An LMI-Based Stabilization Criterion of Sliding Mode Control for a Class of Bilinear State-Delay Systems

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***Abstract*—This paper focuses on the problem of sliding mode control for delay-dependent stabilization criterion of bilinear state-delay systems. Based on the linear matrix inequality (LMI) method and a parameterized model transformation technique, we provide a sufficient condition for the existence of sliding surface and we give an explicit formula of sliding surface guaranteeing the asymptotic stability of the reduced-order equivalent system dynamics restricted to the sliding surface. The proposed stabilization criterion can also be applied to control the bilinear systems with time-varying delay. Finally, a simulation study shows that our method is effective and useful.**

***Keywords*—sliding mode control, delay-dependent stabilization criterion, bilinear state-delay systems, parameterized model transformation, sliding surface.**

I. INTRODUCTION

In fact, time delays often lead to poor performance and instability of many physical processes. The stabilization for a control system is an important and basic issue arising in control theory. Recently, many authors have used different approaches to study the stabilization problem of linear systems with time delay [1-11]. In a real world, many systems can be adequately approximated by means of the bilinear model. Therefore, bilinear systems have been of great interest in the past years. The application areas include nuclear, thermal, chemical processes, biology, secioeconomics, immunology, etc. The bilinear systems are not only a special set of nonlinear systems but can also be regarded as a practical starting point for the study of other nonlinear systems. Moreover, most results are concerning the study of stabilization problem of bilinear systems without time delay (see [15-17] and the references therein). However, only a few papers [18-20] have focused on the stabilization problem of bilinear systems with time delay. That is, research on the stabilization problem of such a system is always a challenging work.

The sliding mode control (SMC) approach, based on the use of discontinuous control laws (relays), is known to be an efficient alternative way to tackle many challenging problems of robust stabilization. However, so far, almost all the published SMC papers [4-11] only concentrated on the linear systems with/without, single/multiple, constant or time-varying delays. In this paper, we will consider the problem of designing SMC laws for bilinear systems with constant or time-varying delay. In terms of Lyapunov-Krasovskii functional approach and parameterized model technique, we derived sufficient conditions for the existence of a linear sliding surface, guaranteeing asymptotic stability of the reduced-order equivalent systems restricted to the sliding surface. And linear sliding surface is parameterized explicitly by using solution matrices to the LMI condition such that one can easily design SMC laws for bilinear systems with constant or time-varying delay. Additionally, our method has advantages in the computational aspect. Finally, a numerical simulation is given to illustrate the results of the proposed approach.

II. PROBLEM FORMULATION

Consider the bilinear systems with time delay in state, described by

$$\dot{x}(t)=Ax(t)+A_d\,x(t-\tau)+[B+C(x(t))]u(t) \qquad (1a)$$

$$x(t)=\varphi(t),\ t\in[-\tau,\ 0] \qquad (1b)$$

where $x(t)\in R^n$ is the state; $u(t)\in R^m$ is the control input; $C(x(t))=[C_1x(t)\ C_2x(t)\cdots C_m x(t)]$; A , A_d , B and C_i , $i=1,\cdots,m,$ are known constant matrices of appropriate dimensions; $\varphi(t)$ is a continuous vector-valued initial function; τ is a positive constant time delay.

In the next section, the following assumptions are needed.

Assumption 1

The following matching condition holds:

$$C(x(t))=BE(x(t)) \qquad (2)$$

where $E(x(t))=[E_1x(t)\ E_2x(t)\ \ E_3x(t),\cdots,E_mx(t)]$; E_i, $i=1,\cdots,m,$ are constant matrices of appropriate dimensions.

Assumption 2

The matrix $[I_m+E(x(t))]$ is nonsingular, where I_m is the $m\times m$ identity matrix.

Assumption 3

The pair (A, B) is stabilizable and the input matrix B has full rank $m<n$.

The proofs of our main results need the following Lemmas.

Lemma 1 [8]

The motion of the sliding surface is asymptotically stable if there exists a Lyapunov function $V(t)=\frac{1}{2}\sigma^T(t)\,(B^T X^{-1} B)^{-1}\sigma(t)$ such that the following condition holds

$$\sigma^T(t)(B^T X^{-1}B)^{-1}\dot{\sigma}(t)<0$$

where $\sigma(t)$ is the sliding surface; B is the input matrix of system (1); X is a symmetric positive definite matrix.

