Subsequence Based Recovery of Missing Samples in Oversampled Bandlimited Signals

Erchin Serpedin

Abstract—A new approach for recovering an arbitrary finite number of missing samples in an oversampled bandlimited signal is presented. This correspondence also proposes an approach for recovering the original signal’s spectrum from the spectra of a certain number of subsequences, obtained by downsampling the original sequence. Closed-form expressions for the missing samples in terms of the known samples are obtained by exploiting the linear dependence relationship among the spectra of downsampling subsequences.

Index Terms—Bandlimited, downsampling, oversampling, reconstruction.

I. INTRODUCTION

This correspondence describes a new approach for restoring a finite (possibly infinite) number of missing samples in an oversampled bandlimited signal. It is well known [1], [4], [5] that if the discrete-time signal $x[n]$ is obtained by oversampling the continuous-time and bandlimited signal $x_r(t)$, $x[n] = x_r(nT)$, with the sampling frequency $f = 1/T$ satisfying

$$2\pi f = \frac{2\pi}{T} > 2\Omega_N$$

where $\Omega_N$ represents the Nyquist frequency or the signal’s bandwidth, then the samples of the signal $x[n]$ have the property that any finite set of samples can be recovered exactly from the remaining ones.

Many approaches have been proposed for the reconstruction of the missing samples in an oversampled bandlimited signal. Among these, we mention the iterative approaches from [3], [5], and [7] and the noniterative ones from [1] and [4]. In this correspondence, we propose a new noniterative algorithm that allows the reconstruction not only of a finite number of samples but of an infinite number of samples as well, provided that the density per unit time of the known samples is greater than the Nyquist rate.

The organization of this correspondence is as follows. First, it is shown that the spectrum (Fourier transform) of the original signal can be recovered from the spectra of certain subsequences obtained by downsampling the original signal. For any fixed frequency, the value of the signal’s spectrum is obtained by solving a system of linear equations in the frequency domain. Next, it is shown that the spectra of these downsampled subsequences are linearly related. This linear dependence relationship is inverse Fourier transformed in order to obtain a closed-form expression for the missing samples. Finally, an approach for exact reconstruction of the original signal from a finite number of downsampling subsequences with arbitrary sampling frequencies is proposed. For practical implementations, the numerical stability of the present algorithm has to be further analyzed. Apart from this, however, the proposed approach has the advantage of not being sensitive to certain additive periodic perturbations and satisfies a certain minimal condition; the unknown samples are expressed in terms of a minimum number of known samples, as opposed to the approaches from [1] and [4], which make use of all the known samples.

II. RECONSTRUCTION OF MISSING SAMPLES

Let the set $\mathcal{L}$ denote the locations of the unknown samples in the discrete-time signal $x[n]$. $\mathcal{L}$ has cardinal $|\mathcal{L}| = l$, and it consists of a sequence of consecutive integers $\mathcal{L} := \{0, 1, \ldots, l-1\}$. Later, we consider the general case of an arbitrary distribution and of an infinite number of the missing samples. Since $2\pi f = 2\pi / T > 2\Omega_N$, there exists a minimal integer $m$ such that

$$m\frac{2\pi}{(m+l)T} \geq 2\Omega_N.$$  

(2)

Indeed, it is sufficient to note that $\lim_{m \to \infty} (m2\pi)/(m+l)T = \pi/\omega_N$. The minimal value of $m$ that satisfies (2) is $m = \lceil (\Omega_N l)/(\pi f - \Omega_N) \rceil$, with $[x]$ denoting the least integer greater than or equal to $x$. Due to the minimality of $m$, it also holds that

$$\Omega_N > \frac{(m-1)2\pi}{(m+l)T}.$$  

(3)

Define $\omega_N := \Omega_N T$. From (2) and (3), it follows that

$$(m-1)\pi < (m+l-1)\omega_N < (m+l)\omega_N < m\pi.$$  

(4)

Define the following $m$ subsequences $x_k[n], \ k = 0, \ldots, m-1$, obtained by downsampling $x[n]$ by a factor $m+l$

$$x_k[n] := x[k + l + n(m + l)].$$  

(5)

