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Measuring Social Tension from Income Class Segregation

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Abstract

We develop an index that effectively measures the level of social tension generated by income class segregation. We adopt the basic concepts of between-group difference (or alienation) and within-group similarity (or identification) from the income [bi]polarization literature; but we allow for asymmetric degrees of between-group antagonism in the alienation function, and construct a more effective identification function using both the relative degree of within-group clustering and the group size. To facilitate statistical inference, we derive the asymptotic distribution of the proposed measure using results from U-statistics. As the new measure is general enough to include existing income polarization indices as well as the Gini index as special cases, the asymptotic result can be readily applied to these popular indices. Evidence from the Panel Study of Income Dynamics data suggests that, while the level of social tension shows an upward trend over the sample period of 1981 to 2005, government's taxes and transfers have been effective in reducing the level of social tension significantly.

Keywords and phrases: social tension, asymmetric antagonism, relative clustering, income polarization, U-statistics.

JEL classification: C13; D31; D63; I32.

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1 Introduction

In development economics, social cost of political instability generated by conflicts between subgroups has been regarded as the major constraint to economic growth and human development (e.g., Collier, 1999; Fearon and Laitin, 2003). In sociology, measuring the level of social tension is of interest as it is related to crime, collective action, riots, social conflict, or civil war (e.g., D'Ambrosio and Wolff, 2001; Reynal-Querol, 2002). Despite its importance, little attention has been paid to developing a measure that represents the level of tension inherent in a society. Although some studies have contributed to measuring the degree of social segregation or conflict, they are somewhat limited for the purpose of representing the level of social tension because they mostly focus on the statistical aspect of the observation (e.g., income distribution) without considering the psychological aspect. For example, Esteban and Ray (1999) investigate the shape of income distribution that is highly correlated with social conflict; Hutchens (2004) considers social segregation using a generalized entropy index of income; and Echenique and Fryer (2007) develop a measure of segregation based on social interactions.

This paper contributes to this area of studies by developing a direct and effective measure of the level of social tension that is generated by income class segregation. For this purpose, we adopt the basic concepts of between-group difference (or alienation) and within-group similarity (or identification) from the income [bi]polarization literature (e.g., Esteban and Ray, 1994; Wolfson, 1994; Duclos, Esteban and Ray, 2004; Foster and Wolfson, 2010). The new measure is, however, different from existing income polarization indices in that it emphasizes the psychological as well as the statistical (or economic) aspects of an income distribution. More precisely, we allow for asymmetric feeling on the income gap between the rich and the poor, assuming that the poor feel a higher level of alienation against the rich than vice versa. Furthermore, we construct a more effective group identification function by considering both the relative group size and the relative degree of group-specific income clustering, assuming that a person feels stronger identification to the members of her group either when the relative size of her own group gets larger or when her within-group income distribution gets less dispersed relative to the entire distribution. In comparison, existing income polarization indices normally assume symmetric between-group alienation and represent within-group identification using either the absolute level of within-group clustering (e.g., Wolfson, 1994) or the group population proportion (e.g., Esteban and Ray, 1994). Since it is quite plausible to assume that the concept of social tension depends in large part on the psychological aspect, considering both asymmetry and relative identification feeling is important in this context. We note that, if this new measure of social tension is instead interpreted as an income polarization index, it is general enough to include the income polarization indices by Esteban and Ray (1994), Wolfson (1994), and Duclos, Esteban and Ray (2004), as well as the Gini index as special cases. It also satisfies the basic axioms commonly adopted in the income [bi]polarization literature.

To facilitate statistical inferences, we derive the limiting distribution of the proposed measure using results from U-statistics, which generalizes Bishop, Formby and Zheng (1997). As the new measure includes the aforementioned income polarization and the Gini indices as special cases, this limiting distribution result can be naturally extended to the distributions of these popular indices. This also makes an important contribution to the income polarization literature, considering that, despite repeated reports that income distribution has become more polarized recently (e.g., Gradin, 2000; Gradin and Rossi, 2006; Esteban, Gradin and Ray, 2007; Hussain, 2009), few studies provide formal statistical conclusions because of the lack of the distribution results of those indices. Note that the limiting distribution in this paper relies on the asymptotics when the number of groups is fixed but the sample size of each group goes to infinity as the entire sample size increases. This limiting distribution is hence different from that of Duclos, Esteban and Ray (2004) or Anderson (2011), where they consider the asymptotics as the number of groups goes to infinity. Understanding that the latter asymptotics can be useful when the group specification is unclear, we additionally provide the asymptotics of our measure under such a case as well.

The remainder of the paper is organized as follows. Section 2 develops a measure of the level of

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social tension that represents aggregate effective antagonism between two income groups. Detailed explanation is provided on how to incorporate the aforementioned asymmetry into the measure and how to represent the group identification function effectively for our purpose. Section 3 interprets this new measure as a generalized income polarization index and compares it with the indices of Wolfson (1994) and Esteban and Ray (1994). It shows that the new measure combines and extends these two types of indices; it also satisfies the common properties/axioms adopted in the income [bi]polarization literature. Section 4 considers estimation of the new measure and develops its limiting distribution. As an extension, Section 5 deals with the case of increasing number of groups, where the group specification is not required. As an illustration, Section 6 obtains the trajectory of the social tension level in the U.S. using the Panel Study of Income Dynamics data, and compares the levels of social tension before and after adjusting government taxes and transfers. Section 7 concludes the paper with some remarks. All technical proofs and additional results of asymptotic distribution are provided in the Appendix, in which an easy-to-implement jackknife-based variance estimation algorithm is also summarized.

2 Social Tension from Formation of Two Income Classes

We formulate the level of social tension generated by segregation of a population into two income subgroups. The definition of groups can vary depending on context, such as income level, gender, race, political parties, ethnicity, and religion. In this paper, we consider the income level to define groups, whose properties are easier to quantify relative to other characteristics. Moreover, although we will discuss the case of multiple groups in subsequent sections, we primarily focus on the tension between two competing groups. Of course, a two-group representation may not reflect the actual state of the population. However, what counts in the measurement of social tension is how close the population is to the state of formation of two income subgroups, rather than how many groups the population is actually segregated into. This is because, with other things being equal, between-group tension is more likely to increase as the number of groups decreases and the size

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of each group increases (e.g., Esteban and Ray, 1994). Esteban and Ray (1999, p.381) find that the emergence of a 'twin-peaks property' in income distribution is related closely to rising social conflict. Esteban, Gradin and Ray (2007) also observe that their extended income polarization measure is maximized with two groups in their empirical example with the Luxemburg Income Study data. In addition, Bossert and Schworm (2008, p.1171) note that the fundamental axioms that capture the spirit of income polarization provided by Wolfson (1994) and Esteban and Ray (1994) lose their intuitive appeal when allowing for more than two groups.

More specifically, using two income groups, the rich and the poor, we develop a measure of the level of social tension based on the between-group difference (or alienation) and the within-group similarity (or identification) concepts. This approach is in a similar vein to the income [bi]polarization indices of Esteban and Ray (1994) and Wolfson (1994). Unlike these income polarization indices, however, we incorporate the psychological aspect of the economic agents into the measure so that it can gauge the level of social tension more effectively.

We let T_k denote the degree of 'group-antagonism' of group k against the other, where the group index $k = 1$ stands for the poor and $k = 2$ for the rich. We assume T_k as some differentiable function of Δ and ϕ_k:

$$T_k = T_k(\Delta, \phi_k) \text{ for } k = 1, 2,$$

where Δ represents the between-group income distance and ϕ_k is the level of within-group identification of group k. We further suppose $\partial T_k/\partial\Delta > 0$ and $\partial T_k/\partial\phi_k > 0$ for both $k = 1, 2$, implying that the group-antagonism increases with between-group distance and within-group identification, respectively. However, the magnitudes of $\partial T_1/\partial\Delta$ and $\partial T_2/\partial\Delta$ can be quite different. Psychologically, it is plausible to assume $\partial T_1/\partial\Delta > \partial T_2/\partial\Delta$, which implies that, as the income distance between the two groups increases, the antagonism the poor have against the rich increases by a greater extent than the antagonism the rich have against the poor. One possible interpretation of this asymmetry assumption is based on between-group differences in the expectation of future income mobility: As the income gap increases, the poor lower their expectation of future upward

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income mobility and hence have higher relative deprivation feeling, resulting in an increased level of antagonism against the rich. At the same time, although the rich also raise their level of antagonism against the poor in the fear that the poor have more incentive to commit individual or collective crime against the rich, its increment is likely to be smaller than that of the poor. It is because the increased income distance makes the rich lower their expectation of future downward income mobility and hence they feel their life-time income is more secured.

Allowing that, for a given income distance, the poor feel a higher level of antagonism against the rich than vice versa, we formulate the social tension measure as follows:

$$\mathcal{S}(\alpha,\theta) = \pi_1\pi_2\{T_1 + T_2\} \equiv \pi_1\pi_2\{(1-\theta)\Delta\phi_1 + \theta\Delta\phi_2\}, \tag{1}$$

in which the between-group income distance and the level of within-group identification interactively raise the level of social tension. Here, θ is some weight such that $0 \leq \theta \leq 0.5$, and π_1 and π_2 are group population proportions. The parameter θ captures the asymmetric psychological aspect between the two groups. Because of this asymmetry, even when the levels of within-group identification are the same (i.e., $\phi_1 = \phi_2$), the poor contribute more to the level of social tension than do the rich. Note that $\pi_1\pi_2$ captures the distributional effect of relative group size on the level of social tension: with other things being held constant, the overall tension is maximized when the population is equally distributed between two groups (i.e., $\pi_1 = \pi_2 = 0.5$).

In order to further specify the between-group income gap Δ and the level of within-group identification ϕ_k, we consider a set of individual income data $\{y_i\}_{i=1}^n$ that is a random sample from an underlying distribution $F(y)$ whose support is $[y_{\min}, y_{\max}]$ with $0 < y_{\min} < y_{\max} < \infty$. We consider two income groups $A_1 = [y_{\min}, y^*]$ and $A_2 = (y^*, y_{\max}]$ for some cutoff point y^*, where y^* is given exogenously. For each group A_k, we define the population fraction π_k, the group mean μ_k, and the group mean difference δ_{kk} as

$$\pi_k = \int_{A_k} dF(y), \ \mu_k = \frac{1}{\pi_k}\int_{A_k} ydF(y), \ \text{and} \ \delta_{kk} = \frac{1}{\pi_k^2}\int_{A_k}\int_{A_k} |x-y|\, dF(x)\, dF(y) \tag{2}$$

for $k = 1, 2$, where $\pi_1, \pi_2 > 0$ and $\pi_1 + \pi_2 = 1$. The overall mean is given as $\mu = \int ydF(y) =$

$\pi_1\mu_1 + \pi_2\mu_2$, and $\mu_1 < \mu_2$ by construction. We also let $\delta = \int\int |x - y|\, dF(x)\, dF(y)$ be the overall mean difference. Using these expressions, we specify that

$$\Delta = \frac{\left|\mu_2 - \mu_1\right|}{\mu} \tag{3}$$

and

$$\phi_k = \left(\frac{\pi_k}{\delta_{kk}/\delta}\right)^{\alpha} \quad \text{for } k = 1, 2 \tag{4}$$

for some $\alpha > 0$, where α is similar to the parameter introduced by Esteban and Ray (1994) that makes their polarization measure distinguished from the inequality measures. The between-group distance Δ is defined as the (normalized) average income gap between two groups, though it can be obtained using other centrality measures such as group-specific median income. The *within-group identification* function ϕ_k is assumed to be positively affected by the *relative group size* π_k but inversely related to the *relative level of within-group income dispersion* δ_{kk}/δ. We could consider other dispersion measures such as variance or the Gini index instead of the mean difference. An individual, when the population proportion of her group gets larger, views income class separation more as a social structural problem and thus feels a higher level of identification to her group members. At the same time, an individual feels stronger group identification when income levels within the group are more clustered relative to the level of overall clustering. Note that we measure the group-specific clustering effect using the relative dispersion δ_{kk}/δ instead of the absolute dispersion δ_{kk}. This specification intends to reflect the following psychological aspect of an income distribution: With the level of income dispersion of the poor given, they would feel stronger within-group identification when the income distribution of the rich were more dispersed and so were the entire distribution.