Lemma 2 [12]

Given matrix $K\in R^{n\times n}$ and scalar δ satisfying $0<\delta<1$, if there exists a symmetric positive definite matrix M such that

$$\begin{bmatrix} -\delta M & \tau K^T M \\ \tau MK & -M \end{bmatrix}\le 0$$

then the operator D: $C_0[-\tau,\ 0]\to R^n$ with

$$D(x_t) = x(t) + K\int_{t-\tau}^{t} x(s)ds$$

is stable for any constant time delay $\tau>0$.

Lemma 3 [1]

Given any vectors z, y of appropriate dimensions and scalar $\varepsilon>0$, then the following inequality is true

$$2z^T y\le\varepsilon z^T z+\varepsilon^{-1}y^T y$$

Lemma 4 [13]

The following linear matrix inequality:

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix}>0,$$

where $Q(x)=Q^T(x)$, $R(x)=R^T(x)$, and $S(x)$ depend affinity on x, is equivalent to

$$R(x)>0,\quad Q(x)-S(x)R^{-1}(x)\,S^T(x)>0.$$

III. MAIN RESULTS

The sliding surface is defined as $\sigma(t)=Sx(t)=0$, where $S\in R^{m\times n}$ is a constant matrix which should be chosen so that SB is nonsingular and the $(n-m)$ reduced-order equivalent system restricted to the sliding surface is asymptotically stable.

Consider the following LMIs

$$N=\begin{bmatrix} \tilde{B} & 0 & 0 \\ 0 & \tilde{B} & 0 \\ 0 & 0 & \tilde{B} \end{bmatrix}^T \begin{bmatrix} N_{11} & N_{12} & N_{13} \\ N_{21} & -N_{22} & N_{23} \\ N_{31} & N_{32} & -N_{33} \end{bmatrix} \begin{bmatrix} \tilde{B} & 0 & 0 \\ 0 & \tilde{B} & 0 \\ 0 & 0 & \tilde{B} \end{bmatrix}<0 \quad (3a)$$

$$\begin{bmatrix} -\delta M_1 & \tau\tilde{B}^T X\tilde{B}M_1 \\ \tau M_1\tilde{B}^T X\tilde{B} & -M_1 \end{bmatrix}\le 0 \quad (3b)$$

where

$$N_{11}=AX+XA^T+2X\tilde{B}\tilde{B}^T X+X\tilde{B}Q_1\tilde{B}^T X+\tau X\tilde{B}W_1\tilde{B}^T X \quad (4a)$$

$$N_{12}=A_d X-X\tilde{B}\tilde{B}^T X \quad (4b)$$

$$N_{13}=XA^T\tilde{B}\tilde{B}^T X+X\tilde{B}\tilde{B}^T X\tilde{B}\tilde{B}^T X \quad (4c)$$

$$N_{21}=XA_d^T-X\tilde{B}\tilde{B}^T X \quad (4d)$$

$$N_{22}=X\tilde{B}Q_1\tilde{B}^T X \quad (4e)$$

$$N_{23}=XA_d^T\tilde{B}\tilde{B}^T X-X\tilde{B}\tilde{B}^T X\tilde{B}\tilde{B}^T X \quad (4f)$$

$$N_{31}=X\tilde{B}\tilde{B}^T AX+X\tilde{B}\tilde{B}^T X\tilde{B}\tilde{B}^T X \quad (4g)$$

$$N_{32}=X\tilde{B}\tilde{B}^T A_d X-X\tilde{B}\tilde{B}^T X\tilde{B}\tilde{B}^T X \quad (4h)$$

$$N_{33}=\tau X\tilde{B}W_1\tilde{B}^T X \quad (4i)$$

and M_1, Q_1, W_1, X are symmetric positive definite matrices; δ is a positive scalar and satisfies $0<\delta<1$; $\tilde{B}$ is any basis of the null space of B^T, i.e. $\tilde{B}$ is an orthogonal complement of B. Assume that the sliding surface σ is given by the following explicit formula

$$\sigma(t)=Sx(t)=B^T X^{-1}x(t)=0 \quad (5)$$

where X is a solution to equation (3). Now, let the control strategy be given as follows

$$u(t)=-[I_m+E(x(t))]^{-1}\frac{\sigma(t)}{\|\sigma(t)\|}\Phi(x(t),\ x(t-\tau),\ t) \quad (6)$$

where

$$\Phi(x(t),\ x(t-\tau),\ t)=\|(SB)^{-1}SAx(t)\|+\|(SB)^{-1}SA_d x(t-\tau)\|+\beta \quad (7)$$

and I_m is the $m\times m$ identity matrix; β is a positive scalar.