From the definition of the set of unknown samples $\mathcal{L}$, it follows that the samples of all subsequences $x_k[n], \ k = 0, \ldots, m-1$ are known. Next, it is shown that the spectra of these $m$ subsequences provide enough information for the recovery of the original signal spectrum $X(\exp(j\omega \cdot ))$. Recall from [6] that if the discrete-time signal $x[n]$ is downsampled by a factor of $m+l$, $x[n(m+l)] \forall n$, then the spectra of these two sequences are related by

$$X_k(e^{j\omega}) = \frac{1}{m+l} \sum_{r=0}^{m+l-1} X_l(e^{j\omega(m+l)T/m}) \quad \forall k.$$  

(6)

Note that in writing (6), the frequency axis is normalized with respect to the reduced sampling rate (12 [p. 13], (6) [p. 103]). According to (5) and (6), the spectrum of $x_k[n]$ can be expressed as

$$X_k(e^{j\omega}) = \frac{1}{m+l} \sum_{r=0}^{m+l-1} e^{-j(\omega(m+l)T/m) + j\omega r T/m} X(e^{j\omega(m+l)T/m}) \quad \forall k.$$  

(7)

Since $X(\exp(j\omega \cdot ))$ has the bandwidth equal to $\omega_N := \Omega_N T$, according to (7), $X_k(e^{j\omega \cdot })$ is a sum of aliased components $X(\exp(j\omega - 2\pi f)/(m+l))$, each one of bandwidth $\bar{\omega} := (m+l)\omega_N$. From (4), we deduce that

$$m - 1 < \bar{\omega} < m\pi.$$  

(8)

Note also that

$$X_k(e^{j(\omega(m+l)T/m)}) = X(e^{j(\omega(m+l)T/m - 2\pi f)}) \quad \forall \omega.$$  

(9)
Using the change of variables $r \leftarrow r - m - l$, we rewrite (7) as

$$X_k(e^{i\omega r}) = \frac{1}{m + l} \sum_{r=-(m+l)/2}^{(m+l)/2} e^{i\omega r X_k(r)} e^{i\omega r X_k(r')}, \forall k.$$  \hspace{1cm} (10)

We then have the result.

**Proposition 1:** In (10), the number of aliased spectral components $X(\exp[j(\omega - 2\pi \tau)/(m + l)])$, which are nonzero on $[-\pi, \pi]$, is equal to $m$. These spectral replicas correspond to the indices shown in (11) at the bottom of the page.

**Proof:** Consider the case when $\tau = 0$, and $m = 2p + 1$. The spectral components $X(\exp[j(\omega - 2\pi \tau)/(m + l)])$, $X(\exp[j(\omega + 2\pi \tau)/(m + l)])$, and $X(\exp[j(\omega - 2\pi \tau)/(m + l)])$ are plotted in Fig. 1. Due to the bandwidth constraint (8), it follows that the components that do not cancel on $[0, \pi]$ are $X(\exp[j(\omega - 2\pi \tau)/(m + l)])$, for $r = -(m - 1)/2, \ldots, (m - 1)/2$. Note that $X(\exp[j(\omega + 2\pi \tau)/(m + l)])$ and $X(\exp[j(\omega - 2\pi \tau)/(m + l)])$ are equal to zero on $[-\pi, \pi]$. Similarly, one can treat the case when $m$ is even.

In what follows, it is assumed w.l.o.g. that $m = 2p + 1$ is odd.

According to Proposition 1, it holds that

$$X_k(e^{i\omega r}) = \frac{1}{m + l} \sum_{r=-(m+l)/2}^{(m+l)/2} e^{i\frac{\omega r X_k(k + D)}{m+l}} X(e^{i\frac{\omega r X_k(k + D)}{m+l}}),$$

\hspace{1cm} \forall k \in [-\pi, \pi]. \hspace{1cm} (12)

It follows that the number of spectral components, which contribute on $[-\pi, \pi]$ to $X_k(\exp[j\omega])$ in (12), is equal to $m$, irrespective of the parity of $m$. Equation (12), for $k = 0, \ldots, m - 1$, allows us to solve for the $m$ spectral components $X(\exp[j(\omega - 2\pi \tau)/(m + l)])$, $r = -p, \ldots, p$, which can be plugged back into (12) to obtain $X_k(\exp[j\omega])$ for $k = -l, \ldots, -1$. This idea will be developed next.