By combining (1), (3), and (4), we define the measure of social tension, which we call *S-index*,

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as:

$$\begin{aligned}\mathcal{S}(\alpha,\theta) &= \pi_1\pi_2\left\{(1-\theta)\frac{\left|\mu_1-\mu_2\right|}{\mu}\left(\frac{\pi_1}{\delta_{11}/\delta}\right)^{\alpha}+\theta\frac{\left|\mu_2-\mu_1\right|}{\mu}\left(\frac{\pi_2}{\delta_{22}/\delta}\right)^{\alpha}\right\} \\ &= \frac{\pi_1\pi_2}{2\mu}\left\{\rho_\theta(\mu_1-\mu_2)\left(\frac{\pi_1}{\delta_{11}/\delta}\right)^{\alpha}+\rho_\theta(\mu_2-\mu_1)\left(\frac{\pi_2}{\delta_{22}/\delta}\right)^{\alpha}\right\}\end{aligned} \tag{5}$$

for given $\alpha > 0$ and $0 \leq \theta \leq 0.5$ that are chosen by the researcher, where

$$\rho_\theta(\mu_k-\mu_\ell) = 2(\mu_k-\mu_\ell)(\theta-\mathbb{I}\{\mu_k<\mu_\ell\}) \text{ for } k,\ell = 1,2 \tag{6}$$

with $\mathbb{I}\{\cdot\}$ being the binary indicator. In this formulation, the *between-group alienation*, as represented by $\rho_\theta(\mu_k-\mu_\ell)$, reflects not only the statistical (or economic) aspect of an income distribution (i.e., the income distance) but also the psychological one (i.e., the *asymmetric* feelings of alienation of each group toward the other, with the degree of the asymmetry being determined by the value θ). As we let $0 \leq \theta \leq 0.5$, in particular, for a given income distance, the poor feel more alienated from the rich than vice versa; the asymmetry gets more severe as θ goes to zero. When $\theta = 0$ as an extreme case, the rich do not feel any alienation against the poor (e.g., Yitzhaki, 1979). When $\theta = 0.5$, $\rho_\theta(\mu_k-\mu_\ell) = |\mu_k-\mu_\ell|$ and hence the degree of alienation is symmetric between the groups, which corresponds to the case of the Gini index and the standard income polarization indices (e.g., Esteban and Ray, 1994; Wolfson, 1994).[1] Note that the parameter of asymmetric feeling of alienation, θ, is different from the inequality aversion parameter in Atkinson's (1970) index or generalized entropy indices (Cowell and Kuga, 1981; Shorrocks, 1984), which represents the overall inequality aversion level.

[1]Choice of α and θ can be arbitrary as long as they satisfy $\alpha > 0$ and $0 \leq \theta \leq 0.5$. This can be a drawback of $\mathcal{S}(\alpha,\theta)$ as an index, but it can give the researcher a degree of freedom, which reflects her own view on the importance of within-group identification and between-group asymmetry. Such arbitrariness also appears in most income polarization indices (e.g., Esteban and Ray, 1994; Duclos, Esteban and Ray, 2004; Esteban, Gradin and Ray, 2007). Wolfson (1994) index avoids this issue by placing an equal weight between group distance and within-group clustering so that $\alpha = 1$. Considering that the current measure is developed to represent the level of social tension effectively, we could choose α and θ that best explain outcome variables of social tension empirically.

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3 Comparison with Income Polarization Indices

After the seminal paper by Love and Wolfson (1976), there has been increasing interest in the concept and measurement of income bipolarization. Examples include Wolfson (1994, 1997), Wang and Tsui (2000), Bossert and Schworm (2008), Foster and Wolfson (2010), and Aaberge and Atkinson (2013). These studies consider two aspects of income distribution, between-group income distance and within-group individual income clustering, and measure the level of income bipolarization by focusing on the phenomenon of the disappearing middle class and formation of two segregated income groups, the poor and the rich. These studies specify two groups divided by the median income; their primary interest lies in measuring how a society is close to the state of formation of two segregated income classes, rather than describing the actual state of the income distribution. In this sense, the Wolfson type income bipolarization index could serve as a measure of between-group conflicts, though such discussion was not explicit in these studies.

Alternatively, Esteban and Ray (1994, 1999), Duclos, Esteban and Ray (2004), and Esteban, Gradin and Ray (2007) consider multiple groups in their income polarization measures and combine two similar, though not identical, concepts as in the Wolfson type index to measure the level of income polarization: between-group alienation and within-group identification. With between-group alienation measured by the income gap and within-group identification by the group size, these Esteban-Ray type indices are also closely related to the level of social conflict. In fact, Esteban and Ray (1994, p.820) write:

> " ... why are we interested in polarization? ... the phenomenon of polarization is closely linked to the generation of tensions, to the possibilities of articulated rebellion and revolt, and to the existence of social tension in general."

Understanding that the S-index $\mathcal{S}(\alpha,\theta)$ in (5) also relies on the concepts of between-group difference (or alienation) and within-group similarity (or identification), these income [bi]polarization indices are comparable to the S-index, the measure of social tension. In this section, we compare

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the S-index with the aforementioned two types of income polarization indices (i.e., Wolfson type and Esteban-Ray type indices) to elaborate their similarities and differences. When interpreted as an income polarization index instead of a measure of social tension, the S-index is shown to be general enough to include existing [bi]polarization income indices as its special cases.

We first compare $\mathcal{S}(\alpha,\theta)$ with the Wolfson type indices. Wolfson's (1994) index is defined as $W = (\mu/m_y)\{((\mu_2-\mu_1)/\mu) - G\} = ((\mu_2-\mu_1)/m_y) - (\delta/m_y)$, where $G = \delta/\mu$ is the overall Gini index and m_y is the median income (e.g., Foster and Wolfson, 2010, Proposition 5). Since, for two non-overlapping groups, the overall mean difference δ can be expressed as $\delta = \pi_1^2\delta_{11} + \pi_2^2\delta_{22} + 2\pi_1\pi_2\left|\mu_2-\mu_1\right|$ (e.g., Silber, 1989), we can rewrite the Wolfson's (1994) index as[2]

$$W = \frac{1}{m_y}\left\{\left|\mu_2-\mu_1\right| - \frac{1}{4}\left(\delta_{11}+\delta_{22}+2\left|\mu_2-\mu_1\right|\right)\right\} = \frac{1}{2m_y}\left\{\left|\mu_2-\mu_1\right| - \frac{\delta_{11}+\delta_{22}}{2}\right\}, \tag{7}$$

where $(\mu_2-\mu_1) > 0$ and $\pi_1 = \pi_2 = 0.5$. Comparing (5) with (7), we can find that $\mathcal{S}(\alpha,\theta)$ and W share the same overall structure: both indices increase in between-group distance $\left|\mu_2-\mu_1\right|$ and decrease in within-group dispersion δ_{kk} for $k = 1,2$. In fact, if we define $\mathcal{S}(\alpha,\theta)$ based on two income groups divided by the median income m_y, and let $\alpha = 1$ and $\theta = 0.5$, then we can rewrite $\mathcal{S}(\alpha,\theta)$ as a linear transformation of W:

$$\mathcal{S}(1,0.5) = \frac{\mu_2-\mu_1}{16\mu}\left\{\frac{\delta}{\delta_{11}} + \frac{\delta}{\delta_{22}}\right\} = \left(m_yW+\delta\right)\times\frac{1}{16\mu}\left\{\frac{\delta}{\delta_{11}} + \frac{\delta}{\delta_{22}}\right\} \tag{8}$$

since $\pi_1 = \pi_2 = 0.5$ in this case.

On the other hand, between-group difference and within-group similarity are formulated differently in $\mathcal{S}(\alpha,\theta)$ and W; terms in $\mathcal{S}(\alpha,\theta)$ are more general so that they can measure the level of social tension effectively. In particular, while between-group distance affects W in a symmetric way for both groups, $\mathcal{S}(\alpha,\theta)$ allows for asymmetric effects, which includes the symmetric case when $\theta = 0.5$. In addition, while W measures the level of within-group similarity using absolute mean difference δ_{kk}, $\mathcal{S}(\alpha,\theta)$ uses not only relative mean difference δ_{kk}/δ but also relative group size π_k. Also note that, while W considers within-group clustering by $-\delta_{kk}$, $\mathcal{S}(\alpha,\theta)$ considers it by $1/\delta_{kk}$.

[2] A similar interpretation is also noted by Foster and Wolfson (2010, Proposition 6).

Second, if we consider more than two groups, we can compare $\mathcal{S}(\alpha,\theta)$ with the income polarization index by Esteban and Ray (1994) or Duclos, Esteban and Ray (2004). Though the concept of social conflict or social tension becomes less clear as the number of groups increases, this generalization can be useful to understand how to locate the S-index within the income polarization literature. More precisely, we consider K number of pre-specified income groups $\{A_k\}_{k=1}^K$, where $A_k = (a_{k-1}, a_k]$ for $k = 1, 2, \cdots, K$ and $2 \leq K < n$. Without loss of generality, we let $y_{\min} = a_0 < a_1 < \cdots < a_{K-1} < a_K = y_{\max}$ and define the first interval as $[a_0, a_1]$. The number of intervals K is given and it is assumed to be fixed (i.e., not growing with n). Using the same definitions in (2) with $\pi_k > 0$ for all k, we have $\sum_{k=1}^K \pi_k = 1$ and $\mu = \sum_{k=1}^K \pi_k \mu_k$. The group means are in ascending order by construction so that $\mu_k < \mu_\ell$ if $k < \ell$. The S-index in (5) then can be generalized as

$$\mathcal{S}_K(\alpha,\theta) = \frac{1}{2\mu} \sum_{k=1}^K \sum_{\ell=1}^K \pi_k \pi_\ell \left(\frac{\pi_k}{\delta_{kk}/\delta} \right)^\alpha \rho_\theta(\mu_k - \mu_\ell). \tag{9}$$

Similar to the income polarization index of Esteban and Ray (1994) that is defined as $ER(\alpha) = (1/\mu) \sum_{k=1}^K \sum_{\ell=1}^K \pi_k^{1+\alpha} \pi_\ell \left|\mu_k - \mu_\ell\right|$, $\mathcal{S}_K(\alpha,\theta)$ combines between-group alienation $\rho_\theta(\mu_k - \mu_\ell)$ and within-group identification $(\pi_k/(\delta_{kk}/\delta))^\alpha$, though each term is formulated more generally so that they can incorporate some psychological aspects to measure the level of social tension effectively.

For example, the alienation function, $\rho_\theta(\mu_k - \mu_\ell) = 2(\mu_k - \mu_\ell)(\theta - \mathbb{I}\{\mu_k < \mu_\ell\})$, is more general than that of $ER(\alpha)$, in the sense that it allows for asymmetric feelings of alienation, where the degree of asymmetry is determined by the value θ that is homogenous across groups. Therefore, $\mathcal{S}_K(\alpha,\theta)$ as an income polarization index reflects not only the between-group income distance (i.e., the economic aspect of alienation) but also the asymmetric degrees of feeling of each group against the others (i.e., the psychological aspect of alienation). Figure 1 depicts $\rho_\theta(\mu_k - \mu_\ell)$, where the absolute value of the slope determines the degree of asymmetric alienation of group k towards the others. Moreover, the identification function, $\phi_k(\alpha) = (\pi_k/(\delta_{kk}/\delta))^\alpha$, captures relative degree of identification feeling by considering not only the relative size effect π_k but also the relative clustering effect $(\delta_{kk}/\delta)^{-1}$. Therefore, $\mathcal{S}_K(\alpha,\theta)$ can be understood as a generalized polarization index, which includes $ER(\alpha)$ as a special case. When $\theta = 0.5$ (and hence $\rho_\theta(u) = |u|$) and if we define the within-group identification function as $\phi_k = \pi_k^\alpha$, then $2\mathcal{S}_K(\alpha,\theta)$ becomes $ER(\alpha)$; when $\theta = 0.5$ and $\alpha = 0$, it is simply the Gini index based on grouped data.

Discussions above show that $\mathcal{S}(\alpha,\theta)$, if we see it as an income [bi]polarization index, includes the Wolfson type and Esteban-Ray type indices as its special cases under some specific values of (α,θ). In this regard, it can be meaningful to verify if $\mathcal{S}(\alpha,\theta)$ also satisfies the key characteristics of these income [bi]polarization indices for more general (α,θ). In the following theorem, more precisely, we consider the four most popular properties of income [bi]polarization indices that are studied in the literature (e.g., Wang and Tsui, 2000; Duclos, Esteban and Ray, 2004; Bossert and Schworm, 2008; Foster and Wolfson, 2010; Esteban and Ray, 2012) and verify that $\mathcal{S}(\alpha,\theta)$ indeed satisfies these properties. The proof is provided in the Appendix.

Theorem 1 *For given $\alpha > 0$ and $0 \leq \theta \leq 0.5$, $\mathcal{S}(\alpha,\theta)$ in (5) satisfies the following properties:*
(i) Ordering of $\mathcal{S}(\alpha,\theta)$ is invariant to any population or income rescaling;
(ii) $\mathcal{S}(\alpha,\theta)$ cannot decrease as the between-group distance increases;
(iii) $\mathcal{S}(\alpha,\theta)$ cannot decrease as the level of within-group identification symmetrically increases in both groups;
(iv) $\mathcal{S}(\alpha,\theta)$ cannot increase under symmetric mean-preserving reduction in the spread of income distribution.

Property (i) is the population-invariance principle of Duclos, Esteban and Ray (2004, Axiom 4) that is also standard for the inequality measures (i.e., homotheticity); and the scale compatibility of Wang and Tsui (2000). Property (ii) considers the ‘increased spread’ concept of Foster and Wolfson (2010) and Wang and Tsui (2000), which corresponds to the ‘between-group spread’ axiom in Bossert and Schworm (2008) and the key idea of Axiom 3 in Duclos, Esteban and Ray (2004). Property (iii) considers the ‘increased bipolarity’ concept of Foster and Wolfson (2010) and Wang and Tsui (2000), that is the case of a within-group Pigou-Dalton transfer. It corresponds to the ‘within-group clustering’ axiom in Bossert and Schworm (2008) and the key idea of Axiom

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2 in Duclos, Esteban and Ray (2004). Property (iv) considers Axiom 1 in Duclos, Esteban and Ray (2004), when the income density function gets squeezed toward the global mean but such mean-preserving reduction in the spread happens symmetrically on both sides of the mean. Note that properties (iii) and (iv) consider symmetric changes only; for general types of asymmetric changes (e.g., changes in within-group clustering happen only for one group; or mean-preserving reduction in the spread happens only on one side of the mean), it is not clear how $\mathcal{S}(\alpha, \theta)$ should behave. We could consider such cases under further restrictions but we leave this discussion for the future research.