Theorem 1

Suppose that the LMIs (3a) and (3b) have a feasible solution M_1, Q_1, W_1, X and the sliding surface is given by equation (5). And consider the closed-loop control system of bilinear time-delay system (1) with the control law (6) and satisfying Assumptions 1-3. Then, every solution trajectory is directed towards the sliding surface $\sigma(t)=0$ and once the trajectory hits the sliding surface it remains on the surface for all subsequent time. And the resulting $(n-m)$ reduced-order dynamics of closed-loop system restricted to the sliding surface is asymptotically stable and is given by

$$\dot{v}_1(t)=\bar{A}_{11}v_1(t)+\bar{A}_{12}v_1(t-\tau) \quad (8)$$

where

$$v_1(t)=\tilde{B}^T x(t), \quad (9a)$$

$$\bar{A}_{11}=\tilde{B}^T AX\tilde{B}(\tilde{B}^T X\tilde{B})^{-1}, \quad (9b)$$

$$\bar{A}_{12}=\tilde{B}^T A_d X\tilde{B}(\tilde{B}^T X\tilde{B})^{-1} \quad (9c)$$

Proof

First, we will show that the control law (6) drives the system trajectory into the sliding surface and maintain the trajectory on the sliding surface for all subsequent time, i.e. the reachability condition is satisfied. Then, using the sliding surface (5) and Assumptions 1-2, we obtain

$$\dot{\sigma}(t)=S[(Ax(t)+A_d x(t-\tau)]+SB[I_m+E(x(t))]u(t) \quad (10)$$

According to Lemma 1, the Lyapunov function is chosen as follows

$$V(t)=\frac{1}{2}\sigma^T(t)\,(B^TX^{-1}B)^{-1}\sigma(t) \tag{11}$$

Then the time derivative of $V(t)$ is

$$\dot{V}(t)=\sigma^T(t)\,(SB)^{-1}\dot{\sigma}(t) \tag{12}$$

From (10) and (12), we have

$$\dot{V}(t)\le\|\sigma(t)\|\,[\,\|(SB)^{-1}SAx(t)\|+\|(SB)^{-1}SA_d\,x(t-\tau)\|\,]$$
$$+\sigma^T(t)\,[I_m+E(x(t))]u(t) \tag{13}$$

Substituting (6) and (7) into (13) yields

$$\dot{V}(t)\le-\beta\|\sigma(t)\| \tag{14}$$

Hence, we select $\beta>0$ to guarantee $\dot{V}(t)<0$, which confirms that reachability condition is satisfied.

Now, we will show that the $(n-m)$ reduced-order dynamics restricted to the sliding surface $\sigma(t)=0$ is asymptotically stable and is given by (8). Define a transformation matrix and the associated vector v as follows

$$T=\begin{bmatrix}\tilde{B}^T\\ B^TX^{-1}\end{bmatrix}=\begin{bmatrix}\tilde{B}^T\\ S\end{bmatrix},\ v=\begin{bmatrix}v_1\\ v_2\end{bmatrix}=Tx \tag{15}$$

where $v_1\in R^{n-m}$ and $v_2\in R^m$. Thus we can see that $T^{-1}=[X\tilde{B}(\tilde{B}^TX\tilde{B})^{-1}B(SB)^{-1}]$ and $v_2=\sigma$. Using (15), we obtain

$$\dot{v}(t)=TAT^{-1}v(t)+TA_d\,v(t-\tau)+TB[I_m+E(x(t))]u(t) \tag{16}$$

Also, (16) can be rewritten as follows

$$\dot{v}_1(t)=\tilde{B}^TAX\tilde{A}B(\tilde{B}^TX\tilde{B})^{-1}v_1(t)+\tilde{B}^TA\,B(SB)^{-1}\sigma(t)$$
$$+\tilde{B}^TA_d\,X\tilde{B}(\tilde{B}^TX\tilde{B})^{-1}v_1(t-\tau)$$
$$+\tilde{B}^TA_d\,B(SB)^{-1}\sigma(t-\tau) \tag{17a}$$

$$\dot{\sigma}(t)=SAX\tilde{B}(\tilde{B}^TX\tilde{B})^{-1}v_1(t)+SAB(SB)^{-1}\sigma(t)$$
$$+SA_d\,X\tilde{B}(\tilde{B}^TX\tilde{B})^{-1}v_1(t-\tau)+SA_d\,B(SB)^{-1}\sigma(t-\tau)$$
$$+SB[I_m+E(x(t))]u(t) \tag{17b}$$