Note that for a fixed $\omega$ in $[-\pi, \pi]$, (12) provides a number of $m$ possible equations, with $m$ unknowns $X(\exp[j(\omega - 2\pi \tau)/(m + l)])$, $r = -p, \ldots, p$. When $\omega$ varies on the interval $[-\pi, \pi]$, the argument of $X(\exp[j(\omega + 2\pi \tau)/(m + l)])$ takes values in the interval $[-\pi(2r + 1)/(m + l), -\pi(2r - 1)/(m + l)]$. Since $r = -(m - 1)/2, \ldots, [(m - 1)/2]$, the sum of all these intervals is $2m/(m + l) \geq 2\pi\nu$. Thus, (12) allows reconstruction of the entire spectrum $X(\exp[j\omega])$ for any $\omega \in [-\pi, \pi]$.

Note that once the original signal spectrum is obtained, the missing samples can be simply obtained by inverse Fourier transforming the recovered spectrum. Here, the recovering of the missing samples is performed using an alternative way, which avoids computing explicitly the spectrum $X(\exp[j\nu\omega])$. It is shown that the spectra of any $m + 1$ subsequences $x_k[n]$ verify a certain linear dependence relationship. This relationship is then inverse Fourier transformed in order to recover the unknown samples. The approach that we pursue relies on expressing the spectra of the $m$ subsequences $x_k[n], x_{k+1}[n], \ldots, x_{k+m-1}[n]$ in terms of the aliased spectral components $X(\exp[j(\omega - 2\pi \tau)/(m + l)])$, $r = -p, \ldots, p$ using (12). Then, the spectra of the $m$ subsequences $x_k[n], x_{k+1}[n], \ldots, x_{k+m-1}[n]$ is obtained by eliminating the dependence relationship on the spectral components $X(\exp[j(\omega - 2\pi \tau)/(m + l)])$. In order to rewrite (12) in matrix form, we introduce the notations

$$z_k := e^{-i\frac{\omega kD}{m+l}}, \hspace{1cm} (13)$$

$$t_k := e^{i\frac{\omega kD}{m+l}}, \hspace{1cm} k = 0, \ldots, 2p,$$

$$X_{-\omega p} := [X(e^{i\frac{\omega (2\pi p)}{m+l}}) \cdots X(e^{i\frac{\omega (2\pi (p-1))}{m+l}})]^T.$$  \hspace{1cm} (15)

With `$\cdot$' standing for transposition. In order to simplify the above notations, we have not included the dependence on $\omega$ in the left-hand side terms of (13)–(15). Variable $\omega$ is supposed to have an arbitrary but fixed value in $[-\pi, \pi]$. We rewrite (12) in the form

$$X_k(e^{i\omega r}) = \frac{t_k}{2p + l + 1} [z_k^{-1} \cdots z_k^{-1}] [z_k \cdots z_k^2] X_{-\omega p}, \hspace{1cm} k = 0, \ldots, 2p.$$  \hspace{1cm} (16)

Collecting together (16) for $k = 0, \ldots, 2p$, we obtain

$$X = \frac{1}{2p + l + 1} \text{TZX}_{-\omega p},$$

where $\text{X} := [X_0(\exp[j\omega]) \ X_1(\exp[j\omega]) \cdots X_{2p}(\exp[j\omega])]^T$, $\text{T} := [\begin{array}{c} t_0 \ 0 \ \cdots \ 0 \ t_1 \ \cdots \ 0 \ \vdots \ \vdots \ 0 \ \cdots \ 0 \end{array}]$, $\text{Z} := \begin{bmatrix} z_0 & \cdots & z_{2p} \\
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & t_{2p} \end{bmatrix}.$
Consider an arbitrary index \( i_0 \), \( 0 \leq i_0 \leq L - 1 \) and the subsequence \( x_{i_0 - [n]} \) defined as in (5). Note that the unknown sample \( x[i_0] \) can be expressed as \( x[i_0] = x_{i_0 - [0]} \). A relationship similar to (17) can be written for the set of subsequences \( \{x_{i_0 - [n]}, x_0 [n], \ldots, x_{2p - 1} [n]\} \)

