

Braided Oscillators

A. Yildiz

Feza Gürsey Institute, P.O. Box 6, 81220, Çengelköy, Istanbul, Turkey

February 1, 2008

Abstract

A generalized oscillator algebra is proposed and the braided Hopf algebra structure for this generalized oscillator is investigated. Using the solutions for the braided Hopf algebra structure, two types of braided Fibonacci oscillators are introduced. This leads to two types of braided Biedenharn-Macfarlane oscillators as special cases of the Fibonacci oscillators. We also find the braided Hopf algebra solutions for the three dimensional braided space. One of these, as a special case, gives the Hopf algebra given in the literature.

1 Introduction

The harmonic oscillator has a wide variety of applications from quantum optics to the realizations of the angular momentum algebra and hence the deformations of the oscillator algebra play an important role in q -deformed theories. The realization of the q -deformed angular momentum algebra by Biedenharn-Macfarlane oscillators¹ and the realization of the Hermitian braided matrices by a pair of q -oscillators² are some of the examples. The two parameter deformations and some of their applications can be found in³. Braided group theory (a self contained review can be found in⁴) deforms the notion of tensor product (called braided tensor product) and hence deforms the independence of the objects. Although braided groups arise in the formulation of quantum group covariant structures, the idea of braiding can be used without any reference to quantum groups to generalize the statistics⁵.

The permutation map π ($\pi : A \otimes B \rightarrow B \otimes A$) in the tensor product algebra of boson algebras

$$(a \otimes b)(c \otimes d) = a\pi(b \otimes c)d = ac \otimes bd$$

is replaced by a generalized map called braiding ψ ($\psi : A \otimes B \rightarrow B \otimes A$) such that

$$(a \otimes b)(c \otimes d) = a\psi(b \otimes c)d.$$

This generalization leads to the generalization of the Hopf algebra called braided Hopf algebra⁶

whose axioms in algebraic (not diagrammatic) form read as

$$m \circ (id \otimes m) = m \circ (m \otimes id)$$

$$m \circ (id \otimes \eta) = m \circ (\eta \otimes id) = id$$

$$(id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta$$

$$(\epsilon \otimes id) \circ \Delta = (id \otimes \epsilon) \circ \Delta = id$$

$$m \circ (id \otimes S) \circ \Delta = m \circ (S \otimes id) \circ \Delta = \eta \circ \epsilon$$

$$\psi \circ (m \otimes id) = (id \otimes m) \circ (\psi \otimes id) \circ (id \otimes \psi)$$

$$\psi \circ (id \otimes m) = (m \otimes id) \circ (id \otimes \psi) \circ (\psi \otimes id)$$

$$\begin{aligned}
(id \otimes \Delta) \circ \psi &= (\psi \otimes id) \circ (id \otimes \psi) \circ (\Delta \otimes id) \\
(\Delta \otimes id) \circ \psi &= (id \otimes \psi)(\psi \otimes id) \circ (id \otimes \Delta) \\
\Delta \circ m &= (m \otimes m)(id \otimes \psi \otimes id) \circ (\Delta \otimes \Delta) \\
S \circ m &= m \circ \psi \circ (S \otimes S) \\
\Delta \circ S &= (S \otimes S) \circ \psi \circ \Delta \\
\epsilon \circ m &= \epsilon \otimes \epsilon \\
(\psi \otimes id) \circ (id \otimes \psi) \circ (\psi \otimes id) &= (id \otimes \psi) \circ (\psi \otimes id) \circ (id \otimes \psi)
\end{aligned} \tag{1}$$

where $m : A \otimes A \rightarrow A$ is the multiplication map, $\Delta : A \rightarrow A \otimes A$ is the comultiplication map, $\eta : K \rightarrow A$ is the unit map, $\epsilon : A \rightarrow K$ is the counit map, $S : A \rightarrow A$ is the antipode map and $\psi : A \otimes A \rightarrow A \otimes A$ is the braiding map. The consistency of the relations (1) requires that

$$\Delta(1_A) = 1_A \otimes 1_A, \quad \psi(1_A \otimes a) = a \otimes 1_A, \quad \psi(a \otimes 1_A) = 1_A \otimes a \quad \forall a \in A \tag{2}$$

where 1_A is the identity of the algebra A . From now on we will drop the subscript and write 1 for the identity of the algebra. All these maps are linear. Note that in the limit $\psi \rightarrow \pi$ the braided Hopf algebra axioms reduce to the ordinary Hopf algebra axioms. The braided Hopf algebra axioms⁷ reduce to the axioms given above when the counit map (ϵ) is an algebra homomorphism. The $*$ -structure for a braided algebra B is different⁸ from the non-braided one such that

$$\begin{aligned}
\Delta \circ * &= \pi \circ (* \otimes *) \circ \Delta \\
S \circ * &= * \circ S \\
(a \otimes b)^* &= b^* \otimes a^*, \quad \forall a, b \in B.
\end{aligned} \tag{3}$$