From (17), we can see that, if $\sigma(t-\tau)=\sigma(t)=\dot{\sigma}(t)=0$ then the reduced-order dynamics restricted to the sliding surface $\sigma(t)=0$ is given by (8). By using the parameterized model transformation [14], the reduced-order equivalent system (8) can be written as

$$\frac{d}{dt}[v_1(t)+\bar{K}\int_{t-\tau}^{t}v_1(s)ds]=(\bar{A}_{11}+\bar{K})v_1(t)$$
$$+(\bar{A}_{12}-\bar{K})v_1(t-\tau) \tag{18}$$

where $\bar{K}$ is a matrix parameter to be chosen.

Construct a Lyapunov-Krasovskii functional as

$$V(v_{1t})=V_1(v_{1t})+V_2(v_{1t})+V_3(v_{1t}) \tag{19}$$

where

$$V_1(v_{1t})=L^T(v_{1t})P_1L(v_{1t}),\ L(v_{1t})=v_1(t)+\bar{K}\int_{t-\tau}^{t}v_1(s)ds \tag{20a}$$

$$V_2(v_{1t})=\int_{t-\tau}^{t}v_1^T(s)\,Q_1v_1(s)ds \tag{20b}$$

$$V_3(v_{1t})=\int_{t-\tau}^{t}\int_{\theta}^{t}v_1^T(s)W_1v_1(s)dsd\theta \tag{20c}$$

The time derivative of $V(v_{1t})$ along the trajectory of system (18) is given by

$$\dot{V}(v_{1t})=\dot{V}_1(v_{1t})+\dot{V}_2(v_{1t})+\dot{V}_3(v_{1t})+\dot{V}_4(v_{1t}) \tag{21}$$

where

$$\dot{V}_1(v_{1t})=v_1^T(t)\{P_1[\tilde{B}^TAX\tilde{B}(\tilde{B}^TX\tilde{B})^{-1}+\bar{K}]$$
$$+[\tilde{B}^TAX\tilde{B}(\tilde{B}^TX\tilde{B})^{-1}+\bar{K}]^TP_1\}v_1(t)$$
$$+2v_1^T(t)P_1[\tilde{B}^TA_d\,X\tilde{B}(\tilde{B}^TX\tilde{B})^{-1}-\bar{K}]v_1(t-\tau)$$
$$+\int_{t-\tau}^{t}2v_1^T(s)\bar{K}^TP_1[\tilde{B}^TAX\tilde{B}(\tilde{B}^TX\tilde{B})^{-1}+\bar{K}]v_1(t)ds$$
$$+\int_{t-\tau}^{t}2v_1^T(s)\bar{K}^TP_1[\tilde{B}^TA_dX\tilde{B}(\tilde{B}^TX\tilde{B})^{-1}-\bar{K}]v_1(t-\tau)ds \tag{22a}$$

$$\dot{V}_2(v_{1t})=v_1^T(t)\,Q_1v_1(t)-v_1^T(t-\tau)\,Q_1v_1(t-\tau)$$
$$=\int_{t-\tau}^{t}\frac{1}{\tau}[v_1^T(t)Q_1v_1(t)-v_1^T(t-\tau)Q_1v_1(t-\tau)]\,ds \tag{22b}$$

$$\dot{V}_3(v_{1t})=\tau\,v_1^T(t)\,W_1v(t)-\int_{t-\tau}^{t}v_1^T(s)W_1v_1(s)\,ds$$
$$=\int_{t-\tau}^{t}\frac{1}{\tau}[\tau\,v_1^T(t)W_1v_1(t)-\tau\,v_1^T(s)W_1v_1(s)]\,ds \tag{22c}$$

Let

$$\bar{K}=\tilde{B}^TX\tilde{B}\ ,\ P_1=(\tilde{B}^TX\tilde{B})^{-1} \tag{23}$$

Substituting (22) and (23) into (21) yields

$$\dot{V}(v_{1t})=\int_{t-\tau}^{t}\frac{1}{\tau}[v_1^T(t)(\tilde{B}^TX\tilde{B})^{-1}v_1^T(t-\tau)(\tilde{B}^TX\tilde{B})^{-1}v_1^T(s)(\tilde{B}^TX\tilde{B})^{-1}]$$
$$\times N\begin{bmatrix}(\tilde{B}^TX\tilde{B})^{-1}v_1(t)\\ (\tilde{B}^TX\tilde{B})^{-1}v_1(t-\tau)\\ (\tilde{B}^TX\tilde{B})^{-1}v_1(s)\end{bmatrix}ds \tag{24}$$

Hence, it is easy to see that $\dot{V}(v_{1t})<0$ if $N<0$. It is obvious that $N<0$ if and only if (3a) holds. By using (23a), (3b) and Lemma 2, this implies that operator $L(v_{1t})$ is stable. Therefore, we conclude that systems (8) and (18) are both asymptotically stable. Thus, the proof is completed.

