection of germs $\{\xi_k^j\} \subset \mathcal{C}^\infty(\lambda)$.

Now, let us rearrange the terms in (5). For each $0 \leq k \leq i$, we define $g^i_k$ by adding all terms of the form $\xi_k^j f_j$ to $g^i_k$. Note that all these new terms are $o(f_{n_k})$ and do not change the principal part. Therefore, we obtain the expansion
\[
f(u, \lambda) = \beta_0 g^i_0 + \beta_1 g^i_1 + \cdots + \beta_{i+1} g^i_{i+1} + \phi_{n+1}(u, \lambda),
\]
where $\phi_{n+1} = o(f_{n+1+i})$. This gives (5) for $i + 1$. \hfill \Box

### 3.3. Estimation of cyclicity and bifurcation diagrams.

**Definition 3.13.** Let $f = f(u, \lambda)$ be an unfolding defined on some interval of the form $[0, U)$ or $(-U, U)$. Given $0 < \delta < U$, we define $N_N(\delta)$ to be the number of isolated roots of the function $u \to f(u, \lambda)$ for $0 < |u| < \delta$. Given $\eta > 0$, let $N(\delta, \eta) = \sup\{N_N(\delta) \mid \lambda < \eta\}$. The cyclicity of $f$ is defined by
\[
\text{Cycl}(f) = \lim_{(\delta, \eta) \to (0, 0)} N(\delta, \eta).
\]

The following result gives an estimate for cyclicity in terms of the length of the division of the unfolding $f$.

**Proposition 3.14.** Let $\mathcal{G}$ be a Chebyshev asymptotic scale, and let $f = f(u, \lambda)$ be an unfolding expandable in this scale. Suppose that $f$ has an $\mathfrak{R}$-division of length $k$ in $\mathcal{G}$. Then $\text{Cycl}(f) \leq k$.

**Proof.** This assertion is proved by a simple application of the derivation-division algorithm as described in [R3]. We start from the expression
\[
f(u, \lambda) = \sum_{i=0}^k \beta_i(\lambda) g_i(u, \lambda),
\]
where $\mathfrak{R} = \{n_0 < \cdots < n_k\}$ and $g_i(u, \lambda) = f_{n_i}(u) + o(f_{n_k})$.

Let us define a sequence of unfoldings $F^0, F^1, \ldots, F^k$ as
\[
F^0 = f, \quad F^{j+1} = \left(\frac{F^j}{F^j_i}\right)_{i=0}^{j-1} \quad \text{for } 0 < j < k - 1,
\]
where $F^j(u, \lambda) = \sum_{i=0}^{k-j} \beta_{i+j}(\lambda) F_i^j(u, \lambda)$. It follows from the properties of $\mathcal{G}$ that
\[
1 \equiv F^0_0 < F^1_1 < \cdots < F^j_{k-j} \quad \text{for all } 0 \leq j \leq k.
\]
In particular, $F^k(u, \lambda) = \beta_k(\lambda)(1 + o(1))$ is nonzero. Applying Rolle’s theorem $k$ times, we conclude that $\text{Cycl}(f) \leq k$.

**Definition 3.15.** We say that two unfoldings $f(u, \lambda)$ and $g(v, \mu)$, where $u, v \in (\mathbb{R}^+, 0)$ and $\lambda, \mu \in (\mathbb{R}^p, 0)$ are **topologically contact equivalent** if there exist representative of these unfoldings for $(u, \lambda) \in [0, U) \times \Lambda$ and $(v, \mu) \in [0, V) \times M$ and a homeomorphism $\Phi : \Lambda \rightarrow M$ (with $\Phi(0) = 0$) such that, for each $\lambda \in \Lambda$, there exists a homeomorphism $\varphi_\lambda : [0, U) \rightarrow [0, V)$ (with $\varphi_\lambda(0) = 0$) which maps bijectively the roots of $f(u, \lambda)$ to the roots of $g(v, \Phi(\lambda))$ and preserves sign in the intervals between roots. The roots are counted with multiplicity outside $u = 0$.

Let $\mathfrak{F} = \{f_0, f_1, \ldots\}$ be a Chebyshev asymptotic scale, and let $\mathfrak{N} = \{n_0 < \cdots < n_k\}$ be a sequence of natural numbers. It is well known [M] that any finite unfolding

$$f(u, \alpha) = \alpha_0f_{n_0}(u) + \cdots + \alpha_kf_{n_{k-1}}(u) + f_{n_k}(u)$$

is topologically contact equivalent to the versal polynomial unfolding

$$P_{\pm,k}(u, \beta) = \beta_0 + \beta_1u + \cdots + \beta_{k-1}u^{k-1} \pm u^k,$$

where $\pm$ is the sign of $f_{n_k}(u)$. This result can be generalized to unfoldings with generic division of length $k$.

**Proposition 3.16.** Let $f = f(u, \lambda)$ be an unfolding with a generic division of length $k$. Then $f$ is topologically contact equivalent to the versal polynomial unfolding $P_{\pm,k}(u, \lambda)$, where $\varepsilon = \text{sign}(\beta_k(0))$.

**Proof.** We give only a sketch of the proof. Up to a diffeomorphism in the parameters and division by a positive function of the parameters, we can suppose that $f$ has the expansion

$$f(u, \lambda) = \lambda_0g_0(u, \lambda) + \cdots + \lambda_{k-1}g_{k-1}(u, \lambda) \pm g_k(u, \lambda),$$

where $g_i(u, \lambda) = f_{n_i}(1 + G_i(u, \lambda))$ with $G_i$ expandable in a Chebyshev asymptotic scale for each $0 \leq i \leq k$.

Given an unfolding $h(u, \lambda)$ expandable in some Chebyshev asymptotic scale $\mathfrak{G}$, let $D_jh(u, \lambda)$ denote the unfolding obtained after $j$ steps of the derivation-division procedure (on the scale $\mathfrak{G}$). As a consequence of the properties of $\mathfrak{G}$, we have

$$\frac{\partial}{\partial \lambda_l} D_jG_i = o_u(1)$$

for any $0 \leq i \leq l - 1$ and $j \geq 0$,

where $o_u(1)$ denotes a function which goes to zero as $u$ goes to zero uniformly in $\lambda$.

In the case $G_0 \equiv \cdots \equiv G_k \equiv 0$, we have the affine system (6). For this system, each stratum dimension $d$ in the bifurcation diagram $\Sigma_0$ has a parametrization $\lambda = S(u, \mu)$, where $\mu \in (\mathbb{R}^{d-1}, 0)$.

For the general system (7), each stratum of the bifurcation diagram $\Sigma$ has an equation of the form

$$\lambda = S(u, \lambda, \mu),$$

where $S(u, 0, \mu) = s(u, \mu)$. Moreover, each partial derivative $\partial S/\partial \lambda_l$ has a polynomial expression without constant term in terms of the partial derivatives (8).

As a consequence, each $\partial S/\partial \lambda_l$ is $o_u(1)$, and the implicit function theorem gives
parametrizations $\lambda = \bar{s}(u, \mu)$ for the strata of the bifurcation diagram. It is possible to estimate the distance between $\Sigma$ and $\Sigma_0$ and prove that there exists an homeomorphism which gives the topological contact equivalence between the unfoldings (7) and (6). Consequently, we obtain the topological contact equivalence between (7) and the polynomial unfolding $P_{c,k}$.

### 3.4. Bilateral unfoldings.

In this section, we consider bilateral unfoldings $f(u, \lambda)$, where $u$ belongs to an open interval $(-U, U)$, for some $U > 0$ and $\lambda \in (\mathbb{R}^l, 0)$. We use the same Definition 3.2 but allow $u$ to range over $(-U, U)$.

**Definition 3.17.** A generalized Chebyshev asymptotic scale of functions is an asymptotic scale of functions $\mathcal{F} = \{f_i\}_{i \in \mathbb{N}}$ defined on $(-U, U)$ such that $\{f_i(u)\}_{|u| < U}$ and $\{f_i(-u)\}_{|u| < U}$ are Chebyshev asymptotic scales of functions in the sense of Definition 3.2.

For a function $f_i$ in such a generalized Chebyshev scale, there are two possibilities: first, it can be strictly monotonic on $(-U, U)$; in this case, we say that it is of parity $I$ (for the French “impaire”); secondly, it can be non-monotonic and have precisely one extremum at 0; in this case, we say that it is of parity $P$ (for the French “paire”). To each generalized Chebyshev asymptotic scale $\mathcal{F}$ we associate a sequence of parities $P(\mathcal{F}) \in \{I, P\}^\mathbb{N}$, always beginning with the parity $P$.

The derivation-division procedure shifts this sequence and changes or preserves all the parities.