These four properties are indeed the most popular axioms of the [bi]polarization indices in the literature, and Theorem 1 verifies that $\mathcal{S}(\alpha, \theta)$ satisfies them. Since it is not our main interest to axiomatize the index of social tension, however, we do not further investigate if these axioms are good enough to develop a measure of social tension, or if $\mathcal{S}(\alpha, \theta)$ can be derived from these four axioms. But an interesting point is that the basic structure of $\mathcal{S}(\alpha, \theta)$ is the same as those of Esteban and Ray (1994) or Duclos, Esteban and Ray (2004), in the sense that it is also a weighted sum of $\mathcal{I}^{\alpha}\mathcal{A}$, where $\mathcal{I}$ is some within-group identification function and $\mathcal{A}$ is some between-group alienation function. Understanding that the specific forms of $\mathcal{I}$ and $\mathcal{A}$ are not derived from the axioms in Esteban and Ray (1994) and Duclos, Esteban and Ray (2004), we could see $\mathcal{S}(\alpha, \theta)$ as a generalized income [bi]polarization index for some proper range of the parameter (α, θ).[3]

In sum, the S-index can be interpreted as a generalized income [bi]polarization index when we understand their structural similarities: The S-index shares the basic concepts of between-group difference and within-group similarity that are commonly adopted in both Wolfson and Esteban-Ray type indices. Furthermore, the S-index (especially $\mathcal{S}_K(\alpha, \theta)$) includes both Wolfson (1994) and Esteban and Ray (1994) measures as its special cases, and satisfies the basic intuitive axioms commonly adopted in the income polarization literature.

[3]It is general in the sense that it considers asymmetry in $\mathcal{A}$ and uses more general form of $\mathcal{I}$. These functions nest those of Wolfson type indices (when $\theta = 0.5$, $\alpha = 1$ and median income is used to separate two groups) and Esteban-Ray type indices (when $\theta = 0.5$ and α satisfies some specific range).

On the other hand, unlike Wolfson and Esteban-Ray type indices, the S-index emphasizes psychological aspects of income distribution as additional factors. Most importantly, it allows for the possibility that the poor(er) contribute more to the social tension than do the rich(er). Moreover, in specifying within-group identification, while Esteban and Ray (1994) consider only the group size and Wolfson (1994) considers only within-group clustering,[4] the S-index combines both. In addition, the S-index considers the relative degree of within-group clustering, while Wolfson (1994) considers the absolute one.

4 Statistical Properties

For given α and θ, using the standard sample analogues of π_k, μ_k, and $\delta_{k\ell}$, we can readily obtain an estimator of $\mathcal{S}(\alpha,\theta)$ in (5) as

$$\widehat{\mathcal{S}}(\alpha,\theta) = \frac{\bar{y}_2 - \bar{y}_1}{\bar{y}}\widehat{\pi}_1\widehat{\pi}_2\left[(1-\theta)\left(\frac{\widehat{\pi}_1}{\widehat{\delta}_{11}/\widehat{\delta}}\right)^{\alpha} + \theta\left(\frac{\widehat{\pi}_2}{\widehat{\delta}_{22}/\widehat{\delta}}\right)^{\alpha}\right] \tag{10}$$

for $\mu_1 < \mu_2$ by construction. For each $k, \ell = 1, 2$, we define

$$\begin{aligned}
\widehat{\pi}_k &= \frac{n_k}{n} = \frac{1}{n}\sum\nolimits_{i=1}^{n} \mathbb{I}\{y_i \in A_k\}, \\
\bar{y}_k &= \frac{1}{n_k}\sum\nolimits_{i=1}^{n} y_i \mathbb{I}\{y_i \in A_k\}, \\
\widehat{\delta}_{k\ell} &= \frac{1}{n_k n_\ell}\sum\nolimits_{i=1}^{n}\sum\nolimits_{j=1}^{n} \left|y_i - y_j\right| \mathbb{I}\{y_i \in A_k\}\,\mathbb{I}\left\{y_j \in A_\ell\right\}
\end{aligned}$$

and $\bar{y} = n^{-1}\sum_{i=1}^{n} y_i = \sum_{k=1}^{2}\widehat{\pi}_k\bar{y}_k$ with n_k being the number of observations in the interval A_k. Furthermore, although the mean difference is not 'decomposable' like Gini index (i.e., the overall Gini cannot be expressed as the weighted sum of the subgroup Gini's; e.g., Bourguignon, 1979), it is known that δ can be expressed as $\delta = \sum_{k=1}^{K}\sum_{\ell=1}^{K}\pi_k\pi_\ell\delta_{k\ell}$ for K-number of 'non-overlapping'

[4] The extended polarization index by Esteban, Gradin and Ray (2007) implicitly considers the group clustering effect in addition to the group size, though such a point is not discussed in their paper. Their extended index is defined as $EGR(\alpha,\beta) = ER(\alpha) - \beta E_K$, where E_K is the error in approximating the continuous Lorenz curve by K-piecewise linear functions and $\beta > 0$ is some weight parameter placed on E_K. E_K gets smaller as 'all' the within-group income distributions become more clustered around each group mean.

groups, where $\delta_{k\ell} = (\pi_k \pi_\ell)^{-1} \int_{A_k} \int_{A_\ell} |x-y|\, dF(x)\, dF(y)$ is the between-group mean difference (e.g., Silber, 1989; Lambert and Aronson, 1993; Dagum, 1997). We thus can estimate $\widehat{\delta}$ as $\widehat{\delta} = n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left| y_i - y_j \right| = \sum_{k=1}^{2} \sum_{\ell=1}^{2} \widehat{\pi}_k \widehat{\pi}_\ell \widehat{\delta}_{k\ell}$. Similarly, we can obtain an estimator of $\mathcal{S}_K(\alpha, \theta)$ in (9) as

$$\widehat{\mathcal{S}}_K(\alpha, \theta) = \frac{1}{2\overline{y}} \sum_{k=1}^{K} \sum_{\ell=1}^{K} \widehat{\pi}_k \widehat{\pi}_\ell \left(\frac{\widehat{\pi}_k}{\widehat{\delta}_{kk}/\widehat{\delta}} \right)^{\alpha} \rho_\theta \left(\overline{y}_k - \overline{y}_\ell \right). \tag{11}$$

Note that, in this framework, we suppose the income groups are pre-specified and given exogenously. Since the number of groups is fixed for both $\widehat{\mathcal{S}}(\alpha, \theta)$ and $\widehat{\mathcal{S}}_K(\alpha, \theta)$ and it does not depend on the sample size n, we can assume that $n_k \to \infty$ for each k as $n \to \infty$ without loss of generality. For given (α, θ), the consistency of $\widehat{\mathcal{S}}(\alpha, \theta)$ to $\mathcal{S}(\alpha, \theta)$ and $\widehat{\mathcal{S}}_K(\alpha, \theta)$ to $\mathcal{S}_K(\alpha, \theta)$ thus readily follow respectively by applying the Slutsky's theorem, since all the components in (10) and (11) are consistent to their population counterparts. In what follows, we derive the limiting distribution of $\widehat{\mathcal{S}}_K(\alpha, \theta)$ to facilitate further statistical inferences of the index. $\mathcal{S}(\alpha, \theta)$ is a special case of $\mathcal{S}_K(\alpha, \theta)$ with $K = 2$, so the result below provides the limiting distribution of $\widehat{\mathcal{S}}(\alpha, \theta)$ as well.

We first introduce the following U-statistics for all $k, \ell = 1, 2, \cdots, K$:

$$\begin{aligned} U_{0,k} &= n^{-1} \sum_{i=1}^{n} \mathbb{I}\{y_i \in A_k\}, \\ U_{1,k} &= n^{-1} \sum_{i=1}^{n} y_i \mathbb{I}\{y_i \in A_k\}, \\ U_{2,k\ell} &= n^{-2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left| y_i - y_j \right| \mathbb{I}\{y_i \in A_k\} \mathbb{I}\left\{y_j \in A_\ell\right\}, \end{aligned}$$

that are consistent estimators of their population counterparts $\upsilon_{0,k} < \infty$, $\upsilon_{1,k} < \infty$ and $\upsilon_{2,k\ell} < \infty$ as $n \to \infty$, respectively, where

$$\begin{aligned} \upsilon_{0,k} &= \int_{A_k} dF(y), \\ \upsilon_{1,k} &= \int_{A_k} y\, dF(y), \\ \upsilon_{2,k\ell} &= \int_{A_k} \int_{A_\ell} |y - x|\, dF(x)\, dF(y). \end{aligned}$$

Since $\widehat{\pi}_k = U_{0,k}$, $\bar{y}_k = U_{1,k}/U_{0,k}$, $\widehat{\delta}_{k\ell} = U_{2,k\ell}/U_{0,k}U_{0,\ell}$ for all $k, \ell = 1, 2, \cdots, K$, we can rewrite $\widehat{\mathcal{S}}_K(\alpha, \theta)$ as

$$\begin{aligned}
\widehat{\mathcal{S}}_K(\alpha,\theta) &= \frac{\left(\sum_{k=1}^K \sum_{\ell=1}^K U_{2,k\ell}\right)^\alpha}{\left(\sum_{k=1}^K U_{1,k}\right)} \sum\nolimits_{k=1}^K \left(\frac{U_{0,k}}{U_{2,kk}/U_{0,k}^2}\right)^\alpha \left\{\theta \sum\nolimits_{\ell<k} U_{0,k}U_{0,\ell}\left(\frac{U_{1,k}}{U_{0,k}} - \frac{U_{1,\ell}}{U_{0,\ell}}\right)\right. \\
&\qquad \left. + (1-\theta)\sum\nolimits_{\ell>k} U_{0,k}U_{0,\ell}\left(\frac{U_{1,\ell}}{U_{0,\ell}} - \frac{U_{1,k}}{U_{0,k}}\right)\right\} \\
&= \frac{\left(\sum_{k=1}^K \sum_{\ell=1}^K U_{2,k\ell}\right)^\alpha}{\left(\sum_{k=1}^K U_{1,k}\right)} \sum\nolimits_{k=1}^K \frac{1}{U_{2,kk}^\alpha}\left\{\theta \sum\nolimits_{\ell<k}\left(U_{0,k}^{3\alpha}U_{0,\ell}U_{1,k} - U_{0,k}^{1+3\alpha}U_{1,\ell}\right)\right. \\
&\qquad \left. + (1-\theta)\sum\nolimits_{\ell>k}\left(U_{0,k}^{1+3\alpha}U_{1,\ell} - U_{0,k}^{3\alpha}U_{0,\ell}U_{1,k}\right)\right\}
\end{aligned}$$

using the expression of $\widehat{\delta}$ given above, which is an consistent estimator of $\mathcal{S}_K(\alpha, \theta)$ as $n \to \infty$ provided $\upsilon_{2,kk} > 0$ for all k.

The limiting distribution of $\widehat{\mathcal{S}}_K(\alpha, \theta)$ can be derived using the joint asymptotic distribution of the vector of U-statistics of $U_{0,k}$, $U_{1,k}$ and $U_{2,k\ell}$ for all $k, \ell = 1, 2, \cdots, K$. For representation purposes, however, it suffices to obtain the joint distribution of the 9×1 vector of U-statistics $\mathbf{U}^* \equiv (U_{0,k}, U_{0,\ell}, U_{1,k}, U_{1,\ell}, U_{2,kk}, U_{2,\ell\ell}, U_{2,k\ell}, U_{2,km}, U_{2,hb})'$ for $k \neq \ell \neq m \neq h \neq b$. The following lemma summarizes the asymptotic distribution of $\mathbf{U}^*$ from Theorem 7.1 of Hoeffding (1948). We let $\boldsymbol{\upsilon}^* \equiv (\upsilon_{0,k}, \upsilon_{0,\ell}, \upsilon_{1,k}, \upsilon_{1,\ell}, \upsilon_{2,kk}, \upsilon_{2,\ell\ell}, \upsilon_{2,k\ell}, \upsilon_{2,km}, \upsilon_{2,hb})'$.

Lemma 1 *Let $\{y_i\}_{i=1}^n$ be i.i.d. with continuous distribution $F(y)$ and finite variance. If $\epsilon < \upsilon_{0,k} < 1-\epsilon$ for all $k = 1, 2, \cdots, K$ and for some $\epsilon \in (0, 1)$, then the joint distribution of $\sqrt{n}(\mathbf{U}^* - \boldsymbol{\upsilon}^*)$ tends to the 9-variate normal distribution as $n \to \infty$ with zero mean and covariance matrix Σ^*, which is given by (A.4) in the Appendix.*

Note that Bishop, Formby and Zheng (1997) consider the joint distribution of $(U_{0,1}, U_{1,1}, U_{2,11})$ for the particular case of $K = 2$, and Lemma 1 above extends their result. Using this lemma and Theorem 7.5 of Hoeffding (1948), we obtain the asymptotic distribution of $\widehat{\mathcal{S}}_K(\alpha, \theta)$ as follows. We define $(2K + K(K + 1)/2) \times 1$ vectors $\mathbf{U} \equiv (U_{0,1}, \cdots, U_{0,K}, U_{1,1}, \cdots, U_{1,K}, U_{2,11}, U_{2,12}, \cdots, U_{2,KK})'$ and $\boldsymbol{\upsilon} \equiv (\upsilon_{0,1}, \cdots, \upsilon_{0,K}, \upsilon_{1,1}, \cdots, \upsilon_{1,K}, \upsilon_{2,11}, \upsilon_{2,12}, \cdots, \upsilon_{2,KK})'$.