\[
\mathbf{Z} := \begin{bmatrix} z_0^{-1} & \cdots & z_0^{-1} & 1 & z_0 & \cdots & z_0^{-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
z_{2p}^{-1} & \cdots & z_{2p}^{-1} & 1 & z_{2p} & \cdots & z_{2p}^{-1} \\
\end{bmatrix} .
\]  

(19)

We obtain the desired spectral elimination by eliminating \( \mathbf{X}_{-p:p} \) between (17) and (20). From (17), we infer that \( \mathbf{X}_{-p:p} = (2p + l + 1) \mathbf{Z}^{-1} \mathbf{T}^{-1} \mathbf{X} \), and substituting it into (20), we obtain

\[
\mathbf{X} = \mathbf{T} \mathbf{Z} \mathbf{Z}^{-1} \mathbf{T}^{-1} \mathbf{X} .
\]  

(21)

Due to its Vandermonde structure, matrix \( \mathbf{Z} \) is always invertible. Note also that

\[
\mathbf{Z} = \mathbf{S} \mathbf{Z}'
\]  

(22)

where the matrix \( \mathbf{S} \) has the companion structure

\[
\mathbf{S} := \begin{bmatrix} s_0 & s_1 & \cdots & s_{2p} \\
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 \\
\end{bmatrix}
\]  

(23)

and the scalars \( s_0, s_1, \ldots, s_{2p} \) satisfy the Vandermonde system of equations

\[
\begin{bmatrix} s_0 \\
\vdots \\
\vdots \\
\vdots \\
1 \\
\end{bmatrix} = \begin{bmatrix} z_0^{-1} \\
\vdots \\
\vdots \\
\vdots \\
z_0^{-1} \\
\end{bmatrix} .
\]  

(24)

Making use of Cramer’s rule to solve (24), \( s_k \) can be expressed as the ratio of two Vandermonde determinants. Simple calculations show that

\[
s_k = \frac{z_k^{-p}}{z_0^{-p}} \prod_{r=0, r \neq k}^{2p} \left( \frac{z_0^{-1} - z_r}{z_0^{-1} - z_k} \right) = \frac{2p}{2p} \sin \left( \frac{\pi (k - 1)}{2p} \right) .
\]  

(25)

Substitute now (22) into (21) to obtain

\[
\mathbf{X} = \mathbf{T} \mathbf{S} \mathbf{Z}' \mathbf{T}^{-1} \mathbf{X} .
\]  

(26)

Taking into account that \( \mathbf{T} \) and \( \mathbf{T}^{-1} \) are diagonal matrices and that \( \mathbf{S} \) is a companion matrix, direct multiplications in (26) lead to

\[
X_{i_0 - l}(e^{j\omega}) = t_{i_0 - l} \left[ s_0 t_0^{-1} X_0(e^{j\omega}) + \cdots + s_{2p} t_{2p}^{-1} X_{2p}(e^{j\omega}) \right] \quad \forall \omega \in [-\pi, \pi]
\]  

(27)