The (bilateral) Chebyshev asymptotic scales of functions introduced in [M] are a special case of the generalized Chebyshev asymptotic scales of functions defined above, where only the alternating sequence of parities $\{P, I, P, I, P, \ldots\}$ is allowed. Note that this property is preserved by the division-derivation procedure. An equivalent definition is obtained when all the functions $f_i^1$ are required to be of type $I$. It was shown in [M] that Propositions 3.14 and 3.16 remain valid for the bifurcation diagrams of an unfolding $f$ which expand in a (bilateral) Chebyshev asymptotic scale, but it is not clear if they remain valid for expansions in generalized Chebyshev asymptotic scales. In this section, we prove assertions that can be used instead of Propositions 3.14 and 3.16. First of all, any generalized Chebyshev asymptotic scale admits a minimal embedding into some Chebyshev asymptotic scale.

**Lemma 3.18.** Any generalized Chebyshev asymptotic scale $\mathcal{F}$ embeds in some Chebyshev asymptotic scale $\mathcal{G}$. This scale $\mathcal{G}$ can be chosen minimal in the sense that if $\mathcal{G}'$ is another Chebyshev asymptotic scale such that $\mathcal{F} \subset \mathcal{G}' \subset \mathcal{G}$, then $\mathcal{G}' = \mathcal{G}$.

**Proof.** Let $f_i$ and $f_{i+1}$ be two consecutive functions in $\mathcal{F}$ with the same parity. Consider the function $f_{i(i,i+1)}$ equal to $\sqrt{|f_i f_{i+1}|}$ for $u > 0$ and equal to $\pm \sqrt{|f_i f_{i+1}|}$ for $u < 0$. The sign $\pm$ is chosen so that the parity of $f_{i(i,i+1)}$ is inverse to that of $f_i$ and $f_{i+1}$. Note that the relations $f_i \prec f_{i(i,i+1)} \prec f_{i+1}$ and $f \prec g$ for two functions defined in a neighbourhood of 0 mean that $f(u) \to 0$ as $u \to 0$. A minimal Chebyshev asymptotic scale $\mathcal{G}$ is obtained by adding all these functions $f_{i(i,i+1)}$ to the scale $\mathcal{F}$.

This lemma allows us to give the following definition.
Definition 3.19. Let $\mathfrak{F} = \{f_0, f_1, \ldots \}$ be a generalized Chebyshev asymptotic scale. The effective order $e.o.\mathfrak{F}(k)$ of the function $f_k$ is defined as the order of $f_k$ in any minimal Chebyshev asymptotic scale $\mathfrak{S}$ in which $\mathfrak{F}$ embeds (recall that the order of the first function in a scale is supposed to be equal to 0).

The effective order depends only on the sequence of parities of $\mathfrak{F}$. For instance, if $P(\mathfrak{F}) = \{P, I, I, P, P, P, \ldots \}$, then $e.o.\mathfrak{F}(0) = 0$, $e.o.\mathfrak{F}(1) = 1$, $e.o.\mathfrak{F}(2) = 3$, $e.o.\mathfrak{F}(3) = 4$, $e.o.\mathfrak{F}(4) = 6$, $e.o.\mathfrak{F}(5) = 8$, \ldots. For any sequence of parities of the asymptotic scale $\mathfrak{F}$, we have $e.o.\mathfrak{F}(k) \leq 2k$.

The notions of expansion and $\mathcal{R}$-division for a bilateral unfolding $f$ can be extended to a generalized Chebyshev asymptotic scales. If a bilateral unfolding $f = f(u, \lambda)$ admits an $\mathcal{R}$-division of length $k$, then its cyclicity has a rough bound of $2k$, which is obtained by applying proposition 3.14 for $u \geq 0$. This bound can be improved by using the notion of effective order.

Proposition 3.20. Let $f = f(u, \lambda)$ be a bilateral unfolding admitting an $\mathcal{R}$-division of length $k$

$$f(u, \lambda) = \sum_{i=0}^{k} \alpha_i(\lambda)f_i(u, \lambda),$$

where the functions $f_i(u) = f_i(u, 0)$, $i = 0, \ldots, k$, are the $k+1$ first functions in a generalized Chebyshev scale $\mathfrak{F}$. Then $\text{Cycl}(f) \leq e.o.\mathfrak{F}(k)$.

Proof. We set $K = e.o.\mathfrak{F}(k)$ and let $\mathfrak{S}$ be a Chebyshev scale in which $\mathfrak{F}$ embeds in a minimal way. Adding terms with trivial coefficients identical to 0 to the $\mathcal{R}$-division of $f$ in $\mathfrak{S}$, we obtain an $\mathcal{R}$-division of $f$ in $\mathfrak{S}$ of length $K$:

$$f(u, \lambda) = \sum_{i=0}^{K} \beta_i(\lambda)g_i(u, \lambda),$$

where $g_i(u) = g_i(u, 0)$, $i = 0, \ldots, K$, are the $K+1$ first functions in $\mathfrak{S}$. By construction, $g_i(u, \lambda) - g_i(u) = o(g_i(u))$ and, as a consequence, the functions $u \to g_i(u, \lambda)$ for $i = 0, \ldots, K$ are also the $K+1$ first functions of a Chebyshev scale depending on $\lambda$. Consider the enlarged unfolding

$$F(u, \lambda, \beta) = \sum_{i=0}^{K} \beta_i g_i(u, \lambda),$$

where $\beta = (\beta_0, \ldots, \beta_K)$ is considered as an independent parameter in $\mathbb{R}^{K+1}$. According to Mardesic’s result about bilateral Chebyshev asymptotic scales, the cyclicity of $F = F(u, \lambda, \beta)$ is bounded by $K$. It is the same for the unfolding $f(u, \lambda)$, which is induced from $F$ by the parameter mapping $\lambda \to (\lambda, (\beta, \lambda))$. \qed

We can define the notion of generic division as in the preceding section. It seems likely (although we do not prove this) that any unfolding $f(u, \lambda)$ with a generic division $f(u, \lambda) = \sum_{i=0}^{k} \alpha_i(\lambda)f_i(u, \lambda)$ in a generalized Chebyshev scale is topologically contact equivalent to a polynomial unfolding $\beta_0 + \beta_1 u^{\tau_1} + \cdots + \beta_{k-1} u^{\tau_{k-1}} \pm u^{\tau_k}$, $\tau_i \in \mathbb{N}$, such that the sequence of the monomial parities is the same as the parity sequence of $\{f_0, \ldots, f_k\}$. We can choose the sequence of $\tau_i$
to be minimal, for instance, the sequence \( \{0, 1, 3, 4, 6, 8, \ldots \} \) for the sequence of parities \( \{P, I, I, P, P, P, \ldots \} \). In this paper, we prove only the following assertion.

**Proposition 3.21.** Let \( f = f(u, \lambda) \) be a bilateral unfolding admitting an \( \mathcal{R} \)-division of length \( k \) satisfying the same conditions as in Proposition 3.20. Suppose that the \( \mathcal{R} \)-division is generic. Then \( \text{Cycl}(f) = e.o.3(k) \).

**Proof.** If the division is generic, we can assume that

\[
f(u, \lambda) = \sum_{i=0}^{k-1} \lambda_i f_i(u, \lambda) + f_k(x, \lambda)
\]

for \( \lambda = (\lambda_0, \ldots, \lambda_{k-1}) \in (\mathbb{R}^k, 0) \). To prove that the cyclicity is equal to \( K = e.o.3(k) \), we shall take an arbitrary \( \epsilon > 0 \) and construct a parameter value \( \lambda = (\lambda_0, \ldots, \lambda_{k-1}) \) with \( 0 < |\lambda_0| < \cdots < |\lambda_{k-1}| < \epsilon \) such that the function \( u \mapsto f(u, \lambda) \) has \( K \) distinct zeros. We shall choose the values \( \lambda_i \) by recursion, beginning with \( \lambda_{k-1} \).

The recurrence step is as follows. Suppose that we have already chosen the values \( \epsilon > |\lambda_{k-1}| > |\lambda_{j+1}| > 0 \) and obtained the function \( F_j(u) = f(u, (0, \ldots, 0, \lambda_{j+1}, \ldots, \lambda_{k-1})) \) whose set of zeros is the union of \( \{0\} \) and a set contained in the complement of some interval \( I_j = (-U_j, U_j) \). The parity of \( F_j \) near 0 coincides with that of \( F_{j+1}(u) = f_{j+1}(u, 0) \).

We have two different cases. If the parity of \( f_j \) is different from that of \( f_{j+1} \), we can choose a value \( \lambda_j \), with \( 0 < |\lambda_j| < |\lambda_{j+1}| \) such that \( F_{j-1}(u) = f(u, (0, \ldots, 0, \lambda_j, \lambda_{j+1}, \ldots, \lambda_{k-1})) \) has one new zero \( a \) inside \( I_j \setminus \{0\} \) and its other zeros are very close to the zeros of \( F_j \). The sign of \( \lambda_j \) depends on the sign of \( f_j \) and \( f_{j-1} \) for \( u > 0 \), and the point \( a \) may be on the left or on the right of \( 0 \). If the parity of \( f_j \) coincides with that of \( f_{j+1} \), we can choose \( \lambda_j \) in such a way that \( F_{j-1} \) have two zeros more than \( F_j \); these zeros lie in \( I_j \), one on each side from 0. Clearly, after \( k \) steps of this induction, we shall obtain an unfolding with \( K \) distinct zeros. \( \square \)

Note that, in generic divisions, cyclicity may exceed the dimension of the parameter space. For instance, for a generic division of length 2 and a sequence of parities \( \{P, I, I, \ldots\} \), cyclicity is equal to 3. In this generic case, it is easy to see that the bifurcation diagram depends only on the sequence of parities of the generalized Chebyshev scale \( \mathring{F} \). The unfolding is then topologically equivalent to that of the polynomial family \( P_2(u, \beta) = \beta_0 + \beta_1 x + x^3 \).