Theorem 2 *Suppose $\upsilon_{2,kk} > 0$ for all k. Under the same conditions as Lemma 1 and for a fixed K, we have $\sqrt{n}(\widehat{\mathcal{S}}_K(\alpha,\theta) - \mathcal{S}_K(\alpha,\theta)) \to_d \mathcal{N}(0, V_K(\alpha,\theta))$ as $n \to \infty$, where $V_K(\alpha,\theta) = [\nabla\mathcal{S}_K(\alpha,\theta)]'\Sigma_K[\nabla\mathcal{S}_K(\alpha,\theta)]$, $\nabla\mathcal{S}_K(\alpha,\theta)$ is the $(2K + K(K+1)/2) \times 1$ vector of partial derivatives of $\mathcal{S}_K(\alpha,\theta)$ with respect to $\boldsymbol{\upsilon}$, and Σ_K is the $(2K + K(K+1)/2) \times (2K + K(K+1)/2)$ asymptotic variance matrix of $\mathbf{U}$ whose elements can be obtained from Σ^* in Lemma 1. The specific form of $\nabla\mathcal{S}_K(\alpha,\theta)$ is given in the Appendix.*

As we discussed in the previous section, $\mathcal{S}_K(\alpha,\theta)$ is general enough to include various inequality measures and income polarization indices as its special cases. For example, $2\mathcal{S}_K(\alpha, 0.5)$ with $\phi_k(\alpha) = \pi_k^\alpha$ becomes the income polarization index of Esteban and Ray (1994); and $2\mathcal{S}_K(0, 0.5)$ becomes the Gini index for grouped data. Therefore, their limiting distributions can be obtained from Theorem 2. When $K = 2$, we can readily obtain the limiting distribution of $\widehat{\mathcal{S}}(\alpha,\theta)$ as we summarize in the Appendix, which can be used to derive the limiting distribution of Wolfson's (1994) income bipolarization index based on (8). Limiting distributions of some poverty indices can be also derived similarly, which include Bishop, Formby and Zheng (1997), Xu (2007), and Barrett and Donald (2009).

5 When Group Specification is Unavailable

It is important to note that the asymptotic results in the previous section (i.e., consistency and asymptotic normality in Theorem 2) are very different from those of Duclos, Esteban and Ray (2004) or Anderson (2011), where they consider the asymptotics when the number of groups goes to infinity (i.e., $K \to \infty$). In fact, in their case, we can regard $K = n$ without loss of generality so that each group includes only one agent; the level of social tension is then formulated as the sum of individual antagonism. In comparison, Theorem 2 relies on $n \to \infty$ asymptotics with a fixed number of groups K, where each group size increases as $n \to \infty$. One of the key assumption of Theorem 2 is, however, that the K groups are pre-specified so that the number of groups and the cutoff points are given. This approach is useful when we have prior information about group specification that a research question already determines, or we have some clear justification of how to specify groups. The S-index $\mathcal{S}(\alpha,\theta)$ is an example as it is motivated by the overall tension between two groups. Wolfson's (1994) income bipolarization index is another example, which is motivated by the phenomenon of the disappearing middle class and formation of two income subgroups. When the group specification is uncertain, on the other hand, we can consider $K \to \infty$ asymptotics so that the unknown group specification can be nonparametrically approximated.

Though it is not conceptually clear to consider the increasing number of groups in the context of social tension generated by formation of subgroups, we can further generalize the S-index to this case as

$$\mathcal{S}_\infty(\alpha,\theta) = \frac{\delta^\alpha}{2\mu} \int \int f(x)^\alpha \rho_\theta(x-y)\, dF(x) dF(y), \tag{12}$$

where $f(\cdot)$ is the density function of y_i. We can understand that $\mathcal{S}_\infty(\alpha,\theta)$ corresponds to $\mathcal{S}_K(\alpha,\theta)$ when $K \to \infty$, which is equivalent to the case that each group includes only one agent and hence $\mu_i = y_i$ for each i. Therefore, it is no longer meaningful to consider within-group (relative) dispersion in the identification function, so we omit δ_{kk} in $\mathcal{S}_\infty(\alpha,\theta)$. By normalizing δ as one, $2\mathcal{S}_\infty(\alpha,\theta)$ becomes the income polarization index of Duclos, Esteban and Ray (2004), $(1/\mu) \iint f(x)^\alpha |x-y|\, dF(x)dF(y)$, when $\theta = 0.5$; whereas $2\mathcal{S}_\infty(\alpha,\theta)$ becomes the standard Gini index, $(1/\mu) \iint |x-y|\, dF(x)dF(y)$, when $\theta = 0.5$ and $\alpha = 0$.

A consistent estimator $\widehat{\mathcal{S}}_\infty(\alpha,\theta)$ of $\mathcal{S}_\infty(\alpha,\theta)$ in (12) is obtained by replacing $F(\cdot)$, μ, and $f(x)$ by the empirical distribution $\widehat{F}(\cdot)$, the sample mean $\bar{y}$, and the kernel density estimator $\widehat{f}(\cdot)$ using a proper kernel function and a bandwidth parameter, respectively. With (α,θ) given, moreover, we can use the same argument as Duclos, Esteban and Ray (2004) to show that $\sqrt{K}(\widehat{\mathcal{S}}_\infty(\alpha,\theta) - \mathcal{S}_\infty(\alpha,\theta))$ has a limiting distribution as follows, which covers the result of Duclos, Esteban and Ray (2004) as a special case.

Theorem 3 *Let $\{y_i\}_{i=1}^n$ be i.i.d. with continuous distribution $F(y)$ and finite variance, whose density $f(y)$ is bounded above zero and twice continuously differentiable. Under the same conditions in*

Duclos, Esteban and Ray (2004), $\sqrt{K}(\widehat{\mathcal{S}}_\infty(\alpha,\theta)-\mathcal{S}_\infty(\alpha,\theta)) \to_d \mathcal{N}(0, V_\infty(\alpha,\theta))$ *as* $K \to \infty$*, where* $V_\infty(\alpha,\theta)$ *is given in the Appendix.*

We remark that, though $\mathcal{S}_\infty(\alpha,\theta)$ seems more general than $\mathcal{S}(\alpha,\theta)$ (or $\mathcal{S}_K(\alpha,\theta)$ more generally), we cannot consider $\mathcal{S}_\infty(\alpha,\theta)$ as the better measure of social tension than $\mathcal{S}(\alpha,\theta)$; they have their own usefulness so that they complement each other. As we emphasized in Section 2, a two-group representation of $\mathcal{S}(\alpha,\theta)$ is more proper for measuring the level of social tension generated by income class separation than $\mathcal{S}_\infty(\alpha,\theta)$. In addition, though we need individual observations to obtain $\widehat{\mathcal{S}}_\infty(\alpha,\theta)$, we can obtain $\widehat{\mathcal{S}}(\alpha,\theta)$ or $\widehat{\mathcal{S}}_K(\alpha,\theta)$ using the group-level (aggregate) observations only. Such difference can be also found when comparing the standard Gini index (as a special case of $\mathcal{S}_\infty(\alpha,\theta)$) and the Gini index from grouped data (as a special case of $\mathcal{S}(\alpha,\theta)$ or $\mathcal{S}_K(\alpha,\theta)$; e.g., Gastwirth, Nayak and Kreiger, 1986). Furthermore, the within-group dispersion (e.g., δ_{kk} or δ_{kk}/δ) can be considered only for $\mathcal{S}(\alpha,\theta)$ or $\mathcal{S}_K(\alpha,\theta)$, which plays an important role in measuring the level of social tension more effectively.

More importantly, though it is true that $\mathcal{S}_\infty(\alpha,\theta)$ ignores the group specification issue by construction and hence it can be useful when the group specification is uncertain, this approach does not completely eliminate the group selection issue *in practice*. This is because its estimator $\widehat{\mathcal{S}}_\infty(\alpha,\theta)$ involves kernel density estimation of $f(\cdot)$, in which bandwidth selection can be interpreted as selecting the number of groups. Note that the bandwidth corresponds to the group size and it thus determines the number of groups over the support $[y_{\min}, y_{\max}]$; the same comment applies to Duclos, Esteban and Ray (2004).

6 Empirical Illustration

Using the Panel Study of Income Dynamics (PSID) data, we illustrate how the level of social tension, as measured by the S-index in (5), has evolved over the period from 1981 to 2005. We use the sample from 1981, in which information on government's taxes and transfers is available, and consider only odd years for consistency of the survey frequency. (Since each year's survey contains the total family income for the previous year, the income observations run every other year from 1980 through 2004.) We analyze two income variables at the family level: the total family income and the total family income adjusted by government actions. The former is defined by the sum of family labor earnings, family asset income, family private transfers, and family private retirement income. The latter is defined by subtracting household taxes from the total family income and adding public transfer income and social security pensions.

Figures 2 through 4 display how the level of social tension has evolved over the sample period for different values of θ. In each figure, the line connecting rectangular points shows the S-index based on the total family income (TFI); and the line connecting triangular points displays the S-index based on the adjusted family income (AFI). The poor and the rich groups are separated by the mean income, which corresponds to the cutoff point of two groups from the optimal grouping algorithm by Aghevli and Mehran (1981). Division of the population by the median income is not preferable for the current purpose because it fixes the relative group size as 0.5 so eliminates its effect on within-group identification. For each series of the level of social tension, a (pointwise) confidence interval at the 95% level is depicted in dashed lines that is obtained using the jackknife method summarized in the Appendix. For all figures, α is set to be 1.6, though the current results are found to be quite robust with respect to different values of α in a qualitative sense. A value of θ smaller than 0.5 implies that the poor feel more alienated against the rich than the rich do against the poor. As we move from Figures 2 to 4, a greater weight is placed on the poor group (i.e., $\theta = 0.5$, 0.25, and 0.0, respectively) so that the degree of this asymmetry gets stronger.

Interesting results emerge from comparison of these figures. First, for all values of θ and for both income variables, the level of social tension has generally increased over the sample period, with noticeable increment since the late 1990's. Second, for both income variables, as we change the value of θ from 0.5 to 0.0, placing a greater weight on the poor group, the level of social tension generally increases. From the form of the S-index in (5), this finding also implies that the level of within-group identification is higher for the poor than the rich. To put it differently, the level of social tension tends to be understated under the symmetry assumption (i.e., when $\theta = 0.5$) that is adopted in the income polarization and inequality literatures. Third, for each value of θ, the level of social tension measured by the adjusted family income is lower than that by the total family income, implying that the government's taxes and transfers have been effective in mitigating the level of social tension. Our calculation shows that, when averaged over the sample period, government's taxes and transfers reduced the level of social tension by 30.4%, 31.3%, and 31.8% for θ values of 0.5, 0.25, and 0.0, respectively. These reductions are also statistically significant. Lastly, although not reported in detail for brevity, during the sample period, the S-index has increased much more than have Wolfson's (1994), Esteban and Ray's (1994), and the Gini indices.

7 Concluding Remarks

This paper considers measuring the level of social tension inherent in an income distribution. Measuring the level of social tension and monitoring its trends are of great importance, understanding the substantial costs of social instability. While various inequality measures (e.g., Gini index) and income polarization indices (e.g., Esteban and Ray, 1994; Wolfson, 1994) have long been served for that purpose, they are not primarily designed to measure the level of social tension. For the purpose of measuring more effectively the level of social tension generated by income class segregation, we adopt the basic concepts of between-group alienation and within-group identification from the income [bi]polarization literature. But we allow for asymmetric degrees of between-group antagonism in the alienation function, and present a more effective identification function than existing income polarization measures. The salient feature of the new measure is that it is designed to reflect psychological aspects of the economic agents and it is general enough to include existing income polarization and inequality indices as its special cases.

One important question is empirical evaluation of relative effectiveness of the indices discussed in this paper. For example, which measure –inequality or polarization– better explains consequences of social tension, represented by individual or collective crimes? In the same context, does consideration of psychological elements significantly raise explanatory power? If so, what values of α and θ best explain social consequences generated by economic segregation? Answers to these questions are to be used for developing a more effective measure of social tension.

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Appendix

A.1 Mathematical proofs

Proof of Theorem 1 First, for property (i), note that population or income rescaling can change μ_1, μ_2, μ, δ_{11}, δ_{22}, and δ; but π_1 and π_2 remain unchanged. However, the changes are proportional and hence all canceled out because these parameters appear as ratios in the index (i.e., μ_1/μ, μ_2/μ, δ_{11}/δ, δ_{22}/δ). The property thus readily follows.

