Local convergence analysis of Inexact Newton method with relative residual error tolerance under majorant condition in Riemannian Manifolds

T. Bittencourt * O. P. Ferreira†

January 9, 2014

Abstract

A local convergence analysis of Inexact Newton’s method with relative residual error tolerance for finding a singularity of a differentiable vector field defined on a complete Riemannian manifold, based on majorant principle, is presented in this paper. We prove that under local assumptions, the inexact Newton method with a fixed relative residual error tolerance converges $Q$-linearly to a singularity of the vector field under consideration. Using this result we show that the inexact Newton method to find a zero of an analytic vector field can be implemented with a fixed relative residual error tolerance. In the absence of errors, our analysis retrieve the classical local theorem on the Newton method in Riemannian context.

Keywords: Inexact Newton’s method, majorant principle, local convergence analysis, Riemannian manifold.

---

*IME/UFG, CP-131, CEP 74001-970 - Goiânia, GO, Brazil (Email: tiberio.b@gmail.com). This author was supported by CAPES.
†IME/UFG, CP-131, CEP 74001-970 - Goiânia, GO, Brazil (Email: orizon@mat.ufg.br). This author was supported by CNPq Grants 302024/2008-5, 480101/2008-6 and 473756/2009-9, PRONEX-Optimization(FAPERJ/CNPq) and FUNAPE/UFG.
Our goal is to prove in Riemannian manifold context the following version of Inexact Newton method with relative residual error tolerance under majorant condition.

**Theorem 1.** Let $M$ be a Riemannian manifold, $\Omega \subseteq M$ an open set and $X : \Omega \to TM$ a continuously differentiable vector field. Let $p_\ast \in \Omega$, $R > 0$ and $\kappa := \text{sup}\{t \in [0, R) : B_t(p_\ast) \subset \Omega\}$. Suppose that $X(p_\ast) = 0$, $\nabla X(p_\ast)$ is invertible and there exists an $f : [0, R) \to \mathbb{R}$ continuously differentiable such that
\[
\|\nabla X(p_\ast)^{-1}[P_{\zeta_0, 1, 0} \nabla X(p) - P_{\zeta, 0} \nabla X(\zeta(\tau))P_{\zeta_0, 1, \tau}]\| \leq f'(d(p_\ast, p)) - f'(\tau d(p_\ast, p)),
\]
for all $\tau \in [0, 1]$, $p \in B_\kappa(p_\ast)$, where $\zeta : [0, 1] \to M$ is a minimizing geodesic from $p_\ast$ to $p$ and

- **h1)** $f(0) = 0$ and $f'(0) = -1$;
- **h2)** $f'$ is strictly increasing.

Let $0 \leq \vartheta < 1/K_{p_\ast}$, $\nu := \text{sup}\{t \in [0, R) : f'(t) < 0\}$, $\rho := \text{sup}\{\delta \in (0, \nu) : [(1 + \vartheta)|t - f(t)/f'(t)|/t + \vartheta] < 1/K_{p_\ast}, t \in (0, \delta)\}$ and
\[r := \text{min}\{\kappa, \rho, r_{p_\ast}\}.
\]

Then the sequence generated by the Inexact Newton method for solving $X(p) = 0$ with starting point $p_0 \in B_r(p_\ast) \setminus \{p_\ast\}$ and residual relative error tolerance $\theta$,
\[p_{k+1} = \exp_{p_k}(S_k), \quad \|X(p_k) + \nabla X(p_k)S_k\| \leq \theta\|X(p_k)\|, \quad k = 0, 1, \ldots,
\]
\[0 \leq \text{cond}(\nabla X(p_k))\theta \leq \vartheta/\left[2/|f'(d(p_\ast, p_0))| - 1\right],
\]
is well defined (for any particular choice of each $S_k \in T_{p_k}M$), the sequence $\{p_k\}$ is contained in $B_r(p_\ast)$ and converges to the point $p_\ast$ which is the unique zero of $X$ in $B_\kappa(p_\ast)$, where $\sigma := \text{sup}\{t \in (0, \kappa) : f(t) < 0\}$, and we have that:
\[d(p_\ast, p_{k+1}) \leq K_{p_\ast} \left[(1 + \vartheta)\left|\frac{d(p_\ast, p_k) - f(d(p_\ast, p_k))}{f'(d(p_\ast, p_k))}\right| + \vartheta\right] d(p_\ast, p_k), \quad k = 0, 1, \ldots
\]
and $\{p_k\}$ converges linearly to $p_\ast$. If, in additional, the function $f$ satisfies the following condition
- **h3)** $f'$ is convex,

then there holds
\[d(p_\ast, p_{k+1}) \leq K_{p_\ast} \left[(1 + \vartheta)\left|\frac{d(p_\ast, p_0) - f(d(p_\ast, p_0))}{f'(d(p_\ast, p_0))}\right| d^2(p_\ast, p_0) + \vartheta\right] d(p_\ast, p_k), \quad k = 0, 1, \ldots
\]
as a consequence, the sequence \( \{p_k\} \) converges to \( p^* \) with linear rate as follows

\[
d(p^*, p_{k+1}) \leq K_p \left[ (1 + \vartheta) \left| \frac{d(p^*, p_0)}{d(p^*, p_{k+1})} \right| \right] \left[ d(p^*, p_0) - \frac{f(d(p^*, p_0))}{F(d(p^*, p_0))} \right] + \vartheta \left[ d(p^*, p_k) \right], \quad k = 0, 1, \ldots
\]