### 4. Unfolding the Cuspidal Loop

In this section, after defining the codimension of a cuspidal loop and reducing the cusp unfolding to an adequate Loray normal form, we establish the properties of the difference map \( \Delta_1 \), which is defined in Section 4.1. We use these properties to obtain sharp estimates for the cyclicity of a cuspidal loop unfolding (See Theorems 4.11 and 4.12).

#### 4.1. Codimension of a cuspidal loop

Consider a smooth vector field \( X \) defined on some open subset \( W \subset \mathbb{R}^2 \). Suppose that \( X \) has a cuspidal singularity \( s \) and...
that the two local separatrices of $s$ are contained in the same trajectory. The union $L$ of $s$ and such a trajectory is a simple closed curve, called a cuspidal loop.

As in Section 2, we choose two small sections $\Gamma^1$ and $\Gamma^2$ transversal to the local stable and unstable separatrices, respectively. The flow of $X$ defines the Dulac map $D: \Gamma^1 \rightarrow \Gamma^2$.

We choose parametrizations $u$ and $v$ of $\Gamma^1$ and $\Gamma^2$, respectively, such that $u = 0$ ($v = 0$) corresponds to the point $p_1 = \Gamma^1 \cap L$ (respectively, $p_2 = \Gamma^2 \cap L$). The orientation is chosen in such a way that the semi-infinite intervals $\Gamma^1_{\pm} = \{u > 0\}$ and $\Gamma^2_{\pm} = \{v > 0\}$ are in the interior of the cusp (we suppose that the cuspidal loop is of the form shown in Fig. 1, where the interior is the side of the disk bounded by the cuspidal loop). By $\Gamma^1_{\pm} = \{u < 0\}$ and $\Gamma^2_{\pm} = \{v < 0\}$ we denote the complements of the sections in the exterior of the cusp. Let

$$ D_+: \Gamma^1_+ \rightarrow \Gamma^2_-, \quad D_-: \Gamma^1_- \rightarrow \Gamma^2_+ $$

denote the corresponding restrictions of the Dulac map, and let $R: \Gamma^1 \rightarrow \Gamma^2$ be the flow transition of the vector field $-X$ along the regular arc of the loop $L$, between $p_1$ and $p_2$.

Consider the difference maps $\Delta_+(u) = D_+(u) - R(u)$ and $\Delta_-(u) = D_-(u) - R(u)$. Note that

$$ \Delta_{\pm} \circ R^{-1}(v) = P_{\pm}(v) - v, \quad (9) $$

where $P_{\pm}$ is the Poincaré mapping for $L$ relatively to $\Gamma^2_{\pm}$.

**Proposition 4.1.** Suppose that the difference maps $\Delta_{\pm}(u)$ are not flat at $u = 0$. Then there exists a natural number $n \geq 1$ and a constant $\gamma_+, \gamma_-$, or $c \neq 0$ such that the principal parts of $\Delta_{\pm}$ have one of the following types:

- the regular type $\Delta_{\pm}(u) = cu^n + o(u^n)$, where $n \geq 1$;
- the $\gamma$-type $\Delta_{\pm}(u) = \gamma_{\pm}u^{n-1/6} + o(u^{n-1/6})$, where $n \geq 2$;
- the $\beta$-type $\Delta_{\pm}(u) = \gamma_{\pm}u^{n+1/6} + o(u^{n+1/6})$, where $n \geq 1$.

The two maps $\Delta_+$ and $\Delta_-$ have the same order but the constants $\gamma_+$ and $\gamma_-$ may be different. The type of the difference maps and the value of $n$ is independent of the choice of the transversal sections and their parametrizations.

**Proof.** First, we prove the independence from the choice of sections and parametrizations. Suppose that $\Delta_+(u) = \gamma u^\tau + o(u^\tau)$, for some $\tau > 1$. According to (9), this is equivalent to $P(v) - v = \gamma' v^\tau + o(v^\tau)$. A change of sections and parametrizations corresponds to conjugation of $P(v)$ by a diffeomorphism. It is easy to see that any conjugation preserves the order $\tau$ of $P(v) - v$.

Let us choose the sections $\Gamma^1$ and $\Gamma^2$ and parametrizations as in Section 2. Then either $D_+(u) - u$ is flat at $u = 0$ or there exists a positive rational number $\tau$ and nonzero constants $\gamma_+$ and $\gamma_-$ such that

$$ D_{\pm}(u) = u + \gamma_{\pm}u^\tau + o(u^\tau), $$

where $\tau = s + 1/6$ for some natural number $s \geq 1$, or $\tau = l - 1/6$ for some natural number $l \geq 2$. 
On the other hand, either $R(u) - u$ is flat or there exists a natural number $m \geq 1$ and a nonzero constant $c$ such that

$$R(u) = u + cu^m + o(u^m).$$

Since the maps $\Delta_{\pm}$ are not flat, $D_{\pm}$ and $R$ cannot be flat simultaneously. It follows immediately that $\Delta_{\pm}$ has one of the three types specified above, with $n$ equal to one of the three numbers $s$, $l$, or $m$. □

**Remark 4.2.** The regular type occurs when the principal term of the difference map comes from the regular transition map $R$. The $\alpha$ or $\beta$ types occur when the principal term of the difference map comes from the Dulac map $D$.

It follows from Proposition 4.1 that the type of the difference map is an intrinsic characteristic. It depends only on the germ $(X, L)$ of the vector field along the cuspidal loop $L$. For brevity, we shall say that such a germ is, respectively, regular, of $\alpha$-type, or of $\beta$-type.

Consider the sequence $S = \{1, 7/6, 11/6, 2, 13/6, \ldots \}$ of rational numbers which appear in the above proposition. We have

$$S = \{\tau \in \mathbb{Q} : \tau = s + 1/6, \tau = l - 1/6 \text{ or } \tau = m \},$$

where $s, l, m \in \mathbb{N}$, $s, m \geq 1$, and $l \geq 2$. Let us enumerate the elements of $S$ in ascending order $\{\tau_n\}_{n \geq 1}$. We say that $n$ is the order of $\tau_n$ (that is, 1 has order 1, 7/6 has order 2, and so on).

**Definition 4.3.** If $\Delta(u)$ is flat at $u = 0$, we say that the germ $(X, L)$ has infinite codimension. Otherwise, the codimension of the germ $(X, L)$, denoted by $\text{cod}(X, L)$, is defined to be the order of the exponent $\tau$ of the principal part of $\Delta(u)$ in the sequence $S$ (beginning with 1).

**Remark 4.4.** Notice that the codimension of a cusp (see Definition 2.8) determines the codimension of the germ $(X, L)$ when this germ is of $\alpha$ or $\beta$ type. In all cases, the codimension of the cuspidal loop $\text{cod}(X, L)$ is always larger than the codimension of the cusp.

### 4.2. Loray normal form for cusp unfoldings

Consider a $\text{cusp-preserving } C^\infty$ unfolding $(X_\lambda, L)$ of $(X, L)$ with a parameter $\lambda \in (\mathbb{R}^p, 0)$. This means that, for each $\lambda$, $X_\lambda$ has a cuspidal singularity $s_\lambda$. It is easy to see that, for such an unfolding, there exists a neighbourhood $V \subset \mathbb{R}^2$ of $s = s_0$ such that, for each $\lambda$, the point $s_\lambda$ belongs to $V$ and is a unique singular point of $X_\lambda$. The function $\lambda \to s_\lambda$ is $C^\infty$; we shall assume that $s_\lambda \equiv s$ up to $C^\infty$ conjugation. In this section, we shall consider the restriction of the unfolding to the neighbourhood $V$. We shall call it the cusp unfolding (associated to the cuspidal loop unfolding).

Consider the family of 1-forms $\omega_\lambda$ which is dual to $X_\lambda$. The arguments of [CM] can be easily adapted to prove that there exists a coordinate change (depending smoothly on the parameters) in the neighbourhood $V$ of $s = s_0$ (chosen small enough) such that, in the new coordinates,

$$\omega_\lambda = d(y^2 - x^3) + g(x, y, \lambda)(3y \, dx - 2x \, dy),$$
where \( g(x, y, \lambda) \) is a \( C^\infty \) function. This is a consequence of the smooth dependence of the local separatrices on the parameters. Note that, in these coordinates, the stable separatrix of \( X_\lambda \) is given by \( \{ y = x^{3/2}, x \geq 0 \} \) and the unstable separatrix is given by \( \{ y = -x^{3/2}, x \geq 0 \} \).

More specifically, the theorem of Loray (see \([L]\)) can be applied to the unfolding \( \omega_\lambda \).