For property (ii), we suppose $(\mu_2 - \mu_1)$ increases, where π_1, π_2, δ_{11}, δ_{22}, and μ are all fixed. For two non-overlapping groups, it is known that the overall mean difference δ can be expressed as (e.g., Silber, 1989; Lambert and Aronson, 1993; Dagum, 1997)[5]

$$\delta = \pi_1^2\delta_{11} + \pi_2^2\delta_{22} + 2\pi_1\pi_2\left|\mu_2 - \mu_1\right|, \tag{A.1}$$

from which we have

$$\frac{\delta}{\delta_{11}} = \pi_1^2 + \pi_2^2\frac{\delta_{22}}{\delta_{11}} + 2\pi_1\pi_2\frac{(\mu_2-\mu_1)}{\delta_{11}} \quad \text{and} \quad \frac{\delta}{\delta_{22}} = \pi_2^2 + \pi_1^2\frac{\delta_{11}}{\delta_{22}} + 2\pi_1\pi_2\frac{(\mu_2-\mu_1)}{\delta_{22}} \tag{A.2}$$

since $\mu_2 > \mu_1$ by construction. It thus follows that we can rewrite $\mathcal{S}(\alpha,\theta)$ in (5) as

$$\begin{aligned}\mathcal{S}(\alpha,\theta) &= \frac{\mu_2-\mu_1}{\mu}\pi_1\pi_2\left\{(1-\theta)\pi_1^\alpha\left(\frac{\delta}{\delta_{11}}\right)^\alpha + \theta\pi_2^\alpha\left(\frac{\delta}{\delta_{22}}\right)^\alpha\right\} \\ &= (1-\theta)\pi_1^{\alpha+1}\pi_2\frac{\mu_2-\mu_1}{\mu}\left(\pi_1^2 + \pi_2^2\frac{\delta_{22}}{\delta_{11}} + 2\pi_1\pi_2\frac{(\mu_2-\mu_1)}{\delta_{11}}\right)^\alpha \\ &\quad +\theta\pi_1\pi_2^{\alpha+1}\frac{\mu_2-\mu_1}{\mu}\left(\pi_2^2 + \pi_1^2\frac{\delta_{11}}{\delta_{22}} + 2\pi_1\pi_2\frac{(\mu_2-\mu_1)}{\delta_{22}}\right)^\alpha,\end{aligned} \tag{A.3}$$

in which it is straightforward to see $\mathcal{S}(\alpha,\theta)$ increases with $(\mu_2 - \mu_1)$ because $\alpha > 0$, $0 \le \theta \le 0.5$, and all other elements are strictly positive and fixed.

For property (iii), we suppose δ_{kk} changes to $\lambda\delta_{kk}$ for some $0 < \lambda < 1$ for both $k = 1, 2$, where π_1, π_2, μ_1, μ_2, and μ are all fixed. A possible example is Pigou-Dalton transfer within each group so that the density of each group has local squeeze about its own mean. In this case, even though the overall mean difference δ decreases, both the ratios δ/δ_{11} and δ/δ_{22} increase, which can be easily verified from (A.2). Therefore, it is straightforward to see $\mathcal{S}(\alpha,\theta)$ increases in this case as $\alpha > 0$.[6]

[5] In general, for $K \ge 2$ number of non-overlapping groups, δ can be expressed as

$$\delta = \sum_{k=1}^K\sum_{\ell=1}^K \pi_k\pi_\ell\delta_{k\ell} = \sum_{k=1}^K \pi_k^2\delta_{kk} + \sum_{k=1}^K\sum_{\ell\neq k}\pi_k\pi_\ell\left|\mu_k - \mu_\ell\right|,$$

where $\delta_{k\ell} = (\pi_k\pi_\ell)^{-1}\int_{A_k}\int_{A_\ell}|x-y|\,dF(x)\,dF(y)$ is the between-group mean difference.

[6] This result can be also derived similarly as for the proof of (iv). For this purpose, we consider non-overlapping local densities $f_k(y)$ of groups $k = 1, 2$ and define the *local squeeze* of $f_k(y)$ about corresponding group mean μ_k as $f_k^\lambda(y) = \lambda^{-1} f_k((y - (1-\lambda)\mu_k)/\lambda)$. Then, with π_1, π_2, μ_1, μ_2, and μ are kept fixed, δ_{kk} changes to $\lambda\delta_{kk}$ for each $k = 1, 2$

Finally, for property (iv), we consider the *global squeeze* of a density $f(y)$ as Duclos, Esteban and Ray (2004): for some $0 < \lambda < 1$, the global squeeze of a density $f(y)$ about its mean is defined as $f^\lambda(y) = \lambda^{-1} f((y - (1-\lambda)\mu)/\lambda)$. Under this squeeze, where π_1, π_2, and μ are kept fixed, μ_1, μ_2, δ_{11}, δ_{22}, and δ are respectively changed to μ_1^λ, μ_2^λ, δ_{11}^λ, δ_{22}^λ, and δ^λ as follows:

$$\mu_1^\lambda = \frac{1}{\pi_1}\int_{-\infty}^{\mu} y f^\lambda(y)\,dy = \frac{1}{\pi_1}\int_{-\infty}^{\mu} (\lambda u + (1-\lambda)\mu)\, f(u)\,du = \lambda\mu_1 + (1-\lambda)\mu$$

by change-of-variables and similarly $\mu_2^\lambda = \pi_2^{-1}\int_\mu^\infty y f^\lambda(y)\,dy = \lambda\mu_2 + (1-\lambda)\mu$;

$$\begin{aligned}\delta_{11}^\lambda &= \frac{1}{\pi_1^2}\int_{-\infty}^{\mu}\int_{-\infty}^{\mu} |x-y|\, f^\lambda(x)\, f^\lambda(y)\,dxdy \\ &= \frac{1}{\pi_1^2}\int_{-\infty}^{\mu}\int_{-\infty}^{\mu} \left|(\lambda u + (1-\lambda)\mu_k) - (\lambda v + (1-\lambda)\mu_k)\right| f(u)\, f(v)\,dudv = \lambda\delta_{11}\end{aligned}$$

by change-of-variables and similarly $\delta_{22}^\lambda = \pi_2^{-2}\int_\mu^\infty\int_\mu^\infty |x-y|\, f^\lambda(x)\, f^\lambda(y)\,dxdy = \lambda\delta_{22}$; and

$$\delta^\lambda = \pi_1^2\delta_{11}^\lambda + \pi_2^2\delta_{22}^\lambda + 2\pi_1\pi_2\left(\mu_2^\lambda - \mu_1^\lambda\right) = \lambda\left\{\pi_1^2\delta_{11} + \pi_2^2\delta_{22} + 2\pi_1\pi_2(\mu_2-\mu_1)\right\} = \lambda\delta$$

from (A.1). It thus follows that after the global squeeze, $\mathcal{S}(\alpha,\theta)$ changes to

$$\mathcal{S}^\lambda(\alpha,\theta) = \frac{\mu_2^\lambda - \mu_1^\lambda}{\mu}\pi_1\pi_2\left\{(1-\theta)\,\pi_1^\alpha\left(\frac{\delta^\lambda}{\delta_{11}^\lambda}\right)^\alpha + \theta\pi_2^\alpha\left(\frac{\delta^\lambda}{\delta_{22}^\lambda}\right)^\alpha\right\} = \lambda\mathcal{S}(\alpha,\theta),$$

which shows that $\mathcal{S}(\alpha,\theta)$ decreases for $\lambda < 1$. □

Proof of Lemma 1 From Theorem 7.1 of Hoeffding (1948), the asymptotic variance Σ^* can be obtained as

$$\Sigma^* = \begin{pmatrix} \Sigma_{11}^* & \Sigma_{12}^* \\ \Sigma_{12}^{*\prime} & \Sigma_{22}^* \end{pmatrix}, \tag{A.4}$$

and thus δ/δ_{kk} increases and so does $\mathcal{S}(\alpha,\theta)$. For example, for $k = 1$

$$\frac{\lambda\pi_1^2\delta_{11} + \lambda\pi_2^2\delta_{22} + \pi_1\pi_2(\mu_2-\mu_1)}{\lambda\delta_{11}} = \frac{\delta}{\delta_{11}} + \left(\frac{1-\lambda}{\lambda}\right)\frac{\pi_1\pi_2(\mu_2-\mu_1)}{\delta_{11}} \geq \frac{\delta}{\delta_{11}}.$$

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where $\underset{(4\times4)}{\Sigma_{11}^*}$, $\underset{(4\times5)}{\Sigma_{12}^*}$ and $\underset{(5\times5)}{\Sigma_{22}^*}$ are given as

$$\begin{pmatrix} \upsilon_{0,k}(1-\upsilon_{0,k}) & -\upsilon_{0,k}\upsilon_{0,\ell} & \upsilon_{1,k}(1-\upsilon_{0,k}) & -\upsilon_{0,k}\upsilon_{1,\ell} \\ & \upsilon_{0,\ell}(1-\upsilon_{0,\ell}) & -\upsilon_{0,\ell}\upsilon_{1,k} & \upsilon_{1,\ell}(1-\upsilon_{0,\ell}) \\ & & \xi_{1,k}-\upsilon_{1,k}^2 & -\upsilon_{1,k}\upsilon_{1,\ell} \\ & & & \xi_{1,\ell}-\upsilon_{1,\ell}^2 \end{pmatrix},$$

$$\begin{pmatrix} 2(1-\upsilon_{0,k})\upsilon_{2,kk} & -2\upsilon_{0,k}\upsilon_{2,\ell\ell} & (1-2\upsilon_{0,k})\,\upsilon_{2,k\ell} & (1-2\upsilon_{0,k})\,\upsilon_{2,km} & -2\upsilon_{0,k}\upsilon_{2,hb} \\ -2\upsilon_{0,\ell}\upsilon_{2,kk} & 2\upsilon_{2,\ell\ell}(1-\upsilon_{0,\ell}) & (1-2\upsilon_{0,\ell})\,\upsilon_{2,k\ell} & -2\upsilon_{0,\ell}\upsilon_{2,km} & -2\upsilon_{0,\ell}\upsilon_{2,hb} \\ 2\left(\xi_{2,kk}-\upsilon_{1,k}\upsilon_{2,kk}\right) & -2\upsilon_{1,k}\upsilon_{2,\ell\ell} & \xi_{2,k\ell}-2\upsilon_{1,k}\upsilon_{2,k\ell} & \xi_{2,km}-2\upsilon_{1,k}\upsilon_{2,km} & -2\upsilon_{1,k}\upsilon_{2,hb} \\ -2\upsilon_{1,\ell}\upsilon_{2,kk} & 2\left(\xi_{2,\ell\ell}-\upsilon_{1,\ell}\upsilon_{2,\ell\ell}\right) & \xi_{2,\ell k}-2\upsilon_{1,\ell}\upsilon_{2,k\ell} & -2\upsilon_{1,\ell}\upsilon_{2,km} & -2\upsilon_{1,\ell}\upsilon_{2,hb} \end{pmatrix},$$

$$\begin{pmatrix} 4\left(\xi_{3,kkk}-\upsilon_{2,kk}^2\right) & -4\upsilon_{2,\ell\ell}\upsilon_{2,kk} & 2\left(\xi_{3,kk\ell}-2\upsilon_{2,kk}\upsilon_{2,k\ell}\right) & 2\left(\xi_{3,kkm}-2\upsilon_{2,kk}\upsilon_{2,km}\right) & -4\upsilon_{2,kk}\upsilon_{2,hb} \\ & 4\left(\xi_{3,\ell\ell\ell}-\upsilon_{2,\ell\ell}^2\right) & 2\left(\xi_{3,\ell\ell k}-2\upsilon_{2,\ell\ell}\upsilon_{2,k\ell}\right) & -4\upsilon_{2,\ell\ell}\upsilon_{2,km} & -4\upsilon_{2,\ell\ell}\upsilon_{2,hb} \\ & & \xi_{3,\ell kk}+\xi_{3,k\ell\ell}-4\upsilon_{2,k\ell}^2 & \xi_{3,k\ell m}-4\upsilon_{2,k\ell}\upsilon_{2,km} & -4\upsilon_{2,k\ell}\upsilon_{2,hb} \\ & & & \xi_{3,mkk}+\xi_{3,kmm}-4\upsilon_{2,km}^2 & -4\upsilon_{2,km}\upsilon_{2,hb} \\ & & & & \xi_{3,bhh}+\xi_{3,hbb}-4\upsilon_{2,hb}^2 \end{pmatrix},$$

respectively, with

$$\begin{aligned} \xi_{1,k} &= \int_{A_k} y^2 dF(y) \\ \xi_{2,k\ell} &= \int_{A_k}\int_{A_\ell} y\,|y-x|\,dF(x)\,dF(y) \\ \xi_{3,k\ell m} &= \int_{A_k}\left\{\int_{A_\ell}|y-x|\,dF(x)\right\}\left\{\int_{A_m}|y-x|\,dF(x)\right\}dF(y). \end{aligned}$$

Though most of the terms are standard, deriving covariance terms involving $U_{2,k\ell}$ needs some extra care. For example, the leading term of the asymptotic variance of $U_{2,k\ell}$ can be obtained from

$$\begin{aligned} & \int\left(\int_{A_k}|x-y|\,\mathbb{I}\{y\in A_\ell\}\,dF(x)-\upsilon_{2,k\ell}\right)^2 dF(y) \\ & +2\int\left(\int_{A_k}|x-y|\,\mathbb{I}\{y\in A_\ell\}\,dF(x)-\upsilon_{2,k\ell}\right)\left(\int_{A_\ell}|x-y|\,\mathbb{I}\{y\in A_k\}\,dF(x)-\upsilon_{2,\ell k}\right)dF(y) \\ & +\int\left(\int_{A_\ell}|x-y|\,\mathbb{I}\{y\in A_k\}\,dF(x)-\upsilon_{2,\ell k}\right)^2 dF(y) \\ = & \int_{A_\ell}\left(\int_{A_k}|x-y|\,dF(x)\right)^2 dF(y)+\int_{A_k}\left(\int_{A_\ell}|x-y|\,dF(x)\right)^2 dF(y)-4\upsilon_{2,\ell k}^2 \end{aligned}$$

since $\upsilon_{2,k\ell} = \upsilon_{2,\ell k}$ and $\int\left(\int_{A_k}|x-y|\,\mathbb{I}\{y\in A_\ell\}\,dF(x)\right)\left(\int_{A_\ell}|x-y|\,\mathbb{I}\{y\in A_k\}\,dF(x)\right)dF(y) = 0$ for $\mathbb{I}\{y\in A_\ell\}\,\mathbb{I}\{y\in A_k\}=0$ with $k\neq\ell$. The other terms can be obtained similarly. □