We infer that for any oversampled bandlimited signal, there exists a linear dependence relationship among the spectra of downsampled subsequences \( x_k[n] \). We remark also that relation (27) can be tested by plugging (12) into (27), equating the coefficients of \( X(e^{j\omega}) = \frac{2\pi}{(m + l)} \), \( r = -p, \ldots, p \) of both sides of (27), and taking into account (24). Considering the notation \( a_k := (i_0 - k - l)/((2p + l + 1)) \), \( \forall k \) integer and observing that \( t_{i_0 - l}(e^{j\omega}) = \exp(j\omega a_k) \), we can express (27) under the equivalent forms

\[
X_{i_0 - l}(e^{j\omega}) = s_0 e^{j\omega a_0} X_0(e^{j\omega}) + \cdots + s_{2p} e^{j\omega a_{2p}} X_{2p}(e^{j\omega})
\]

\[
\times \sum_{n=-\infty}^{\infty} x_0[n + (m + l)] e^{-jn\omega}
\]

\[
= s_0 \sum_{n=-\infty}^{\infty} x[l + n(m + l)] e^{j\omega (a_0 - n)} + \cdots
\]

\[
+ s_{2p} \sum_{n=-\infty}^{\infty} x[2p + l + n(m + l)] e^{j\omega (a_{2p} - n)}
\]  

(28)

where \( X_k(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[k + l + n(m + l)] \exp(-jn\omega) \).

Integrating (28) from \( \pi \) to \( -\pi \) and subtracting \( X[i_0] \), we obtain

\[
x[i_0] = \sum_{k=0}^{m-1} \left( \frac{2\pi}{2p} \sin \left( \frac{\pi (k - 1)}{2p} \right) \right)
\]  

(29)

In (29), the unknown samples \( x[i_0], 0 \leq i_0 \leq L - 1 \), are expressed in terms of only the samples \( x[n] \) with \( n \equiv k + l \mod m + 1 \) and \( 0 \leq k \leq m - 1 \). It follows that the recovery of the missing samples \( x[i_0], 0 \leq i_0 \leq L - 1 \) is not sensitive to periodic perturbations that affect only the samples \( x[n] \), with \( n \equiv k + l \mod m + 1 \), \( -l \leq k \leq -1 \) since these samples are not used in the recovery of \( x[i_0] \) (29). Note that even an infinite number of samples can be recovered. All the samples \( x[n] \), with \( n \equiv k + l \mod m + 1 \), \( -l \leq k \leq -1 \) can be reconstructed along similar lines. Equation (29) suggests that the algorithm is numerically unstable, i.e., small perturbations in the amplitudes of the original signal and/or truncation of the infinite sum in (29) may give rise to large variations in the reconstructed samples. This is due to the fact that the dependence on samples in (29) is proportional to sinc\( (a_k - n) \sim n^{-1} \), which may not guarantee numerical stability. The stabilization of this algorithm (i.e., finding a dependence relationship proportional to \( n^{-\alpha} \) with \( \alpha > 1 \)) is an open problem.

Next, we want to show that the reconstruction of the missing samples can be performed assuming an arbitrary distribution of the missing samples. The following result establishes a sufficient condition for the perfect reconstruction of an oversampled bandlimited signal from a set of subsequences obtained by downsampling the original sequence with arbitrary rates.

**Proposition 2:** Given \( m \) disjoint subsequences \( x_k[n], k = 0, 1, \ldots, m - 1 \) obtained by downsampling by a factor \( M_k \), the oversampled and bandlimited signal \( x[n] \), then the original signal \( x[n] \) can be recovered from the samples of these subsequences, provided that the following condition holds:

\[
2\pi (f_0 + f_1 + \cdots + f_{m-1}) \geq 2f_N
\]  

(30)

where \( f_k \) represents the sampling rate of subsquence \( x_k[n] \), \( f_k = 1/M_kT \), \( k = 0, \ldots, M - 1 \), and \( f_N \) denotes the Nyquist frequency.