**Proposition 4.5.** For each \( m \in \mathbb{N} \), there exists a smooth change of coordinates defined in a neighbourhood of \( s \) in \( \mathbb{R}^2 \) (depending on \( \lambda \)) such that

\[
\omega_\lambda = d\tilde{H} + \left[ b_0(\lambda)x + \sum_{k=1}^{m} (a_k(\lambda) + xb_k(\lambda))\tilde{H}^k + G(x, y, \lambda) \right] (3y\, dx - 2x\, dy),
\]

where \( \tilde{H} = y^2 - x^3 \). Here \( a_k(\lambda), b_k(\lambda) \), and \( G(x, y, \lambda) \) are smooth functions and \( G(x, y, \lambda) = 0(\{ |x|^3 + |y|^{2m+1} \}) \).

**Proof.** This is essentially the prenormal form theorem from \([L]\) for \( (p, q) = (2, 3) \). In this theorem, the dependence of the change of coordinates and the normal form on the parameters is not explicit. However, it is easy to extract this dependence from Loray’s proof.

Indeed, let us look more closely at the recurrence step in the lemma of Section II (p. 167) in \([L]\). Suppose that we have already established that \( \omega_\lambda \) has the form given in the statement of the proposition up to order \( n \).

Let \( \Delta(x, y, \lambda) \) denote the term of quasi-homogeneous order \( n + 1 \) in \( G(x, y, \lambda) \). We look for a change of coordinates of the form \( X = x(1 + P)^2, Y = y(1 + P)^3 \), where \( P \) is quasi-homogeneous of degree \( n \), in order to reduce \( \omega_\lambda \) to normal form up to order \( n + 1 \). This gives the equation (see \([L]\, p. 167])

\[
\Delta - \left( 2y \frac{\partial}{\partial x} + 3x^2 \frac{\partial}{\partial y} \right) \cdot P = Q,
\]

where \( Q = \Delta_0(\tilde{H}) + x\Delta_1(\tilde{H}) \) is some polynomial of quasi-homogeneous degree \( n + 1 \). In this equation, \( P \) and \( Q \) are unknown polynomials depending on \( \Delta \). In the proof of the lemma, Loray shows that this equation has a (possibly non-unique) solution \( P, Q \) whose coefficients are linear functions of the coefficients of \( \Delta \).

Repeating this argument, we find a polynomial change of coordinates of the form specified in the statement of the proposition whose coefficients depend polynomially on the coefficients of the \( 6m \)-quasi-homogeneous jet of the unfolding \( \omega_\lambda \). The same holds for the coefficients of the resulting normal form of order \( m \).

At this point, we know that the remainder \( G_m(x, y, \lambda) \) is smooth and of order \( O((|x| + |y|)^{2(m+1)}) \). To prove that this reminder is in fact of order \( O((|x|^3 + |y|^{2m+1}) \), let us write a normal form at order \( N - 1 \) with \( 2N > 3(m+1) \).

Every term \( (a_k(\lambda) + xb_k(\lambda))\tilde{H}^k \) in the principal part, where \( k \geq m+1 \), is of order \( O((|x|^3 + |y|^{2m+1}) \). The remainder \( G_{N-1}(x, y, \lambda) \) is of order \( O((|x| + |y|)^{2N}) \), and hence of order \( O((|x|^3 + |y|^{2m+1}) \). Finally,

\[
G_m(x, y, \lambda) = \sum_{k=m+1}^{N-1} ((a_k(\lambda) + xb_k(\lambda))\tilde{H}^k + G_{N-1}(x, y, \lambda)) (3y\, dx - 2x\, dy)\]
is of order \( O((|x|^3 + |y|^2)^{m+1}) \).

\[ \square \]

4.3. The Dulac expansion of the \( \Delta_\lambda \)-unfolding. Consider the Dulac maps and difference maps for the cusp-preserving unfolding \( X_\lambda \).

We use the notation of Section 4.1. Consider the Dulac unfoldings \( D(u, \lambda) \) and \( D_\pm(u, \lambda) \), the regular unfoldings \( R(u, \lambda) \), and the difference unfoldings \( \Delta(u, \lambda) \) and \( \Delta_\pm(u, \lambda) \). These maps are defined, respectively, for \( u \in (0, U) \), \( u \in (-U, 0) \), or \( u \in (-U, U) \) for some \( U > 0 \).

**Theorem 4.6.** The unfoldings \( \Delta_+(u, \lambda) \) and \( \Delta_-(u, \lambda) \) and, hence, the unfolding \( \Delta(u, \lambda) \) are expandable in the generalized Chebyshev asymptotic scale

\[ \mathfrak{F} = \{|u|^r \log^q |u| \}, \]

where the pairs \((r, q)\) are such that \( r \in \frac{1}{6} \mathbb{N} = \{0, \frac{1}{6}, \ldots, \frac{7}{6}, \ldots\} \) and for each \( r \) the integer \( q \) belongs to the interval \( \{0, \ldots, N(r)\} \), where \( N(r) \) is some map \( N: \frac{1}{6} \mathbb{N} \rightarrow \mathbb{N} \) with \( N(0) = 0 \).

**Proof.** The simplest way to prove the theorem is to look at the real \((2, 3)\)-quasi-homogeneous blowing-up \( x = r^2 \tilde{x}, y = r^3 \tilde{y} \), where \((r, (\tilde{x}, \tilde{y})) \in \mathbb{R}^+ \times S^1 \).

This transformation desingularizes \( \omega_\lambda \) in a single step (see [DRS]). In the exceptional divisor \( D = \{r = 0\} \), there are two hyperbolic saddles, with hyperbolic ratios \( 1/6 \) and \( 6 \) (independently of \( \lambda \)).

The unfoldings \( D_\pm(u, \lambda) \) can be represented as the compositions of the local Dulac maps at these saddles and \( C^\infty \) regular maps defined in a neighbourhood of \( D \). It follows from the theory of Dulac [D] that each of these maps has an asymptotic expansion of the form

\[ \sum_{r \in \frac{1}{6} \mathbb{N}} P_r(\log |u|, \lambda) |u|^r, \]

where each \( P_r \) is a polynomial in \( \log |u| \) with coefficients depending smoothly on \( \lambda \). Therefore, \( D_\pm(u, \lambda) \) also has an asymptotic expansion of this form. We can choose a function \( N(r) \) as in the statement of the theorem such that \( D_\pm(u, \lambda) \) is expandable in the corresponding asymptotic scale \( \mathfrak{F} \).

On the other hand, it is easy to see that the \( C^\infty \) unfolding \( R(u, \lambda) \) is also expandable in \( \mathcal{F} \), and the same holds for the difference maps \( \Delta_\pm(u, \lambda) = D_\pm(u, \lambda) - R(u, \lambda) \).

\[ \square \]

4.4. The division property of the \( \Delta_\lambda \)-unfolding. Consider the sections \( \Gamma^1 \) and \( \Gamma^2 \) and their parametrizations \( u \) and \( v \) specified in Section 2. In these coordinates, the Dulac maps \( v = D_+(u, \lambda) \) and \( v = D_-(u, \lambda) \) have the following property.

**Lemma 4.7.** Let \( \lambda_0 \) be a fixed parameter value, and let \( \{a_i(\lambda_0), b_i(\lambda_0)\} \) be the sequence of coefficients which appear in the normal form of Proposition 4.5. Then there exists a sequence of nonzero constants \( \{A_n, B_n\} \) such that

(i) if \( n \geq 1 \) and \( b_0(\lambda_0) = a_1(\lambda_0) = b_1(\lambda_0) = \cdots = b_{n-1}(\lambda_0) = 0 \), then

\[
D_+(u, \lambda_0) = u - A_n a_n(\lambda_0) u^{n+5/6} + o(u^{n+5/6}) \quad \text{and} \quad
D_-(u, \lambda_0) = u + (-1)^n \sqrt{3} A_n a_n(\lambda_0) |u|^{n+5/6} + o(|u|^{n+5/6});
\]
(ii) if \( n \geq 0 \) and \( b_0(\lambda_0) = a_1(\lambda_0) = b_1(\lambda_0) = \cdots = a_n(\lambda_0) = 0 \) (this is the empty condition if \( n = 0 \)), then
\[
D_+(u, \lambda_0) = u - \overline{B}_n b_n(\lambda_0) u^{n+7/6} + o(u^{n+7/6}) \quad \text{and} \\
D_-(u, \lambda_0) = u + (-1)^n \sqrt{3} \overline{B}_n b_n(\lambda_0) |u|^{n+7/6} + o(|u|^{n+7/6}).
\]

Proof. We prove the lemma for the expansion of \( D_+(u, \lambda_0) \) in case (i) (the other cases are treated similarly). By the assumption of (i), \( \omega_{\lambda_0} \sim \alpha_n \), and we can apply Theorem 2.17. It follows that
\[
D_+(u, \lambda_0) = u - \overline{A}_n u^{n+5/6} + o(u^{n+5/6})
\]
for some nonzero constant \( \overline{A}_n \). Thus, we only have to prove the linear dependence of this constant on \( \alpha_n(\lambda_0) \), i.e., that
\[
\overline{A}_n = \overline{A}_n(\lambda_0) \quad \text{for some} \quad \overline{A}_n \in \mathbb{R}.
\]