Proof of Theorem 2 The result follows directly from Lemma 1 of this paper and Theorem 7.5 of Hoeffding (1948). In this proof, we summarize the elements of $\nabla\mathcal{S}_K(\alpha,\theta)$. We first note that, for each $h = 1, 2, \cdots, K$, we have

$$\begin{aligned}
&\frac{\partial \mathcal{S}_K(\alpha,\theta)}{\partial \upsilon_{0,h}} \\
= \;& C_K(\alpha) \sum_{k=1}^{K} \frac{1}{\upsilon_{2,kk}^{\alpha}} \Big\{ \mathbb{I}\{k > h\}\, \theta \upsilon_{0,k}^{2\alpha} \upsilon_{1,k} + \mathbb{I}\{k < h\}(1-\theta)\left(-\upsilon_{0,k}^{2\alpha}\upsilon_{1,k}\right) \\
& + \mathbb{I}\{k = h\} \Big[\theta \sum_{\ell<k} \left(2\alpha \upsilon_{0,k}^{2\alpha-1}\upsilon_{0,\ell}\upsilon_{1,k} - (1+2\alpha)\upsilon_{0,k}^{2\alpha}\upsilon_{1,\ell}\right) \\
& + (1-\theta)\sum_{\ell>k}\left((1+2\alpha)\upsilon_{0,k}^{2\alpha}\upsilon_{1,\ell} - 2\alpha\upsilon_{0,k}^{2\alpha-1}\upsilon_{0,\ell}\upsilon_{1,k}\right)\Big]\Big\} \\
= \;& C_K(\alpha) \sum_{k=1}^{K} \left(\frac{\upsilon_{2,kk}}{\upsilon_{0,k}^2}\right)^{-\alpha} \\
& \times \left\{ \left(\theta \mathbb{I}\{k>h\} - (1-\theta)\mathbb{I}\{k<h\}\right)\upsilon_{1,k} + \mathbb{I}\{k=h\}\left(\theta \sum_{\ell<k}\gamma_{0,k\ell}(\alpha) - (1-\theta)\sum_{\ell>k}\gamma_{0,k\ell}(\alpha)\right)\upsilon_{0,\ell} \right\},
\end{aligned}$$

where $C_K(\alpha) = (\sum_{k=1}^{K}\sum_{\ell=1}^{K}\upsilon_{2,k\ell})^{\alpha}/(\sum_{k=1}^{K}\upsilon_{1,k})$ and $\gamma_{0,k\ell}(\alpha) = 2\alpha(\upsilon_{1,k}/\upsilon_{0,k}) - (1+2\alpha)(\upsilon_{1,\ell}/\upsilon_{0,\ell})$ for all $k \neq \ell$. Similarly, for each $h = 1, 2, \cdots, K$,

$$\begin{aligned}
&\frac{\partial \mathcal{S}_K(\alpha,\theta)}{\partial \upsilon_{1,h}} \\
= \;& -\frac{\mathcal{S}_K(\alpha,\theta)}{\sum_{k=1}^{K}\upsilon_{1,k}} \\
& + C_K(\alpha)\sum_{k=1}^{K}\frac{1}{\upsilon_{2,kk}^{\alpha}}\Big\{\mathbb{I}\{k>h\}\,\theta\left(-\upsilon_{0,k}^{1+2\alpha}\right) + \mathbb{I}\{k<h\}(1-\theta)\upsilon_{0,k}^{1+2\alpha} \\
& + \mathbb{I}\{k=h\}\Big[\theta\sum_{\ell<k}\upsilon_{0,k}^{2\alpha}\upsilon_{0,\ell} + (1-\theta)\sum_{\ell>k}\left(-\upsilon_{0,k}^{2\alpha}\upsilon_{0,\ell}\right)\Big]\Big\} \\
= \;& -\frac{\mathcal{S}_K(\alpha,\theta)}{\sum_{k=1}^{K}\upsilon_{1,k}} + C_K(\alpha)\sum_{k=1}^{K}\left(\frac{\upsilon_{2,kk}}{\upsilon_{0,k}^2}\right)^{-\alpha} \\
& \times \left\{\left(-\theta\mathbb{I}\{k>h\} + (1-\theta)\mathbb{I}\{k<h\}\right)\upsilon_{0,k} + \mathbb{I}\{k=h\}\left(\theta\sum_{\ell<k} - (1-\theta)\sum_{\ell>k}\right)\upsilon_{0,\ell}\right\}.
\end{aligned}$$

The derivatives with respect to $\upsilon_{2,k\ell}$ can be readily obtained as

$$\frac{\partial \mathcal{S}_K(\alpha,\theta)}{\partial \upsilon_{2,kk}} = \frac{\alpha \mathcal{S}_K(\alpha,\theta)}{\sum_{k=1}^{K}\sum_{\ell=1}^{K}\upsilon_{2,k\ell}} - \frac{\alpha C_K(\alpha) D_k(\alpha,\theta)}{\upsilon_{2,kk}^{\alpha+1}}$$

for each $k = 1, 2, \cdots, K$, where

$$D_k(\alpha,\theta) = \theta \sum\nolimits_{\ell<k} \left(\upsilon_{0,k}^{2\alpha}\upsilon_{0,\ell}\upsilon_{1,k} - \upsilon_{0,k}^{1+2\alpha}\upsilon_{1,\ell}\right) + (1-\theta) \sum\nolimits_{\ell>k} \left(\upsilon_{0,k}^{1+2\alpha}\upsilon_{1,\ell} - \upsilon_{0,k}^{2\alpha}\upsilon_{0,\ell}\upsilon_{1,k}\right). \quad \square$$

In order to prove Theorem 3, we need the following lemma.

Lemma 2 *We can rewrite*

$$\mathcal{S}_\infty(\alpha,\theta) = \frac{\delta^\alpha}{\mu} \int f(x)^\alpha \{(1-\theta)\mu + x(F(x) - (1-\theta)) - \mu^*(x)\} dF(x) \tag{A.5}$$

and

$$\delta = 2 \int (xF(x) - \mu^*(x)) dF(x),$$

where $\mu^*(x) = \int^x y dF(y)$ *and* $\mu = \int y dF(y)$.

Proof of Lemma 2 First note that

$$\begin{aligned} & \int\int f(x)^\alpha (x-y)\{\theta - \mathbb{I}\{x<y\}\} dF(x)dF(y) \\ = \ & \theta \int\int f(x)^\alpha (x-y) dF(x)dF(y) - \int\int_{-\infty}^{y} f(x)^\alpha (x-y) dF(x)dF(y). \end{aligned}$$

It is straightforward to write the first term as

$$\theta \int\int f(x)^\alpha (x-y) dF(x)dF(y) = \theta \int x f(x)^\alpha dF(x) - \theta\mu \int f(x)^\alpha dF(x)$$

as $\mu = \int y dF(y)$. For the second term, by the integration by parts, we have

$$\begin{aligned} & \int\int_{-\infty}^{y} f(x)^\alpha x dF(x)dF(y) \\ = \ & \int_{-\infty}^{y} x f(x)^\alpha dF(x)F(y)\Big|_{y=-\infty}^{\infty} - \int y f(y)^\alpha F(y)dF(y) = \int x f(x)^\alpha (1-F(x)) dF(x) \end{aligned}$$

and

$$\begin{aligned} & \int\int_{-\infty}^{y} f(x)^\alpha y dF(x)dF(y) \\ = \ & \int_{-\infty}^{y} f(x)^\alpha dF(x) \int_{-\infty}^{y} x dF(x)\Big|_{y=-\infty}^{\infty} - \int f(y)^\alpha dF(y) \int_{-\infty}^{y} x dF(x) = \int f(x)^\alpha (\mu - \mu^*(x)) dF(x), \end{aligned}$$

where $\mu^*(x) = \int^x y dF(y)$. By combining these results, it thus follows that

$$\begin{aligned} & \int\int f(x)^\alpha (x-y)\{\theta - \mathbb{I}\{x<y\}\} dF(x)dF(y) \\ = \ & 2\int f(x)^\alpha \{(1-\theta)\mu + x(F(x) - (1-\theta)) - \mu^*(x)\} dF(x), \end{aligned}$$

which yields the desired result. Note that the integration by parts also gives

$$\delta = \iint |x-y|\, dF(x)\, dF(y) = \iint \rho_{0.5}(x-y)\, dF(x)\, dF(y) = 2\int (xF(x) - \mu^*(x))\, dF(x). \quad \square$$

Proof of Theorem 3 We sketch the proof similarly as the working paper version of Duclos, Esteban and Ray (2004). Note that without loss of generality, we can regard $K = n$ so that each group only includes one agent. Therefore, in what follows, we do not distinguish the index k from i. We also suppose $\{y_i\}_{i=1}^n$ are ordered such that $y_1 \leq y_2 \leq \cdots \leq y_n$.

We first denote that $P(\alpha,\theta) = \int f(x)^\alpha \{(1-\theta)\mu + x(F(x) - (1-\theta)) - \mu^*(x)\}\, dF(x)$ and $Q = \int (xF(x) - \mu^*(x))\, dF(x)$ so that $\mathcal{S}_\infty(\alpha,\theta) = (2Q)^\alpha P(\alpha,\theta)/\mu$ from Lemma 2. By deriving the joint limiting distribution of $(P(\alpha,\theta), Q, \mu)'$, therefore, we can obtain the limiting distribution of $\mathcal{S}_\infty(\alpha,\theta)$ using the delta method:

$$\sqrt{K}\left(\widehat{\mathcal{S}}_\infty(\alpha,\theta) - \mathcal{S}_\infty(\alpha,\theta)\right) \to_d \mathcal{N}(0, V_\infty(\alpha,\theta)),$$

where $V_\infty(\alpha,\theta) = [\nabla\mathcal{S}_\infty(\alpha,\theta)]'\Sigma_\infty[\nabla\mathcal{S}_\infty(\alpha,\theta)]$ with

$$\nabla\mathcal{S}_\infty(\alpha,\theta) = \begin{pmatrix} \partial\mathcal{S}_\infty(\alpha,\theta)/\partial P(\alpha,\theta) \\ \partial\mathcal{S}_\infty(\alpha,\theta)/\partial Q \\ \partial\mathcal{S}_\infty(\alpha,\theta)/\partial\mu \end{pmatrix} = \begin{pmatrix} (2Q)^\alpha/\mu \\ 2\alpha(2Q)^{\alpha-1}P(\alpha,\theta)/\mu \\ -(2Q)^\alpha P(\alpha,\theta)/\mu^2 \end{pmatrix}.$$

and

$$\sqrt{K}\begin{pmatrix} \widehat{P}(\alpha,\theta) - P(\alpha,\theta) \\ \widehat{Q} - Q \\ \widehat{\mu} - \mu \end{pmatrix} \to_d \mathcal{N}(0, \Sigma_\infty).$$

Σ_∞ can be derived as follows.

First note that using the empirical distribution $\widehat{F}(\cdot)$, sample mean $\widehat{\mu} = \bar{y}$, and the kernel density estimator $\widehat{f}(\cdot)$, we define $\widehat{P}(\alpha,\theta)$ as $\widehat{P}(\alpha,\theta) = \int \widehat{f}(x)^\alpha \widehat{a}_\theta(x)\, d\widehat{F}(x)$, where $\widehat{a}_\theta(x) = (1-\theta)\widehat{\mu} + x(\widehat{F}(x) - (1-\theta)) - \widehat{\mu}^*(x)$. Then, similarly as Duclos, Esteban and Ray (2004), by collecting the dominating terms, we can approximate

$$\begin{aligned}\widehat{P}(\alpha,\theta) - P(\alpha,\theta) \quad &\simeq \quad \int \left(\widehat{f}(x)^\alpha - f(x)^\alpha\right) a_\theta(x)\, dF(x) \\ &\quad + \int f(x)^\alpha (\widehat{a}_\theta(x) - a_\theta(x))\, dF(x) + \int f(x)^\alpha a_\theta(x)\, d(\widehat{F} - F)(x)\end{aligned}$$

under proper integrability of each term and under the standard conditions for the kernel density estimator (e.g., a second-order kernel that is positive and integrable; a bandwidth parameter dimin-

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ishing to zero at the proper rate). The first and the last term can be shown to satisfy

$$\int \left(\widehat{f}(x)^{\alpha} - f(x)^{\alpha}\right) a_{\theta}(x)\, dF(x) \simeq \frac{\alpha}{n} \sum\nolimits_{i=1}^{n} \{f(y_i)^{\alpha} a_{\theta}(y_i) - P(\alpha, \theta)\},$$
$$\int f(x)^{\alpha} a_{\theta}(x)\, d(\widehat{F} - F)(x) = \frac{1}{n} \sum\nolimits_{i=1}^{n} \{f(y_i)^{\alpha} a_{\theta}(y_i) - P(\alpha, \theta)\}.$$

For the second term, we note that

$$\begin{aligned}
& \int f(x)^{\alpha} (\widehat{a}_{\theta}(x) - a_{\theta}(x))\, dF(x) \\
= & \int f(x)^{\alpha} \left\{(1-\theta) \frac{1}{n} \sum\nolimits_{i=1}^{n} y_i + x\left(\frac{1}{n} \sum\nolimits_{i=1}^{n} \mathbb{I}\{y_i < x\} - (1-\theta)\right) \right. \\
& \left. - \frac{1}{n} \sum\nolimits_{i=1}^{n} y_i \mathbb{I}\{y_i < x\} - a_{\theta}(x)\right\} dF(x) \\
= & \frac{1}{n} \sum\nolimits_{i=1}^{n} \left\{ y_i (1-\theta) \int f(x)^{\alpha}\, dF(x) + \int_{y_i} x f(x)^{\alpha}\, dF(x) \right. \\
& \left. - (1-\theta) \int x f(x)^{\alpha}\, dF(x) - y_i \int_{y_i} f(x)^{\alpha}\, dF(x) - P(\alpha, \theta) \right\}.
\end{aligned}$$