Remark 1

If the time-delay system (1) satisfying Assumption 3 is linear and not bilinear, i.e. system (1) becomes

$$\dot{x}(t)=Ax(t)+A_d\,x(t-\tau)+Bu(t) \tag{25}$$

then control law is given by

$$u(t)=-\frac{\sigma(t)}{\|\sigma(t)\|}\Phi(x(t),\ x(t-\tau),\ t) \tag{26}$$

where $\Phi(x(t),\ x(t-\tau),\ t)$ is the same as in (7), makes the sliding surface $\sigma(t)=0$ stable and globally attractive in finite time. And the reduced-order system, which is the same as in (8), restricted to the sliding surface $\sigma(t)=0$ is asymptotically stable for any constant time delay τ satisfying LMIs (3a) and (3b), The proof is similar to the proof of Theorem 1.

Theorem 1 can be easily extended to the time-varying delay case. Consider the following bilinear system

$$\dot{x}(t)=Ax(t)+A_d\,x(t-\tau(t))+[B+C(x(t))]u(t) \tag{27a}$$

$$x(t)=\Psi(t),\ t\in[-h,\ 0] \quad (27b)$$

where the time-varying delay satisfies $0\leq\tau(t)\leq h$ and $\dot{\tau}(t)\leq d<1$.

In this case, we have the following result.

Theorem 2

Let Assumptions 1-3 hold. Then, given a scalar $\beta>0$, with the chosen sliding surface (5) and the control law

$$u(t)=-[I_m+E(x(t))]^{-1}\frac{\sigma(t)}{\|\sigma(t)\|}$$

$$\times[\beta+\|(SB)^{-1}SAx(t)\|+\|(SB)^{-1}SA_d\|\sup_{\theta\in[-h,0]}\|x(t+\theta)\|] \quad (28)$$

system (27) achieves the desired sliding mode in finite time. And the resulting $(n-m)$ reduced-order dynamics in the sliding mode is described by

$$\dot{v}_1(t)=\bar{A}_{11}v_1(t)+\bar{A}_{12}v_1(t-\tau(t)) \quad (29)$$

where $\bar{A}_{11}$ and $\bar{A}_{12}$ are the same as in (9b), and is asymptotically stable for any time delay $\tau(t)$ satisfying $0\leq\tau(t)\leq h$ and $\dot{\tau}(t)\leq d<1$ if there exist a symmetric positive definite matrix *X*, and positive scalars μ_1, μ_2 solving the following LMI:

$$\Omega^T\begin{bmatrix} U_{11} & A_d-X\tilde{B}\tilde{B}^TX & hX\tilde{B} & hX\tilde{B} \\ XA_d^T-X\tilde{B}\tilde{B}^TX & -U_{22} & 0 & 0 \\ h\tilde{B}^TX & 0 & -h\mu_1 I_{n-m} & 0 \\ h\tilde{B}^TX & 0 & 0 & -h\mu_2 I_{n-m} \end{bmatrix}\Omega<0 \quad (30)$$

where

$$\Omega=\begin{bmatrix} \tilde{B} & 0 & 0 & 0 \\ 0 & \tilde{B} & 0 & 0 \\ 0 & 0 & I_{n-m} & 0 \\ 0 & 0 & 0 & I_{n-m} \end{bmatrix} \quad (31a)$$

$$U_{11}=AX+XA^T+2X\tilde{B}\tilde{B}^TX+h\mu_1 XA^T\tilde{B}\tilde{B}^TAX+X \quad (31b)$$

$$U_{12}=(1-d)\tilde{B}^TX\tilde{B}-\frac{h\mu_2}{1-d}\tilde{B}^TXA_d^T\tilde{B}\tilde{B}^TA_dX\tilde{B} \quad (31c)$$

and $\tilde{B}$ is an orthogonal complement of *B*; I_{n-m} is the $(n-m)\times(n-m)$ identity matrix.