From the considerations of Section 2, it is clear that \( \overline{A}_n \) depends only on the quasi-homogeneous jet of order \( 6n \) of \( \omega_{\lambda_0} \). Therefore, it suffices to prove the following statement. Consider the one-parameter family of differential forms
\[
\alpha_{n,\mu} = d(y^2 - x^3) + \mu (y^2 - x^3)^n (3y \, dx - 2x \, dy), \quad \text{with} \quad \mu \in \mathbb{R}
\]
and the transversal sections \( \Gamma^1 \) and \( \Gamma^2 \) with parametrizations \( u \) and \( v \) mentioned above. There exists a nonzero constant \( \overline{A}_n \) such that, for each \( \alpha_{n,\mu} \), the corresponding Dulac map \( v = D_+(u, \mu) \) has the asymptotic expansion
\[
v = D_+(u, \mu) = u - \overline{A}_n \mu u^{n+5/6} + o(u^{n+5/6}).
\]

To prove this, we first note that the linear map \( (x, y) \to (\bar{x}, \bar{y}) = (c^2 x, c^3 y) \) conjugates (up to division by a constant) \( \alpha_n = \alpha_{n,1} \) with \( \alpha_{n,c^6-1} \). This map implements a linear transformation of the transversal sections \( \Gamma^1 \) and \( \Gamma^2 \) and their parameterizations. That is, the Dulac map \( \bar{v} = D_+(\bar{u}, c^{6n-1}) \) can be obtained from the Dulac map \( v = D_+(u, 1) \) by taking left and right compositions with the mappings \( u = c^6 \bar{u} \) and \( \bar{v} = c^{-3} v \). A simple computation gives
\[
\bar{v} = D_+(\bar{u}, c^{6n-1}) = \bar{u} - \overline{c_6} \overline{A}_n \bar{u}^{n+5/6} + o(\bar{u}^{n+5/6}),
\]
which shows that the coefficient of \( \bar{u}^{n+5/6} \) depends linearly on \( c^{6n-1} \).

In the rest of this section, we assume that the unfolding \( \omega_{\lambda} \) is such that \( \omega_0 \) has finite codimension \( s \). Take an integer \( m \) having order strictly larger than \( s \) in the sequence \( S_0 = \{0\} \cup S \), where \( S \) is defined by (10). Let us write the Taylor expansion
\[
R(u, \lambda) = u + \sum_{i=0}^{m} c_i(\lambda) u^i + O(u^{m+1})
\]
of the regular map \( R(u, \lambda) \) up to order \( m \) and consider the Loray normal form up to order \( m \):
\[
\omega_{\lambda} = d\tilde{H} + \left[ b_0(\lambda) x + \sum_{k=1}^{m} (a_k(\lambda) + \varepsilon_k(\lambda)) \tilde{H}^k + G(x, y, \lambda) \right] (3y \, dx - 2x \, dy).
\]
Let us order the coefficients of $\overline{A}_i a_i(\lambda)$, $\overline{B}_i b_i(\lambda)$, and $-c_i(\lambda)$ as
\[-c_0(\lambda), -c_1(\lambda), -\overline{B}_0 b_0(\lambda), -\overline{A}_1 a_1(\lambda), -c_2(\lambda), -\overline{B}_1 b_1(\lambda), \ldots, -\overline{A}_i a_i(\lambda), -c_{i+1}(\lambda), -\overline{B}_i b_i(\lambda), \ldots\]
and denote them by $\gamma_j$, $j \geq 0$. Consider the list of parameters $\gamma_0 = -c_0$, $\gamma_1 = -c_1$, $\gamma_2 = -\overline{B}_0 b_0$, $\ldots$. Let $\gamma(\lambda) = (\gamma_0(\lambda), \ldots, \gamma_{s-1}(\lambda))$ denote the first elements in this list. Since the codimension of the cuspidal loop is equal to $s$, we have $\gamma(0) = (0, \ldots, 0)$ and $\gamma_{s}(0) \neq 0$. Using the standard gluing procedure, we can define a new unfolding $\Omega_{\mu,\lambda}$, where $\mu \in (R^+, 0)$, such that
- $\omega_\lambda = \Omega_{\mu=\gamma(\lambda),\lambda}$ (in particular, $\Omega_{0,0}$ has codimension $s$);
- the above procedure applied to the unfolding $\Omega_{\mu,\lambda}$ gives the list of parameters $\gamma(\mu, \lambda) = (\mu_0, \ldots, \mu_{s-1})$.

Let $\overline{R}(u, \mu, \lambda), \overline{D}_\pm(u, \mu, \lambda)$, and $\overline{\Delta}_\pm(u, \mu, \lambda)$ be the unfoldings associated to $\Omega_{\mu,\lambda}$. The corresponding unfoldings for $\omega_\lambda$ are induced by the map $\lambda \rightarrow \mu = \gamma(\lambda)$. In particular, we have
\[
\overline{\Delta}_\pm(u, \lambda) = \Delta_\pm(u, \gamma(\lambda), \lambda);
\]
so, it suffices to study the unfoldings $\Delta_\pm(u, \mu, \lambda)$.

First, consider $\overline{\Delta}_+$. It is expandable in the Chebyshev asymptotic scale $\mathfrak{g}^+ = \{f^+_0 = 1, f^+_1 = u^{1/6} \log^N(1/6) u, \ldots\}$, which is the restriction to $u > 0$ of the generalized Chebyshev asymptotic scale $\mathfrak{g}$ introduced in Theorem 4.6. The sequence of monomials $u^\tau$ for $\tau \in S_0$ is a subscale $\mathfrak{g}^+_0$ of $\mathfrak{g}^+$. Let $n_i, i \in \mathbb{N}$, be the sequence of indices of elements of $\mathfrak{g}^+_0$ as they appear in $\mathfrak{g}^+$: $n_0 = 0$, $\ldots$. As a direct consequence of the definition of $\Omega_{\mu,\lambda}$ and of Lemma 4.7, we have the following assertion.

**Proposition 4.8.** For any $(\mu, \lambda)$, $\overline{\Delta}_+(u, \mu, \lambda) = \mu_0 + o(1)$, and if $\mu_0 = \cdots = \mu_j = 0$ for some $j$ such that $0 \leq j \leq s - 1$, then
\[
\overline{\Delta}_+(u, \mu, \lambda) = \mu_{j+1} f^+_{n_{j+1}} + o(f^+_{n_{j+1}}).
\]

Being coordinates in the parameter space, the functions $\mu_i, 0 \leq i \leq s - 1$, have the radicality property and, moreover, $\gamma_i(0) \neq 0$. Applying Proposition 3.11 to the unfolding $\overline{\Delta}_+$, we obtain the division
\[
\overline{\Delta}_+(u, \mu, \lambda) = \sum_{i=0}^{s-1} \mu_i g^+_i(u, \mu, \lambda) + \gamma_s(\lambda) g^+_s(u, \mu, \lambda)
\]
of this unfolding, where $g^+_i(u, \mu, \lambda) = f^+_{n_i}(u) + o(f^+_{n_i})$ for $i = 0, \ldots, s$. Using the induction formula (11) and taking $g^+_i(u, \lambda) = \tilde{g}^+_i(u, \gamma(\lambda), \lambda)$ for $i = 0, \ldots, s$, we immediately obtain a division for the unfolding $\Delta_+$ itself. Thus, the smooth unfoldings $g^+_i(u, \lambda) = f^+_{n_i} + \cdots$ are expandable in $\mathfrak{g}^+$, where the $f^+_{n_i}, i = 0, \ldots, s$, are the first $s + 1$ monomials in the sequence $\mathfrak{g}^+_0$, and we can write the division
\[
\Delta_+(u, \lambda) = \sum_{i=0}^{s} \gamma_i(\lambda) g^+_i(u, \lambda).
\]
We can write a similar expansion for the unfolding $\Delta_-(u, \lambda)$. For this purpose, starting again with the asymptotic scale $\mathcal{F} = \{|u|^n \log^m |u|\}$ for $u < 0$, we change the elements of order $n_i$, $i \in \mathbb{N}$; namely, we replace the monomials $|u|^{n+5/6}$, $n \geq 1$, by $(-1)^{n+1} \sqrt{3}|u|^{n+5/6}$, the monomials $|u|^{n+7/6}$, $n \geq 0$, by $(-1)^{n+1} \sqrt{3}|u|^{n+7/6}$, and the monomials $|u|^n$ by $(-1)^n |u|^n$ for $n > 0$. Let $\mathcal{F}^-\lambda$ and $\mathcal{F}_0^-$ be the new asymptotic scales for $u < 0$. Then there exist smooth unfoldings expandable in $\mathcal{F}^-$ $(g_i^-(u, \lambda) = f_{n_i}^- + \ldots$, where the $f_{n_i}^-, i = 0, \ldots, s$, are the first $s+1$ monomials in the sequence $\mathcal{F}_0^-)$ such that

$$\Delta_-(u, \lambda) = \sum_{i=0}^s \gamma_i(\lambda) g_i^-(u, \lambda)$$

(13)

for $u < 0$ with sufficiently small $|u|$.