It thus follows that

$$\begin{aligned}
& \widehat{P}(\alpha, \theta) - P(\alpha, \theta) \\
\simeq & \frac{1}{n} \sum\nolimits_{i=1}^{n} \left[(1+\alpha) f(y_i)^{\alpha} a_{\theta}(y_i) + y_i (1-\theta) \int f(x)^{\alpha}\, dF(x) + \int_{y_i} (x - y_i) f(x)^{\alpha}\, dF(x)\right] \\
& - \left[(1+\alpha) P(\alpha, \theta) + (1-\theta) \int x f(x)^{\alpha}\, dF(x) + P(\alpha, \theta)\right]
\end{aligned}$$

which satisfies $\sqrt{K}(\widehat{P}(\alpha, \theta) - P(\alpha, \theta)) = \sqrt{n}(\widehat{P}(\alpha, \theta) - P(\alpha, \theta)) \to_d \mathcal{N}\left(0, \sigma_P^2\right)$ as $K(= n) \to \infty$ under the proper conditions for CLT, where

$$\sigma_P^2 = var\left((1+\alpha) f(y_i)^{\alpha} a_{\theta}(y_i) + y_i (1-\theta) \int f(x)^{\alpha}\, dF(x) + \int_{y_i} (x - y_i) f(x)^{\alpha}\, dF(x)\right).$$

Second, using a similar procedure, we can also show that

$$\widehat{Q} - Q \simeq \int \left\{\left(x\widehat{F}(x) - \widehat{\mu}^*(x)\right) - (xF(x) - \mu^*(x))\right\} dF(x) + \int (xF(x) - \mu^*(x))\, d(\widehat{F} - F)(x),$$

where the second term is simply

$$\int (xF(x) - \mu^*(x))\, d(\widehat{F} - F)(x) = \frac{1}{n} \sum\nolimits_{i=1}^{n} \{y_i F(y_i) - \mu^*(y_i) - Q\}$$

and the first term can be shown to satisfy

$$\begin{aligned}
&\int \left\{\left(x\widehat{F}(x) - \widehat{\mu}^*(x)\right) - (xF(x) - \mu^*(x))\right\} dF(x) \\
=\ &\int \left\{\left(\frac{x}{n}\sum\nolimits_{i=1}^{n} \mathbb{I}\{y_i < x\} - \frac{1}{n}\sum\nolimits_{i=1}^{n} y_i \mathbb{I}\{y_i < x\}\right) - (xF(x) - \mu^*(x))\right\} dF(x) \\
=\ &\frac{1}{n}\sum\nolimits_{i=1}^{n}\left\{\int_{y_i} x dF(x) - y_i \int_{y_i} dF(x) - Q\right\}
\end{aligned}$$

yielding

$$\widehat{Q} - Q = \frac{1}{n}\sum\nolimits_{i=1}^{n}\left[y_i F(y_i) - \mu^*(y_i) + \int_{y_i}(x - y_i) dF(x)\right] - 2Q.$$

Therefore, $\sqrt{K}(\widehat{Q} - Q) \to_d \mathcal{N}\left(0, \sigma_Q^2\right)$ as $K \to \infty$, where

$$\sigma_Q^2 = var\left(y_i F(y_i) - \mu^*(y_i) + \int_{y_i}(x - y_i) dF(x)\right).$$

Third, it is straightforward to show that $\widehat{\mu} = n^{-1}\sum_{i=1}^{n} y_i$ satisfies $\sqrt{K}(\widehat{\mu} - \mu) \to_d \mathcal{N}\left(0, \sigma^2\right)$, where $\sigma^2 = var(y_i)$. Finally, we can similarly obtain the expressions of the covariances as

$$\begin{aligned}
\sigma_{PQ} &= cov\left(\sqrt{K}(\widehat{P}(\alpha,\theta) - P(\alpha,\theta)),\ \sqrt{K}(\widehat{Q} - Q)\right) \\
&= cov\left((1+\alpha) f(y_i)^\alpha a_\theta(y_i) + y_i(1-\theta)\int f(x)^\alpha dF(x) + \int_{y_i}(x - y_i) f(x)^\alpha dF(x),\right. \\
&\qquad \left. y_i F(y_i) - \mu^*(y_i) + \int_{y_i}(x - y_i) dF(x)\right),
\end{aligned}$$

$$\begin{aligned}
\sigma_{P\mu} &= cov\left(\sqrt{K}(\widehat{P}(\alpha,\theta) - P(\alpha,\theta)),\ \sqrt{K}(\widehat{\mu} - \mu)\right) \\
&= cov\left((1+\alpha) f(y_i)^\alpha a_\theta(y_i) + y_i(1-\theta)\int f(x)^\alpha dF(x) + \int_{y_i}(x - y_i) f(x)^\alpha dF(x), y_i\right),
\end{aligned}$$

and

$$\sigma_{Q\mu} = cov\left(\sqrt{K}(\widehat{Q} - Q),\ \sqrt{K}(\widehat{\mu} - \mu)\right) = cov\left(y_i F(y_i) - \mu^*(y_i) + \int_{y_i}(x - y_i) dF(x), y_i\right).$$

By combining all these results, therefore, we can find the 3×3 matrix Σ_∞ as

$$\Sigma_\infty = \begin{pmatrix} \sigma_P^2 & \sigma_{PQ} & \sigma_{P\mu} \\ & \sigma_Q^2 & \sigma_{Q\mu} \\ & & \sigma_\mu^2 \end{pmatrix}$$

and obtain the limiting variance of $\sqrt{n}(\widehat{\mathcal{S}}_\infty(\alpha,\theta) - \mathcal{S}_\infty(\alpha,\theta))$ as described above. □

A.2 Limiting distribution of $\widehat{\mathcal{S}}(\alpha,\theta)$

This Appendix summarizes the limiting distribution of $\widehat{\mathcal{S}}(\alpha,\theta)$ as a special case of Theorem 2. When $K = 2$, we define two income groups $[y_{\min}, y^*]$ and $(y^*, y_{\max}]$ for some cutoff point y^* and consider seven U-statistics:

$$\mathbf{U}_2 \equiv \begin{bmatrix} U_{0,1} \\ U_{1,1} \\ U_{2,11} \\ U_{0,2} \\ U_{1,2} \\ U_{2,22} \\ U_{2,21} \end{bmatrix} = \begin{bmatrix} n^{-1}\sum_{i=1}^{n}\mathbb{I}\{y_i \le y^*\} \\ n^{-1}\sum_{i=1}^{n} y_i\mathbb{I}\{y_i \le y^*\} \\ n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|y_i - y_j\right|\mathbb{I}\{y_i \le y^*\}\mathbb{I}\left\{y_j \le y^*\right\} \\ n^{-1}\sum_{i=1}^{n}\mathbb{I}\{y_i > y^*\} \\ n^{-1}\sum_{i=1}^{n} y_i\mathbb{I}\{y_i > y^*\} \\ n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|y_i - y_j\right|\mathbb{I}\{y_i > y^*\}\mathbb{I}\left\{y_j > y^*\right\} \\ n^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|y_i - y_j\right|\mathbb{I}\{y_i > y^*\}\mathbb{I}\left\{y_j \le y^*\right\} \end{bmatrix}$$

which are consistent estimators of the following population counterparts as $n \to \infty$, respectively:

$$\boldsymbol{\upsilon}_2 \equiv \begin{bmatrix} \upsilon_{0,1} \\ \upsilon_{1,1} \\ \upsilon_{2,11} \\ \upsilon_{0,2} \\ \upsilon_{1,2} \\ \upsilon_{2,22} \\ \upsilon_{2,21} \end{bmatrix} = \begin{bmatrix} \int_{-\infty}^{y^*} dF(y) = F(y^*) \\ \int_{-\infty}^{y^*} y\,dF(y) \\ \int_{-\infty}^{y^*}\int_{-\infty}^{y^*} |y-x|\,dF(y)\,dF(x) \\ \int_{y^*}^{\infty} dF(y) = 1 - F(y^*) \\ \int_{y^*}^{\infty} y\,dF(y) \\ \int_{y^*}^{\infty}\int_{y^*}^{\infty} |y-x|\,dF(y)\,dF(x) \\ \int_{-\infty}^{y^*}\int_{y^*}^{\infty} |y-x|\,dF(y)\,dF(x) \end{bmatrix}.$$

Then we can rewrite $\widehat{\mathcal{S}}(\alpha,\theta)$ as

$$\widehat{\mathcal{S}}(\alpha,\theta) = \frac{(U_{1,2}U_{0,1} - U_{1,1}U_{0,2})(U_{2,11} + U_{2,22} + 2U_{2,21})^{\alpha}}{(U_{1,1} + U_{1,2})}\left\{(1-\theta)\left(\frac{U_{0,1}^3}{U_{2,11}}\right)^{\alpha} + \theta\left(\frac{U_{0,2}^3}{U_{2,22}}\right)^{\alpha}\right\}$$

that is an consistent estimator of $\mathcal{S}(\alpha,\theta)$ as $n \to \infty$ by the Slutsky's theorem, provided $\upsilon_{2,kk} > 0$ for $k = 1, 2$. Under the same condition of Lemma 1 of this paper, Theorem 7.1 of Hoeffding (1948) gives asymptotic normality of $\sqrt{n}(\mathbf{U}_2 - \boldsymbol{\upsilon}_2)$ with asymptotic variance Σ_2 given as

$$\begin{pmatrix} \upsilon_{0,1}(1-\upsilon_{0,1}) & \upsilon_{1,1}(1-\upsilon_{0,1}) & 2\upsilon_{2,11}(1-\upsilon_{0,1}) & -\upsilon_{0,1}\upsilon_{0,2} & -\upsilon_{0,1}\upsilon_{1,2} & -\upsilon_{0,1}\upsilon_{2,22} & -\upsilon_{0,1}\upsilon_{2,21} \\ & \zeta_{[1]} & 2\zeta_{[2]} & -\upsilon_{1,1}\upsilon_{0,2} & -\upsilon_{1,1}\upsilon_{1,2} & -\upsilon_{1,1}\upsilon_{2,22} & -\upsilon_{1,1}\upsilon_{2,21} \\ & & 4\zeta_{[3]} & -\upsilon_{2,11}\upsilon_{0,2} & -\upsilon_{2,11}\upsilon_{1,2} & -\upsilon_{2,11}\upsilon_{2,22} & -\upsilon_{2,11}\upsilon_{2,21} \\ & & & \upsilon_{0,2}(1-\upsilon_{0,2}) & \upsilon_{1,2}(1-\upsilon_{0,2}) & 2\upsilon_{2,22}(1-\upsilon_{0,2}) & \upsilon_{2,21}(1-\upsilon_{0,2}) \\ & & & & \zeta_{[4]} & 2\zeta_{[5]} & 2\upsilon_{2,21}(1-\upsilon_{1,2}) \\ & & & & & 4\zeta_{[6]} & 4\zeta_{[7]} \\ & & & & & & 4\zeta_{[8]} \end{pmatrix},$$

where

$$\begin{aligned}
\zeta_{[1]} &\equiv \xi_{1,1} - \upsilon_{1,1}^2 = \int_{-\infty}^{y^*} y^2 dF(y) - \upsilon_{1,1}^2 \\
\zeta_{[2]} &\equiv \xi_{2,11} - \upsilon_{2,11}\upsilon_{1,1} = \int_{-\infty}^{y^*}\int_{-\infty}^{y^*} y\,|y-x|\,dF(y)\,dF(x) - \upsilon_{2,11}\upsilon_{1,1} \\
\zeta_{[3]} &\equiv \xi_{3,111} - \upsilon_{2,11}^2 = \int_{-\infty}^{y^*}\left\{\int_{-\infty}^{y^*}|y-x|\,dF(x)\right\}^2 dF(y) - \upsilon_{2,11}^2 \\
\zeta_{[4]} &\equiv \xi_{1,2} - \upsilon_{1,2}^2 = \int_{y^*}^{\infty} y^2 dF(y) - \upsilon_{1,2}^2 \\
\zeta_{[5]} &\equiv \xi_{2,22} - \upsilon_{2,22}\upsilon_{1,2} = \int_{y^*}^{\infty}\int_{y^*}^{\infty} y\,|y-x|\,dF(y)\,dF(x) - \upsilon_{2,22}\upsilon_{1,2} \\
\zeta_{[6]} &\equiv \xi_{3,222} - \upsilon_{2,22}^2 = \int_{y^*}^{\infty}\left\{\int_{y^*}^{\infty}|y-x|\,dF(x)\right\}^2 dF(y) - \upsilon_{2,22}^2 \\
\zeta_{[7]} &\equiv \xi_{3,221} - \upsilon_{2,22}\upsilon_{2,21} = \int_{y^*}^{\infty}\left\{\int_{y^*}^{\infty}|y-x|\,dF(y)\right\}\left\{\int_{-\infty}^{y^*}(y-x)dF(y)\right\} dF(x) - \upsilon_{2,22}\upsilon_{2,21} \\
\zeta_{[8]} &\equiv \xi_{3,211} - \upsilon_{2,21}^2 = \int_{y^*}^{\infty}\left\{\int_{-\infty}^{y^*}(y-x)dF(x)\right\}^2 dF(y) - \upsilon_{2,21}^2.
\end{aligned}$$