**Proof:** Let \( M \) be the least common multiple of \( M_0, M_1, \ldots, M_{m-1} \). Thus, there exist integers \( N_0, N_1, \ldots, N_{m-1} \) such that

\[
M = M_0 N_0 = \cdots = M_{m-1} N_{m-1}.
\]  

(31)

We further downsample each subsequence \( x_k[n] \) by a factor \( N_k, k = 0, \ldots, m - 1 \). We obtain \( N_k \) new subsequences \( x_{k,a}[n], q = 0, \ldots, N_k - 1 \) for each \( k = 0, \ldots, m - 1 \). Hence, we obtain a total of \( N_0 + N_1 + \cdots + N_{m-1} \) subsequences, each one having the same sampling rate \( 2\pi/(M_kT N_k) = 2\pi/MT \). The spectrum of subsequence \( x_{k,a}[n] \) consists of a sequence of shifted
of subsequences has been presented. However, the stabilization of the spectra of these subsequences. A general condition for the reconstruction of the original signal spectrum from the set of samples for an oversampled bandlimited signal is based on the reconstruction of the classical theorem of Nyquist (Shannon).

III. CONCLUSION

A new approach has been described for the reconstruction of a finite set of samples for an oversampled bandlimited signal. The approach is based on the reconstruction of the original signal spectrum from the spectra of certain subsequences obtained by down sampling the original sequence and on the linear dependence relationship that is satisfied by the spectra of these subsequences. A general condition for the reconstruction of a bandlimited and oversampled signal from an arbitrary set of subsequences has been presented. However, the stabilization of the proposed algorithm remains open for future research.

REFERENCES


Adaptive Subspace Algorithm for Blind Separation of Independent Sources in Convolutive Mixture

Ali Mansour, Christian Jutten, and Philippe Loubaton

Abstract—The advantage of the algorithm proposed in this correspondence is that it reduces a convolutive mixture to an instantaneous mixture by using only second-order statistics (but more sensors than sources). Furthermore, the sources can be separated by using any algorithm applicable to an instantaneous mixture. Otherwise, to ensure the convergence of our algorithm, we assume some classical assumptions for blind separation of sources and some added subspace assumptions. Finally, the assumptions concerning the subspace model and their properties are emphasized in this correspondence.

Index Terms—Cholesky decomposition, convolutive mixture, ICA and blind separation of sources, LMS, subspace methods, Sylvester matrix.

I. INTRODUCTION

In the last five years, only a few methods have been proposed for separating sources from a linear convolutive mixture [1]. Generally, the separation methods are based on higher order statistics (typically fourth-order statistics). However, separation algorithms based on second-order statistics are very efficient if specific assumptions concerning sources or mixtures are satisfied. Generally, the blind separation of sources is based on two basic assumptions: The source signals are statistically independent [2], and only one source has normal distribution [3]. In the following, we assume that these two assumptions are satisfied.

The blind-separation-of-sources problem involves retrieving the p sources from the q observations of unknown mixtures of unknown sources. Let us denote a set of q observed mixtures of p unobservable independent sources \( s(n) = (s_1(n), s_2(n), \ldots, s_p(n))^T \) by \( x(n) = (x_1(n), x_2(n), \ldots, x_q(n))^T \). The channel effect is considered to be a convolutive mixture

\[
x(n) = \sum_{i=0}^{M} A(i) s(n-i) \iff \mathbf{x}(z) = A(z) \mathbf{s}(z)
\]

where \( A(z) \) is the channel transfer matrix. Suppose \( A(z) = (a_{ij}(z)) \) is generically full column rank, and \( M \) is the degree of \( A(z) \). In addition, let \( \mathbf{x}(n) \) and \( \mathbf{s}(n) \) be random vectors defined by

\[
\mathbf{x}(n) = \left( \begin{array}{c} \mathbf{x}_N(n) \\ \mathbf{x}_N(n-N) \end{array} \right)
\]

and

\[
\mathbf{s}(n) = \left( \begin{array}{c} \mathbf{s}(n) \\ \mathbf{s}(n-M-N) \end{array} \right)
\]

Using (1) and (2), we can easily prove that \( \mathbf{x}_N(n) = T_N(A) \mathbf{s}_N(n) \), where \( T_N(A) \) is the Sylvester matrix associated with \( A(z) \), shown in (2a) at the bottom of the next page.

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A. Mansour is with the BMC Research Center (Riken), Nagoya, Japan.
C. Jutten is with INPG-LIB, Grenoble, France.
P. Loubaton is with the University de Marne la Vallée, Noisy-Le-Grand, France.

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