Finally, combining the divisions of $\Delta_+(u, \lambda)$ and $\Delta_-(u, \lambda)$, we obtain a division of $\Delta(u, \lambda)$. We introduce the asymptotic scale of functions $\mathcal{F}_t = \{h_i\}_i$ defined on some interval $[-U, U]$ by gluing together the scales $\mathcal{F}^+$ and $\mathcal{F}^-$ as follows: $h_i(u) = f_i^+(u)$ for $u > 0$ and $h_i(u) = f_i^-(u)$ for $u < 0$. The subscales $\mathcal{F}_t^+$ and $\mathcal{F}_t^-$ a subscale $\mathcal{F}_0 = \{h_n\}$. We shall write $h_n(u) = g_i(u)$.

This scale $\mathcal{F}_0$ is a generalized Chebyshev asymptotic scale. Importantly, it is not a Chebyshev scale. For instance, $g_0(u) = 1, g_1(u) = u, g_2(u) = u^7/6$ for $u \geq 0$, and $g_2(u) = -\sqrt{3}|u|^{7/6}$ for $u \leq 0$.

More generally, looking at the sign of $g_i(u)$ for $u > 0$ and $u < 0$, we easily obtain the following result.

**Lemma 4.9.** The sequence of parities of the scale $\mathcal{F}_0$ is $\mathcal{P}(\mathcal{F}_0) = \{P, I, I, P, P, P, I, I, I, I, \ldots\}$; it consists of the first three terms $P, I, I$ and alternating triples $P, P, P$ and $I, I, I$. The effective order in the scale $\mathcal{F}_0$ is given for $s > 0$ by e.o.$\mathcal{F}_0(s) = (5s - 3)/3$ if $s = 0 \bmod 3$, e.o.$\mathcal{F}_0(s) = (5s - 2)/3$ if $s = 1 \bmod 3$ and e.o.$\mathcal{F}_0(s) = (5s - 1)/3$ if $s = 2 \bmod 3$. This gives the sequence $1, 3, 4, 6, 8, 9, 11, 13, 14, \ldots$.

Consider the unfoldings $g_i(u, \lambda)$ equal to $g_i^+(u, \lambda)$ for $u > 0$ and to $g_i^-(u, \lambda)$ for $u < 0$. Combining the division formulas (12) and (13), we obtain the following result.

**Proposition 4.10.** If a cuspidal loop has finite codimension $s$, then the difference map of any smooth unfolding has $\mathcal{R}$-division

$$\Delta(u, \lambda) = \sum_{i=0}^s \gamma_i(\lambda) g_i(u, \lambda)$$

of length $s$ for $(u, \lambda)$ in some neighbourhood of $(0, 0) \in \mathbb{R} \times \mathbb{R}^p$. The functions $g_i(u, \lambda)$ are expandable in the scale $\mathcal{F}_t$; moreover, $g_i(u, \lambda) = g_i(u) + \cdots$, where the $g_i$ for $i = 0, \ldots, s$ are the first $s+1$ functions in the sequence $\mathcal{F}_0$.

**4.5. Cyclicality estimates.** Consider the difference map $\Delta(u, \lambda)$ of an unfolding $(X_\lambda, L)$ preserving the cuspidal loop $L$. A root $u > 0$ of $\Delta(u, \lambda)$ corresponds to an internal limit cycle of $X_\lambda$ (inside the disk bounded by the cuspidal loop, if this loop is of the form shown in Fig. 1) and a root $u < 0$ corresponds to an external limit cycle. Applying Propositions 3.14 and 3.16 to the internal and the external limit cycles separately, we see that the bifurcation diagram for the internal limit cycles
in a generic unfolding of a cuspidal loop of codimension \( s \) is homeomorphic to the bifurcation diagram of the versal polynomial unfolding \( P_{\pm,s}(u, \beta) \), and the same is true for the external limit cycles. It is not clear what is the bifurcation diagram when the two types of limit cycles are considered simultaneously. Concerning this global problem, we only deduce finite cyclicity results from Propositions 3.20 and 3.21.

**Theorem 4.11.** Suppose that a germ of vector field \((X_0, L)\) along a cuspidal loop \( L \) has finite codimension \( s \geq 1 \) and let \((X_{\lambda}, L)\) be any smooth unfolding of it. Then

\[
\text{Cycl}(X_{\lambda}, L) = \text{Cycl}(\Delta, 0) \leq e.o.\beta_0(s).
\]

For generic unfoldings, the following theorem is valid.

**Theorem 4.12.** If the smooth unfolding \((X_{\lambda}, L)\) is generic in sense that \( d\gamma_0(0) \wedge \cdots \wedge d\gamma_{s-1}(0) \neq 0 \), \( \gamma_0 = \cdots = \gamma_{s-1} = 0 \), and \( \gamma_s(0) \neq 0 \), then

\[
\text{Cycl}(X_{\lambda}, L) = e.o.\beta_0(s).
\]

**Remark 4.13.** The bound \( e.o.\beta_0(s) \) is less than \( \frac{2s}{3} \). It is given explicitly in Lemma 4.9.

As mentioned, a generic unfolding of a loop of codimension 2, the unfolding of the difference map \( \Delta \) is topologically contact equivalent to the polynomial unfolding \( P_2(u, \beta) = \beta_0 + \beta_1 u \pm u^3 \), where \( \beta = (\beta_0, \beta_1) \). The cuspidal loop is attracting if \( \pm \) is \( - \) and repelling if \( \pm \) is \( + \). Fig. 10 shows the bifurcation diagram (we can assume that the parameter \( \lambda \) of the vector field unfolding is equal to \( \beta = (\beta_0, \beta_1) \)). As mentioned, there exist three limit cycles for some parameter values.

**Figure 10.** A generic cusp-preserving unfolding of a repelling loop of codimension 2 (\( e \) and \( i \) stand for external and internal limit cycles, respectively)
5. UNFOLDINGS OF HAMILTONIAN CUSPIDAL LOOPS

Now, consider the class of cusp-preserving unfoldings of Hamiltonian cuspidal loops. For \( \lambda = (\bar{\lambda}, \varepsilon) \in (\mathbb{R}^p, 0) \times (\mathbb{R}, 0) \) and the dual form for \( X_{(\bar{\lambda}, \varepsilon)} \) is given by

\[
\omega_{(\bar{\lambda}, \varepsilon)} = dH + \varepsilon \nu_\lambda + o(\varepsilon),
\]

where \( \nu_\lambda \) is a smooth 1-form unfolding in the parameter \( \bar{\lambda} \) and the remainder is a smooth 1-form unfolding depending on the parameters \( (\bar{\lambda}, \varepsilon) \) which is \( o(\varepsilon) \) uniformly in the variables and other parameters. The above formula is valid in a neighbourhood of \( L \times \{(\bar{\lambda}, \varepsilon) = (0, 0)\} \) in \( \mathbb{R}^2 \times \mathbb{R}^p \times \mathbb{R} \).

We parametrize the two transversals \( \Gamma^1 \) and \( \Gamma^2 \) by the Hamiltonian value \( h = H(x, y) \). We suppose that the level curve \( L_h = \{(x, y) : H(x, y) = h\} \) is an internal cycle for \( h > 0 \) and an external cycle for \( h < 0 \), and that \( L_0 = L \).

It is well known (see [ALGM]) that the difference map \( \Delta(h, \bar{\lambda}, \varepsilon) \) has the asymptotic expansion

\[
\Delta(h, \bar{\lambda}, \varepsilon) = \varepsilon I(h, \bar{\lambda}) + o(\varepsilon), \tag{14}
\]

where the unfolding \( I(h, \bar{\lambda}) \) is the Abelian integral

\[
I(h, \bar{\lambda}) = \int_{L_h} \nu_\lambda.
\]

We shall see that the limit cycles bifurcating from \( L \) can be related to the roots of the Abelian integral bifurcating from \( h = 0 \).

**Proposition 5.1.** The unfolding \( I(h, \bar{\lambda}) \) is expandable in the generalized Chebyshev asymptotic scale \( S_0 = \{g_0 \equiv 1, g_1, \ldots \} \), which is defined in Section 4.4. This means that there exists a sequence of smooth germs \( \{\alpha_0(\bar{\lambda}), \alpha_1(\bar{\lambda}), \ldots \} \) such that, for each \( N \in \mathbb{N} \),

\[
I(h, \bar{\lambda}) = \sum_{i=0}^{N} \alpha_i(\bar{\lambda}) g_i(h) + \phi_N(h, \bar{\lambda}),
\]

as in Definition 3.3.