From this result, Theorem 7.5 of Hoeffding (1948) yields $\sqrt{n}(\widehat{\mathcal{S}}(\alpha,\theta) - \mathcal{S}(\alpha,\theta)) \to_d \mathcal{N}(0, V_2(\alpha,\theta))$ as $n \to \infty$, where $V_2(\alpha,\theta) = [\nabla\mathcal{S}(\alpha,\theta)]'\Sigma_2[\nabla\mathcal{S}(\alpha,\theta)]$ and the 7×1 vector $\nabla\mathcal{S}(\alpha,\theta)$ is given by

$$\frac{\zeta\delta^\alpha}{\mu}\begin{pmatrix} \upsilon_{1,2}/\zeta & 3(1-\theta)(\upsilon_{0,1}^{3\alpha-1}/\upsilon_{2,11}^{\alpha}) \\ -\upsilon_{0,2}/\zeta - 1/\mu & 0 \\ \alpha/\delta & -(1-\theta)(\upsilon_{0,1}^{3\alpha}/\upsilon_{2,11}^{\alpha+1}) \\ -\upsilon_{1,1}/\zeta & 3\theta(\upsilon_{0,2}^{3\alpha-1}/\upsilon_{2,22}^{\alpha}) \\ -\upsilon_{0,1}/\zeta - 1/\mu & 0 \\ \alpha/\delta & -\theta(\upsilon_{0,2}^{3\alpha}/\upsilon_{2,22}^{\alpha+1}) \\ 2\alpha/\delta & 0 \end{pmatrix}\begin{pmatrix} \Phi(\alpha,\theta) \\ \alpha \end{pmatrix}$$

with $\zeta = \upsilon_{1,2}\upsilon_{0,1} - \upsilon_{1,1}\upsilon_{0,2} = (\mu_2 - \mu_1)\pi_1\pi_2$ and $\Phi(\alpha,\theta) = (1-\theta)\left(\upsilon_{0,1}^3/\upsilon_{2,11}\right)^\alpha + \theta\left(\upsilon_{0,2}^3/\upsilon_{2,22}\right)^\alpha$.

The asymptotic variance $V_2(\alpha,\theta)$ can be consistently estimated using the sample counterparts of each element (i.e., the U-statistics), but the calculation is quite cumbersome. To facilitate the variance estimation of $\widehat{\mathcal{S}}(\alpha,\theta)$, we propose a subsampling method, specifically the jackknife variance estimation (e.g., Yitzhaki, 1991; Karagiannis and Kovacevic, 2000), as follows.

1. $\widehat{S}(\alpha,\theta)$:

   (a) Sort the income data in *ascending* order and denote them as $\{y_i\}_{i=1}^n$; the index of y_i also represents its rank r_i.

   (b) Calculate $\bar{y} = (1/n)\sum_{i=1}^n y_i$ and define $L = \sum_{i=1}^n r_i y_i$ and $H_i = \sum_{j=i+1}^n y_j$ for $i = 1, 2, \cdots, n$ with $H_n = 0$. The Gini coefficient is obtained as $\widehat{G} = [(2L)/(\bar{y}n^2) - (n+1)/n]$ and thus the mean difference is given as $\widehat{\delta} = 2\bar{y}\widehat{G}$.

   (c) Group the sorted data into two using a given cutoff point y^*, and let $A_1 = \{y_i | y_i < y^*\}$ and $A_2 = \{y_i | y_i \geq y^*\}$. For each group $k = 1, 2$, let n_k be the number of observations in group k and $\{y_{k,i}\}_{i=1}^{n_k}$ be the sorted income data in group k. Also denote $r_{k,i}$ as the rank of $y_{k,i}$'s in group k.

   (d) Calculate $\widehat{\pi}_k = n_k/n$ and $\bar{y}_k = (1/n_k)\sum_{i=1}^{n_k} y_{k,i}$. Also define $L_k = \sum_{i=1}^{n_k} r_{k,i} y_{k,i}$ and $H_{k,i} = \sum_{j=i+1}^{n_k} y_{k,j}$ for $i = 1, 2, \cdots, n_k$ with $H_{k,n_k} = 0$. The Gini coefficient of group k can be obtained as $\widehat{G}_k = (2L_k)/(\bar{y}_k n_k^2) - (n_k+1)/n_k$ and thus the mean difference is given as $\widehat{\delta}_k = 2\bar{y}_k\widehat{G}_k$.

   (e) Using values obtained above, calculate $\widehat{S}(\alpha,\theta)$ as in (5) for given α and θ.

2. Leave-one-out:

   (a) Omit the i-th observation y_i from the entire sample. For the rest of the observations, do not change their group membership even after omission.

   (b) Using $(n-1)$-number of observations, obtain the new sample mean, the Gini coefficient, and the mean difference as

$$\bar{y}_{(-i)} = \frac{n\bar{y} - y_i}{n-1}, \quad \widehat{G}_{(-i)} = \frac{2(L - r_i y_i - H_i)}{\bar{y}_{(-i)}(n-1)^2} - \frac{n}{n-1}, \quad \text{and } \widehat{\delta}_{(-i)} = 2\bar{y}_{(-i)}\widehat{G}_{(-i)}.$$

   (c) Let

$$\begin{cases} \widehat{\pi}_{1,(-i)} = \frac{(n_1-1)}{n-1}, & y_{1,(-i)} = \frac{(n_1\bar{y}_1 - y_i)}{n_1-1}, & \bar{y}_{2,(-i)} = \bar{y}_2 & \text{if } y_i \in A_1 \\ \widehat{\pi}_{1,(-i)} = \frac{n_1}{n-1}, & y_{1,(-i)} = \bar{y}_1, & \bar{y}_{2,(-i)} = \frac{(n_2\bar{y}_2 - y_i)}{n_2-1} & \text{if } y_i \in A_2 \end{cases},$$

   $\widehat{\pi}_{2,(-i)} = 1 - \widehat{\pi}_{1,(-i)}$, and calculate the Gini coefficients as

$$\begin{cases} \widehat{G}_{1,(-i)} = \frac{2\left(L_1 - r_{1,i}y_i - H_{1,i}\right)}{\bar{y}_{1,(-i)}(n_1-1)^2} - \frac{n_1}{n_1-1}, & \widehat{G}_{2,(-i)} = \widehat{G}_2 & \text{if } y_i \in A_1 \\ \widehat{G}_{1,(-i)} = \widehat{G}_1, & \widehat{G}_{2,(-i)} = \frac{2\left(L_2 - r_{2,i}y_i - H_{2,i}\right)}{\bar{y}_{2,(-i)}(n_2-1)^2} - \frac{n_2}{n_2-1} & \text{if } y_i \in A_2 \end{cases}$$

which yields the mean difference of each group as $\widehat{\delta}_{1,(-i)} = 2\bar{y}_{1,(-i)}\widehat{G}_{1,(-i)}$ and $\widehat{\delta}_{2,(-i)} = 2\bar{y}_{2,(-i)}\widehat{G}_{2,(-i)}$.

(d) Using these values, obtain the leave-one-out estimator $\widehat{\mathcal{S}}_{(-i)}(\alpha, \theta)$ for each i.

(e) Iterate the leave-one-out steps from $i = 1$ to n and recursively calculate

$$\widehat{V}_{2,i} = \widehat{V}_{2,i-1} + \frac{n-1}{n}\left(\widehat{\mathcal{S}}_{(-i)}(\alpha, \theta) - \widehat{\mathcal{S}}(\alpha, \theta)\right)^2$$

with $\widehat{V}_{2,0} = 0$. Then, $\widehat{V}_{2,n}$ is the jackknife variance estimate of $\widehat{\mathcal{S}}(\alpha, \theta)$.

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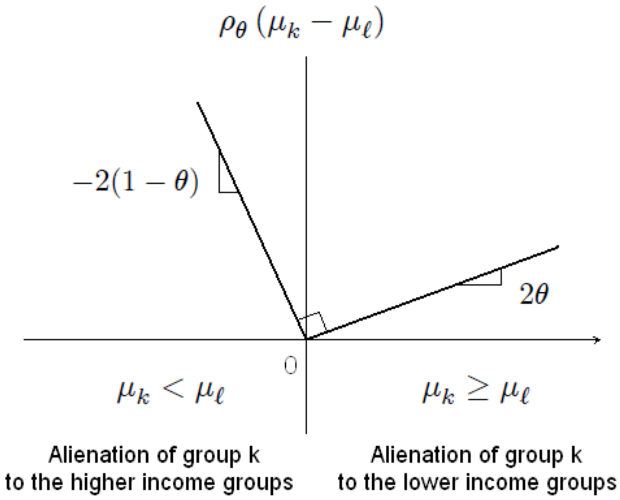

Figure 1: $\rho_\theta(\mu_k - \mu_\ell)$ describes asymmetric alienations of group k towards ℓ

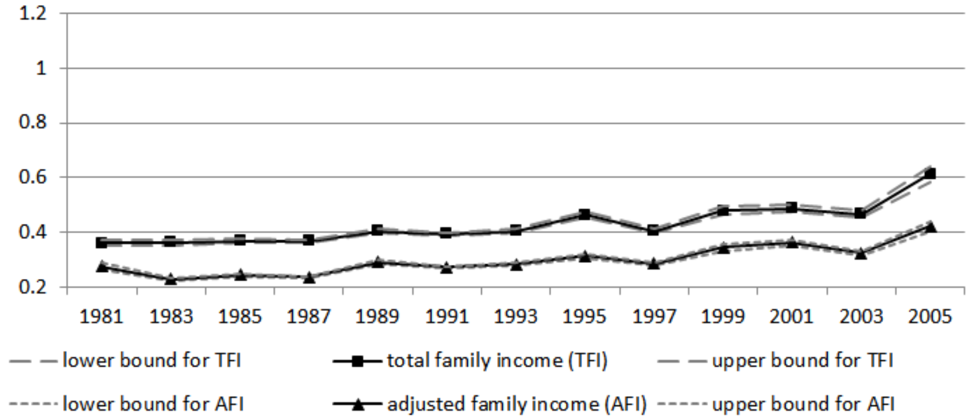

Figure 2: Evolution of the Level of Social Tension: $(\theta, \alpha) = (0.5, 1.6)$. Data source: Panel Study of Income Dynamics. The horizontal axis represents survey years. The 95% confidence level is adopted for (pointwise) confidence intervals.

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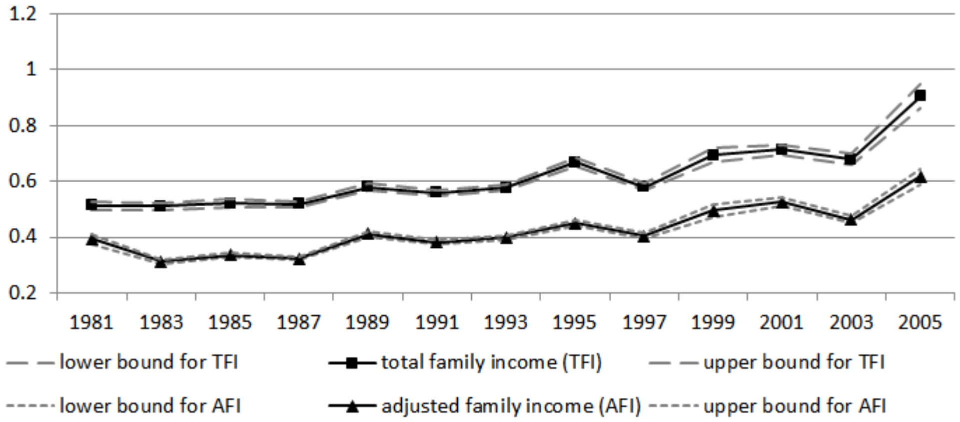

Figure 3: Evolution of the Level of Social Tension: $(\theta, \alpha) = (0.25, 1.6)$.

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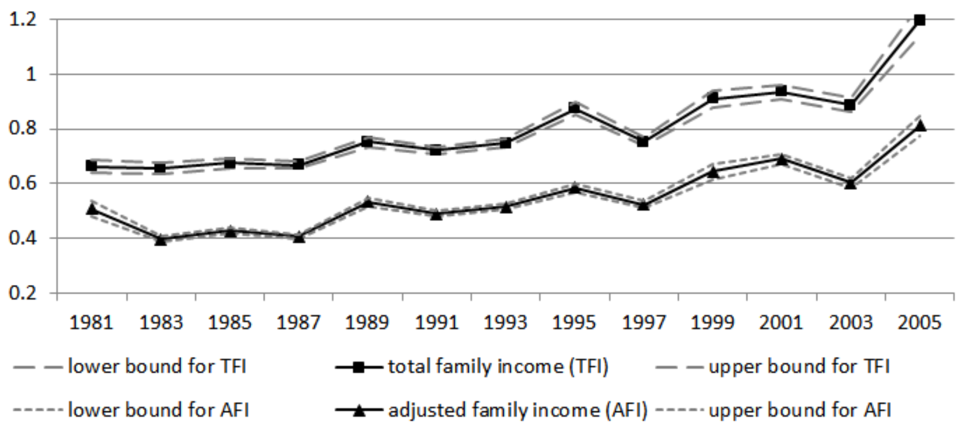

Figure 4: Evolution of the Level of Social Tension: $(\theta, \alpha) = (0.0, 1.6)$.

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