Proof

As previously, the proof is achieved in two parts: attractivity of the sliding surface $\sigma(t)=0$ and stability of the reduced-order equivalent system. The proof of the first part follows the same steps as the proof of Theorem 1 by using the control law (28), so we will not repeat it here. Next, the proof of the second part is the following.

Once in sliding mode, the equations $\sigma(t-\tau(t))=\sigma(t)=\dot{\sigma}(t)=0$ lead to the reduced-order system (29). By using the parameterized model transformation, the reduced-order system (29) can be rewritten in the form

$$\dot{v}_1(t)=(\bar{A}_{11}+\bar{K})v_1(t)+(\bar{A}_{12}-\bar{K})v_1(t-\tau(t))$$

$$-\bar{K}\int_{t-\tau(t)}^{t}[\bar{A}_{11}v_1(s)+\bar{A}_{12}v_1(s-\tau(s))]ds \quad (32)$$

where $\bar{K}$ is a matrix parameter to be chosen.

Let the Lyapunov-Krasovskii functional be

$$V(v_{1t})=V_1(v_{1t})+V_2(v_{1t})+V_3(v_{1t})+V_4(v_{1t}) \quad (33)$$

where

$$V_1(v_{1t})=v_1^T(t)P_1v_1(t) \quad (34a)$$

$$V_2(v_{1t})=\int_{t-\tau(t)}^{t}v_1^T(s)Q_1v_1(s)ds \quad (34b)$$

$$V_3(v_{1t})=\int_{t-\tau}^{t}\int_{\theta}^{t}\mu_1 v_1^T(s)(\bar{A}_{11}+\bar{K})^T(\bar{A}_{11}+\bar{K})v_1(s)dsd\theta \quad (34c)$$

$$V_4(v_{1t})=\int_{t-\tau(t)}^{t}\int_{\theta}^{t}\frac{\mu_2}{1-d}v_1^T(s-\tau(s))(\bar{A}_{12}-\bar{K})^T(\bar{A}_{12}-\bar{K})\times v_1(s-\tau(s))dsd\theta \quad (34d)$$

The time derivative of $V(v_{1t})$ along the trajectory of system (32) is given by

$$\dot{V}(v_{1t})=\dot{V}_1(v_{1t})+\dot{V}_2(v_{1t})+\dot{V}_3(v_{1t})+\dot{V}_4(v_{1t}) \quad (35)$$

where

$$\dot{V}_1(v_{1t})=v_1^T(t)[P_1(\bar{A}_{11}+\bar{K})+(\bar{A}_{11}+\bar{K})^TP_1]v_1(t)+2v_1^T(t)P_1(\bar{A}_{12}-\bar{K})v_1(t-\tau(t))$$
$$-\int_{t-\tau(t)}^{t}2v_1^T(t)P_1\bar{K}\bar{A}_{11}v_1(s)ds-\int_{t-\tau(t)}^{t}2v_1^T(t)P_1\bar{K}\bar{A}_{12}v_1(s-\tau(s))ds \quad (36a)$$

$$\dot{V}_2(v_{1t})=v_1^T(t)Q_1v_1(t)-[1-\dot{\tau}(t)]v_1^T(t-\tau(t))Q_1v_1(t-\tau(t))$$
$$\leq v_1^T(t)Q_1v_1(t)-(1-d)v_1^T(t-\tau(t))Q_1v_1(t-\tau(t)) \quad (36b)$$

$$\dot{V}_3(v_{1t})=h\mu_1v_1^T(t)\bar{A}_{11}^T\bar{A}_{11}v_1(t)-\int_{t-h}^{t}\mu_1v_1^T(s)\bar{A}_{11}^T\bar{A}_{11}v_1(s)ds \quad (36c)$$

$$\dot{V}_4(v_{1t})=\frac{\mu_2\tau(t)}{1-d}v_1^T(t-\tau(t))\bar{A}_{12}^T\bar{A}_{12}v_1(t-\tau(t))-\frac{\mu_2[1-\dot{\tau}(t)]}{1-d}\int_{t-\tau(t)}^{t}v_1^T(s-\tau(s))\bar{A}_{12}^T\bar{A}_{12}v_1(s-\tau(s))ds$$
$$\leq\frac{h\mu_2}{1-d}v_1^T(t-\tau(t))\bar{A}_{12}^T\bar{A}_{12}v_1(t-\tau(t))-\int_{t-\tau(t)}^{t}\mu_2v_1^T(s-\tau(s))\bar{A}_{12}^T\bar{A}_{12}v_1(s-\tau(s))ds \quad (36d)$$