**Proof.** Take some \( m \in \mathbb{N} \), a neighbourhood of the cuspidal singularity \( U \subset \mathbb{R}^2 \) and a local change of coordinates such that \( \omega_\lambda \) is in Loray normal form up to order \( m \). This implies that \( H(x, y) = \tilde{H}(x, y) = y^2 - x^3 \) and \( \nu_\lambda = \nu_0 + G(x, y, \bar{\lambda})(3y dx - 2x dy) \), where \( G(x, y) = O((|x|^3 + |y|^2)^m) \) and

\[
\nu_0 = \left(b_0(\bar{\lambda}) x + \sum_{k=1}^{m} (\bar{a}_k(\bar{\lambda}) + x \bar{b}_k(\bar{\lambda})) \tilde{H}^k \right)(3y dx - 2x dy),
\]

where the \( \bar{a}_k \) and \( \bar{b}_k \) are smooth functions of the parameter \( \bar{\lambda} \). Let us choose transversals \( \Gamma^1, \Gamma^2 \subset U \) as in Section 2.

We write \( I(h, \bar{\lambda}) = I_1(h, \bar{\lambda}) + I_2(h, \bar{\lambda}) \), where \( I_2(h, \bar{\lambda}) \) is the integral along the segment of \( L_h \) contained between the sections \( \Gamma^1 \) and \( \Gamma^2 \) near the regular part of
the loop $L$ and $I_1(h, \lambda)$ is the integral on the complementary part $I_h$ of $L_h$, which passes near the cuspidal singularity.

The integral $I_2(h, \lambda)$ is a smooth function and can be expanded in Taylor series with respect to the variable $h$. To study the integral $\int_{I_h} G(x, y, \lambda) \ (3g \ dx - 2x \ dy)$, we have to desingularize the cuspidal singularity by using the quasi-homogeneous blowing-up $x = \rho^2, \ y = \rho^3 y$, as in [DRS]. The transition near the cuspidal point is lifted to a composition of transitions near two saddle singularities with hyperbolicity ratios $\frac{1}{6}$ and 6, respectively, and situated on the divisor $\{ \rho = 0 \}$.

The function $G$ is lifted to a function of order $O(\rho^{6(m+1)})$. As a consequence, $\int_{I_h} G(x, y, \lambda(3g \ dx - 2x \ dy)) = r(\lambda) + R(h, \lambda)$, where $r(\lambda)$ is smooth and $R(h, \lambda)$ is an unfolding which is $m$-flat and $\mathcal{C}^m$ in $h$. This implies that the expansion of $I_1(h, \lambda)$ up to $o(h^m)$ is given by

$$\int_{I_h} \nu_i^f = J_0(h) \sum_{k=1}^m a_k(\lambda) h^k + J_1(h) \sum_{k=0}^m b_k(\lambda) h^k,$$

where $J_i(h) = \int_{I_h} x^i (3g dx - 2x dy)$.

Each $J_i(h)$ is the integral of a polynomial 1-form along a path on the level curve $H(x, y) = h$ between the two transversal sections. To study it, we consider the complex extension of polynomial function $H$ to $(x, y) \in \mathbb{C}^2$. Each complex level curve $\{ H(x, y) = h \} \subset \mathbb{C}^2$ (for $h \in (\mathbb{C}, 0 \setminus \{0\})$ is a complex Riemann surface homeomorphic to the torus minus one point. The homology of such a surface is generated by two vanishing cycles relative to the cuspidal point. It is well known [AGV] that the integral $J_i(h)$ is the sum of a holomorphic function in $h$ plus a non-trivial linear combination $\delta(h)$ of the integral of the 1-form $x^i (3g dx - 2x dy)$ along these vanishing cycles (this linear combination depends on whether we consider the extension in the complex of the real integral for $h > 0$ or for $h < 0$, as explained in Section 2). Consider the integral $J_i^\sigma(h) = \int_{\sigma(h)} x^i (3g dx - 2x dy)$, where $\sigma(h)$ is a vanishing cycle. The linear map $(x, y) \mapsto (\hat{x}, \hat{y}) = (h^{1/3} x, h^{1/2} y)$ takes $\sigma(h)$ to $\sigma(1)$, and the pull back of $x^i (3g dx - 2x dy)$ is $h^{-(2i+5)/6} x^i (3g dx - 2x dy)$. It follows that

$$J_i^\sigma(h) = J_i^\sigma(1) h^{(2i+5)/6}.$$

Therefore, $I_1(h, \lambda)$ has an expansion of order $o(h^m)$ in the Chebyshev asymptotic scale $\mathcal{D}_0$. The same holds for $I(h, \lambda)$.

Using this result, we can introduce the notion of codimension and generic unfolding for the Abelian integral.

**Definition 5.2.** We say that $I(h, 0)$ has **codimension** $s$ if $\alpha_0(0) = \alpha_1(0) = \cdots = \alpha_{s-1}(0) = 0$ and $\alpha_s(0) \neq 0$. In this case, we say that $I(h, \lambda)$ is a **generic unfolding of codimension** $s$ if the mapping $\lambda \mapsto (\alpha_0(\lambda), \ldots, \alpha_{s-1}(\lambda))$ is a local submersion at $\bar{\lambda} = 0$.

Consider the **reduced difference map** $\Delta(h, \bar{\lambda}, \varepsilon) = \frac{1}{\Delta(h, \bar{\lambda}, \varepsilon)}$. For $\varepsilon \neq 0$, the roots of the function $\Delta(h, \bar{\lambda}, \varepsilon)$ are the intersections of the periodic orbits of $X(\varepsilon, \lambda)$ with $\Gamma^1$.
From (14), we have $\Delta(h, \bar{\lambda}, 0) = I(h, \bar{\lambda})$. As a consequence, when $I(h, 0)$ has finite codimension, $\Delta(h, \bar{\lambda}, \varepsilon)$ has the following division property.

**Proposition 5.3.** Suppose that $I(h, 0)$ has codimension $s$. Then there exist smooth functions $\beta_0(\bar{\lambda}, \varepsilon), \ldots, \beta_s(\bar{\lambda}, \varepsilon)$ and unfoldings $g_0(h, \bar{\lambda}, \varepsilon), \ldots, g_s(h, \bar{\lambda}, \varepsilon)$ smooth for $h \neq 0$ such that

(i) $\beta_i(\bar{\lambda}, 0) = \alpha_i(\bar{\lambda})$ for $0 \leq i \leq s$;

(ii) for each $0 \leq i \leq s$, $g_i(h, \bar{\lambda}, \varepsilon)$ is expandable in the asymptotic scale $\mathfrak{H}$ introduced in Section 4.4 and

$$g_i(h, \bar{\lambda}, \varepsilon) = g_i(h) + o(f_i);$$

(iii) $\Delta(h, \bar{\lambda}, \varepsilon)$ has the division property

$$\Delta(h, \bar{\lambda}, \varepsilon) = \sum_{i=0}^s \beta_i(\bar{\lambda}, \varepsilon) g_i(h, \bar{\lambda}, \varepsilon).$$

**Proof.** Take a natural number $m$ depending on the codimension $s$ as described in Section 4.4. The regular transition map $R(h, \lambda)$ has the Taylor expansion

$$R(h, \lambda) = h + \varepsilon \left( \sum_{i=0}^{m} \tilde{c}_i(\lambda) h^i + O(h^{m+1}) \right).$$

Similarly, the local Loray normal form up to order $m$ is given by

$$\omega_\lambda = d\bar{H} + \varepsilon \left( b_0(\lambda) x + \sum_{k=1}^{m} (a_k(\lambda) + \sum_{k=1}^{m} b_k(\lambda)) \bar{H}^k + G(x, y, \lambda) \right) (3y dx - 2x dy),$$

where $\bar{H}(x, y) = y^2 - x^3$. This expression is a consequence of the fact that (see Section 2) the difference map $\Delta(h, \lambda)$ is identically zero if and only if the Loray normal form for $\omega_\lambda$ is simply $d\bar{H}$.

Using a gluing procedure, we can define a new smooth unfolding $\Omega_{\bar{\mu}, \bar{\lambda}, \varepsilon}$, as in Section 4.4, such that it induces $\omega_\lambda$ and the corresponding Abelian integral $\tilde{I}(h, \bar{\mu}, \bar{\lambda})$ has codimension $s$ and is generic (see Definition 5.2).

Note that, for each $\varepsilon \neq 0$, we obtain the same conditions as in subsection 4.4. As a consequence, we can apply Proposition 3.11 to the reduced difference map associated to $\Omega_{\bar{\mu}, \bar{\lambda}, \varepsilon}$. The division property for the original reduced difference map $\Delta(h, \bar{\lambda}, \varepsilon)$ can be obtained by using the induction map. \hfill $\Box$

The above division formula and Propositions 3.14 and 3.16 immediately imply the following results:

**Theorem 5.4.** Suppose that $(X_{\lambda, \varepsilon}, L)$ is a cusp-preserving Hamiltonian unfolding such that the corresponding Abelian integral $I(h, 0)$ has codimension $s$. Then

$$\text{Cycl}(X_{\lambda}, L) = \text{Cycl}(\Delta, 0) \leq c.o.\mathfrak{s}_0(s).$$

**Theorem 5.5.** Suppose that $(X_{\lambda, \varepsilon}, L)$ is a cusp-preserving Hamiltonian unfolding such that the corresponding Abelian integral $I(h, \bar{\lambda})$ is a generic unfolding of codimension $s$. Then

$$\text{Cycl}(X_{\lambda}, L) = \text{Cycl}(\Delta, 0) = c.o.\mathfrak{s}_0(s).$$
The bifurcation diagram of $(X_{\lambda, \varepsilon}, L)$ for $s = 2$ is homeomorphic to the product $\Sigma \times (-\varepsilon_0, \varepsilon_0)$, where $\Sigma$ is the bifurcation diagram of $(P_2(u, \beta), 0)$ shown in Fig. 10 and $\varepsilon_0$ is sufficiently small.

5.2. Examples of generic unfoldings. In this section, we give polynomial examples of generic cusp-preserving Hamiltonian unfoldings. The simplest polynomial Hamiltonian function with a cuspidal singularity is the quartic polynomial

$$H(x, y) = \frac{1}{2} y^2 + \frac{1}{3} x^3 + \frac{1}{4} x^4.$$  

The level $H = 0$ is a cuspidal loop $L$. Unlike in the preceding section, the regular cycles $L_h$ are internal for $h < 0$ and external for $h > 0$. It is easy to improve this situation, but we choose the Hamiltonian $H$, because it was studied in [ZZ2] and we want to use computations performed therein (see also [ZZ1]). Consider the polynomial forms $\nu_i = yx^i \, dx$ and their Abelian integrals $I_i(h) = \int_{L_h} \nu_i$. It is well known that the $\mathbb{R}[h]$-modulus of polynomial Abelian integrals is generated by the three integrals $I_0$, $I_1$, and $I_2$ for $h > 0$ and $h < 0$ (see, e.g., [R3]). This implies that any polynomial form $\nu$ is equal, up to a relatively exact form $dG + F \, dH$, to a linear combination of the forms $h^i \nu_i$ for $i, j \in \mathbb{N}$. Therefore, it suffices to consider perturbations of $dH$ by such linear combinations $\nu = \sum \alpha_{ij} h^i \nu_j$ ($\alpha_{ij} \in \mathbb{R}$). Moreover, to preserve the cuspidal singularity, such a perturbation must have no component on $\nu_0$. For these reasons, we introduce the ordered sequence of forms $\Omega_0, \Omega_1, \Omega_2, \Omega_3, \Omega_5, \Omega_6, \Omega_7, \Omega_8, \cdots = \nu_1, \nu_2, h\nu_0, h\nu_1, h^2 \nu_0, h^2 \nu_1, h^2 \nu_2, \cdots.$

**Lemma 5.6.** The Abelian integrals $I_0$, $I_1$ and $I_2$ have the following expansions:

(i) $I_0(h) = \frac{1}{4} \sqrt{2\pi} + O(|h|^{5/7});$

(ii) $I_1(h) = -\frac{10}{\pi^2} \sqrt{2\pi} - 2\sqrt{2\pi} h + C_2^+ |h|^{7/6} + O(|h|^{11/6});$

(iii) $I_2(h) = \frac{28}{27\pi^3} \sqrt{2\pi} + \frac{1}{3} \sqrt{2\pi} h + O(|h|^{11/6}),$

where $C_2^+$ is a real non-zero constant depending on the sign of $h$.

**Proof.** All the above assertions, except that $C_2^+ \neq 0$, were proved in [ZZ2]. To prove that $C_2^+ \neq 0$, we choose a local system of analytic coordinates $X = U(x) = x + \cdots, Y = y$ in which the Hamiltonian function takes the form $H(X, Y) = \frac{1}{2} Y^2 - \frac{1}{3} X^3$. In these coordinates, the 1-forms $\nu_i$ become the 1-forms $\tilde{\nu}_i = \nu_i + \cdots$ is a convergent series of forms with quasi-homogeneous degree $\deg_{(2,3)}$ strictly larger than $\deg_{(2,3)}(\nu_i) = 3 + 2(i + 1)$. Repeating the proof of Proposition 5.1, we conclude that there exist non-trivial linear combinations $\delta^\pm(h)$ of vanishing cycles such that the complex continuation of $I_i(h)$, considered for $h > 0$ or $h < 0$, differs from $J_i^\pm(h) = \int_{\delta^\pm(h)} \nu_i$ by an holomorphic function. Thus, $I_i(h) = J_i^\pm(1) h^{\frac{2+2(i+1)}{3}} + o(h^{\frac{2+2(i+1)}{3}}) + R(h)$ (where $R(h)$ is some holomorphic function); in particular, $C_2^+ = J_1^+(1)$. It is well known that the two forms $\nu_0$ and $\nu_1$ generate the relative cohomology of $H$ (see, e.g., [R3] for instance). As a consequence, $J_1^+(h)$ cannot be identically zero, and $C_2^+ = J_1^+(1) \neq 0$. □

The above lemma implies that the coefficient matrix of the expansion of the three integrals $I_1$, $I_2$, and $hI_0$ in the first three functions $1$, $h$, and $|h|^{7/6}$ for $h > 0$
or $h < 0$ is invertible. This remark easily extended to an expansion at any order in the asymptotic scale $\mathcal{F}_0$; namely, the following lemma is valid.

**Lemma 5.7.** For any $n \in \mathbb{N}$, there exists an invertible $n \times n$ real matrix $M_n$ with the following property. Let $\Omega = \sum_{i=0}^{n} \alpha_i \Omega_i$ be any linear combination, and let $I(h) = \int_{L_h} \Omega$ be its integral with expansion $I(h) = \sum_{i=0}^{n} \beta_i f_i + o(f_i)$ at order $n$ in $\mathcal{F}_0$. Then $\beta = M_n \alpha$, where $\alpha$ and $\beta$ are the column-vectors $\alpha = (\alpha_0, \ldots, \alpha_n)$ and $\beta = (\beta_0, \ldots, \beta_n)$.

As a consequence of this lemma, for any $s$, we can choose constants $\gamma_0, \ldots, \gamma_s$ such that the 1-form $\Theta_s = \sum_{i=0}^{s} \gamma_i \Omega_i$ has Abelian integral equivalent to the function $f_s(h) \in \mathcal{F}_0$. Thus, again as a consequence of Lemma 5.7 and Theorem 5.5, we obtain an example of a polynomial generic unfolding of codimension $s$.

**Theorem 5.8.** For the cusp-preserving unfolding of Hamiltonian cuspidal loop

$$\omega_\alpha = d\left(\frac{1}{2} y^2 + \frac{1}{3} x^3 + \frac{1}{4} x^4\right) + \varepsilon \left(\sum_{i=0}^{s-1} \alpha_i \Omega_i \pm \Theta_s\right),$$

where $\alpha = (\alpha_0, \ldots, \alpha_{s-1}) \in (\mathbb{R}^p, 0)$ and the polynomial 1-forms $\Omega_i$ and $\Theta_s$ are defined as above, the Abelian integral is a generic unfolding of codimension $s$. As a consequence, the cyclicity of $\omega_\alpha$ is equal to $c.o.\mathcal{F}_0(s)$.

### 5.3. General analytic unfoldings

Now, consider a real analytic cuspid-preserving unfolding $\omega_\lambda$ of a Hamiltonian cuspidal loop with $\omega_0 = dH$. The Hamiltonian function $H$ is supposed to be defined and analytic in some neighbourhood $U$ of the cuspidal loop $L$, and the unfolding is given by an analytic family of 1-forms defined for $((x, y), \lambda) \in U \times (\mathbb{R}^p, 0)$. We no longer suppose that there exists a special parameter $\varepsilon$. The difference map $\Delta(h, \lambda)$ is expandable in the asymptotic scale $\mathcal{F}_0$. The coefficients of this expansion are analytic functions of the parameter $\lambda$. They generate a Noetherian ideal of analytic germs in $\lambda \in (\mathbb{R}^p, 0)$, which is equal to the Bautin ideal $\mathcal{I}$ of the unfolding (the definition of this ideal is given in [R3]).

Consider a system of generators $\alpha_1(\lambda), \ldots, \alpha_\ell(\lambda)$ for $\mathcal{I}$. Arguing as in the proof of Proposition 3.11 for an unfolding of finite codimension, we can decompose the map $\Delta$ as

$$\Delta(h, \lambda) = \sum_{i=1}^{\ell} \alpha_i(\lambda) g_i(h, \lambda),$$

where functions $g_i(h, \lambda) = h_{n_i}(h) + \cdots$ are expandable in the asymptotic scale $\mathcal{F}_0$ and the $g_i(h) = h_{n_i}(h)$ for $i = 0, \ldots, \ell$ are the first $l + 1$ functions of the scale $\mathcal{F}_0$. Exactly as it was done for a smooth $\mathfrak{M}$-division in Proposition 3.14, we apply formula (15) to prove that the cyclicity of $\Delta$ is bounded by $\ell$ for $h > 0$ and for $h < 0$; thus, in the two-sided case, it is bounded by $2\ell$. We have obtained the following result.

**Theorem 5.9.** Any analytic cusp-preserving unfolding of a cuspidal loop has finite cyclicity.

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References