By using Lemma 3, we have

$$-\int_{t-\tau(t)}^{t}2v_1^T(t)P_1\bar{K}\bar{A}_{11}v_1(s)ds$$
$$\leq\int_{t-\tau(t)}^{t}[\mu_1^{-1}v_1^T(t)P_1\bar{K}\bar{K}^TP_1v_1(t)+\mu_1v_1^T(s)\bar{A}_{11}^T\bar{A}_{11}v_1(s)]ds$$
$$\leq h\mu_1^{-1}v_1^T(t)P_1\bar{K}\bar{K}^TP_1v_1(t)+\int_{t-h}^{t}\mu_1v_1^T(s)\bar{A}_{11}^T\bar{A}_{11}v_1(s)ds \quad (37a)$$

$$-\int_{t-\tau(t)}^{t} 2v_1^T(t)P_1\bar{K}\bar{A}_{12}v_1(s-\tau(s))\,ds$$
$$\le \int_{t-\tau(t)}^{t}[\mu_2^{-1}v_1^T(t)P_1\bar{K}\bar{K}^T P_1 v_1(t)$$
$$+\mu_2 v_1^T(s-\tau(s))\bar{A}_{12}^T\bar{A}_{12}v_1(s-\tau(s))]ds$$
$$\le h\mu_2^{-1}v_1^T(t)P_1\bar{K}\bar{K}^T P_1 v_1(t)$$
$$+\int_{t-\tau(t)}^{t}\mu_2 v_1^T(s-\tau(s))\bar{A}_{12}^T\bar{A}_{12}v_1(s-\tau(s))\,ds \quad (37b)$$

Let

$$\bar{K}=\tilde{B}^T X\tilde{B},\ P_1=Q_1=(\tilde{B}^T X\tilde{B})^{-1} \quad (38)$$

Substituting (9b), (36), (37) and (38) into (35) yields

$$\dot{V}(v_{1t})=[v_1^T(t)\,(\tilde{B}^T X\tilde{B})^{-1}\ v_1^T(t-\tau(t))\,(\tilde{B}^T X\tilde{B})^{-1}] \times\hat{N}\begin{bmatrix}(\tilde{B}^T X\tilde{B})^{-1}v_1(t)\\ (\tilde{B}^T X\tilde{B})^{-1}v_1(t-\tau(t))\end{bmatrix} \quad (39)$$

where

$$\hat{N}=\begin{bmatrix} U & \tilde{B}^T(A_d X-X\tilde{B}\tilde{B}^T X)\tilde{B} \\ \tilde{B}^T(XA_d^T-X\tilde{B}\tilde{B}^T X)\tilde{B} & \begin{pmatrix}-(1-d)\tilde{B}^T X B+\dfrac{h\mu_2}{1-d}\\ \times\tilde{B}^T X A_d^T\tilde{B}\tilde{B}^T A_d X\tilde{B}\end{pmatrix}\end{bmatrix} \quad (40)$$

where

$$U=\tilde{B}^T(AX+XA^T+2X\tilde{B}\tilde{B}^T X+h\mu_1^{-1}X\tilde{B}\tilde{B}^T X$$
$$+h\mu_1^{-1}X\tilde{B}\tilde{B}^T X+h\mu_1 XA^T\tilde{B}\tilde{B}^T AX+X\ )\tilde{B}$$

Therefore, it is easy to see that $\dot{V}(v_{1t})<0$ if $\hat{N}<0$. Moreover, using Lemma 4, $\hat{N}<0$ if and only if (30) holds. Then, we conclude that systems (29) and (32) are both asymptotically stable. Thus, the proof is completed.

IV. EXAMPLE

Consider the following bilinear system with time delay in state and initial condition $x(0)=[1.5\ -1]^T$ and $x(t)=0$ for $t<0$.

$$\dot{x}(t)=Ax(t)+A_d x(t-\tau)+B[u(t)+E(x(t))u(t)]$$

where

$$A=\begin{bmatrix}0&1\\1&0\end{bmatrix},\ A_d=\begin{bmatrix}-1&0\\-1&-1\end{bmatrix},\ B=\begin{bmatrix}0\\2\end{bmatrix},\ \tau=1,$$
$$E(x(t))=0.1x_1(t)+0.5x_2(t)\,.$$

Since a basis of the null space of B^T can be given as $[1\ 0]^T$, we can define $\tilde{B}$ as $\tilde{B}^T=[1\ 0]$. Using the LMI condition (3), we have a feasible solution M_1, Q_1, W_1, X, δ as follows:

$$M_1=\begin{bmatrix}3.2857&0.5727\\0.5727&6.8027\end{bmatrix},\ Q_1=\begin{bmatrix}2.0904&-0.3138\\-0.3138&1.3579\end{bmatrix},$$
$$W_1=\begin{bmatrix}4.2419&2.0425\\2.0425&4.4669\end{bmatrix},\ X=\begin{bmatrix}1&-2\\-2&6\end{bmatrix},\ \delta=0.5\,.$$

Therefore, we can see that there exists a sliding surface guaranteeing the asymptotic stability of the reduced-order dynamics restricted to the sliding surface. According to the explicit formula (5), we can easily get a sliding surface

$$\sigma(t)=Sx(t)=B^T X^{-1}x(t)=[2\ \ 1]x(t)$$

Based on (6), the VSC law can be obtained as follows

$$u(t)=-\frac{\sigma(t)}{\|\sigma(t)\|}[1+E(x(t))]^{-1}[0.5\|SAx(t)\|+0.5\|SA_d x(t-\tau)\|+1] \quad (41)$$

and from (8), we can see that the reduced-order equivalent dynamics restricted to the sliding surface $\sigma(t)=0$ is given by

$$\dot{v}_1(t)=-2v_1(t)-v_1(t-\tau),\ v_1(t)=[1\ \ 0]x(t)$$

which implies that the reduced-order dynamics is asymptotically stable for time delay $\tau=1$. Fig. 1 shows the transient of x_1 and x_2 of the bilinear system controlled by above control law (41). Fig. 2 demonstrates the performance of x_1 and x_2 of the linear system is still well controlled by above control law (41) with $E(x(t))=0$. It is confirmed that the proposed VSC law is suitable for bilinear or linear time-delay system.

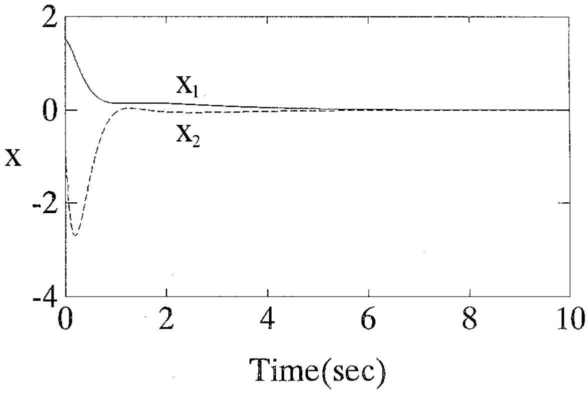

Fig.1 State variable dynamics for $\dot{x}(t)=Ax(t)$ $+A_d x(t-\tau)+B[u(t)+E(x(t))u(t)]$ controlled by the controller (41)

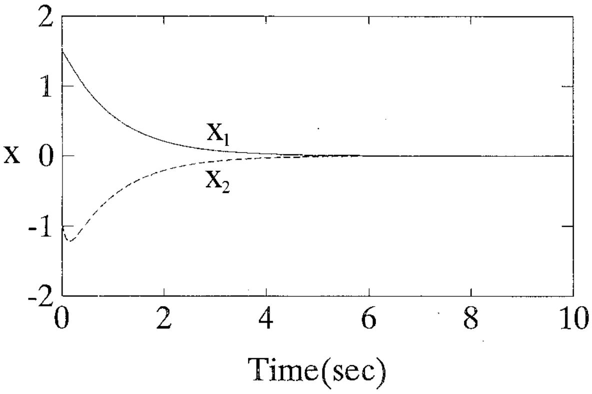

Fig.2 State variable dynamics for $\dot{x}(t)=Ax(t)$ $+A_d x(t-\tau)+Bu(t)$ controlled by the controller (41) with $E(x(t))=0$

V. CONCLUSION

A stabilizing sliding mode control approach has been developed for the bilinear systems with time delay in state. By means of LMI approach and parameterized model transformation technique, we derived new sufficient conditions for the existence of sliding surface guaranteeing asymptotic stability of the reduced-order equivalent system restricted to the sliding surface. In this study, it shows sliding mode controller design to guarantee the global reaching condition of the sliding mode for the bilinear systems with constant or time-varying delay. The main results can also be extended to the bilinear systems with multiple discrete and distributed time delays or nonlinear perturbations. These will be the subjects of further investigations. Finally, we remark that the LMI-based approach combining with parameterized model transformation to SMC design has advantages over the conventional SMC design methods.

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