2 Generalized Oscillator

We propose a generalized oscillator algebra generated by a , a^* , q^N and 1 satisfying

$$\begin{aligned}
aq^N &= qq^N a, \\
q^N a^* &= qa^* q^N, \\
aa^* - Q_1 a^* a &= Q_2 q^{2N} + Q_3 q^N + Q_4
\end{aligned} \tag{4}$$

where q, Q_1, Q_2, Q_3, Q_4 are real constants whose values determine the type of the oscillator. For example, if Q_1 is a free parameter then the $Q_2 = 1, Q_3 = Q_4 = 0$ case and the $Q_3 = 1, Q_2 = Q_4 = 0$ case define two different Fibonacci oscillators. For the $*$ -structure we impose $(q^N)^* = q^N, (a^*)^* = a$. The actions of the generators on the Hilbert space are given by

$$\begin{aligned}
a |n\rangle &= a_n |n-1\rangle \\
a^* |n\rangle &= a_{n+1}^* |n+1\rangle \\
q^N |n\rangle &= q^n |n\rangle.
\end{aligned} \tag{5}$$

Using the fact that for a given algebra the Hopf algebra structure is not unique, we write the general forms of the coproducts

$$\begin{aligned}
\Delta(q^N) &= A_1 q^N \otimes q^N + A_2 a \otimes a^* + A_3 a^* \otimes a + A_4 1 \otimes q^N + A_5 q^N \otimes 1 + A_6 1 \otimes 1, \\
\Delta(a) &= B_1 q^N \otimes a + B_2 a \otimes q^N + B_3 1 \otimes a + B_4 a \otimes 1, \\
\Delta(a^*) &= B_1 a^* \otimes q^N + B_2 q^N \otimes a^* + B_3 a^* \otimes 1 + B_4 1 \otimes a^*,
\end{aligned} \tag{6}$$

the counits

$$\epsilon(q^N) = e_1, \quad \epsilon(a) = \epsilon(a^*) = e_2, \tag{7}$$

the antipodes

$$S(q^N) = k_1 q^N + k_2 a + k_3 a^* + k_4,$$

$$S(a) = m_1 q^N + m_2 a + m_3 a^* + m_4, \quad (8)$$

$$S(a^*) = m_1 q^N + m_2 a^* + m_3 a + m_4$$

and the braidings

$$\begin{aligned} \psi(q^N \otimes q^N) &= g_1 q^N \otimes q^N + g_2 a \otimes a^* + g_3 a^* \otimes a + g_4 1 \otimes q^N + g_5 q^N \otimes 1 + g_6 1 \otimes 1, \\ \psi(q^N \otimes a) &= d_1 a \otimes q^N + d_2 q^N \otimes a + d_3 1 \otimes a + d_4 a \otimes 1, \\ \psi(a^* \otimes q^N) &= d_1 q^N \otimes a^* + d_2 a^* \otimes q^N + d_3 a^* \otimes 1 + d_4 1 \otimes a^*, \\ \psi(q^N \otimes a^*) &= f_1 a^* \otimes q^N + f_2 q^N \otimes a^* + f_3 1 \otimes a^* + f_4 a^* \otimes 1, \\ \psi(a \otimes q^N) &= f_1 q^N \otimes a + f_2 a \otimes q^N + f_3 a \otimes 1 + f_4 1 \otimes a, \\ \psi(a \otimes a) &= z_1 a \otimes a, \quad (9) \\ \psi(a^* \otimes a^*) &= z_1 a^* \otimes a^*, \\ \psi(a \otimes a^*) &= b_1 q^N \otimes q^N + b_2 a \otimes a^* + b_3 a^* \otimes a + b_4 1 \otimes q^N + b_5 q^N \otimes 1 + b_6 1 \otimes 1, \\ \psi(a^* \otimes a) &= c_1 q^N \otimes q^N + c_2 a \otimes a^* + c_3 a^* \otimes a + c_4 1 \otimes q^N + c_5 q^N \otimes 1 + c_6 1 \otimes 1 \end{aligned}$$

where symbols with a subscript are the constants to be determined.

3 Braided Hopf Algebra Structure of the Generalized Oscillator

To find the general braided Hopf algebra structure for the oscillator algebra given by (4) we substitute these general forms into the braided Hopf algebra axioms (1) and find the solutions using the computer programming Maple V. The constants which are the same for all solutions read as

$$Q_4 = A_1 = A_6 = B_1 = B_2 = 0, \quad A_4 = A_5 = B_3 = B_4 = 1, \quad (10)$$

$$k_2 = k_3 = k_4 = m_1 = m_3 = m_4 = e_1 = e_2 = 0, \quad m_2 = -1, \quad (11)$$

$$b_4 = b_5 = b_6 = c_4 = c_5 = c_6 = g_4 = g_5 = g_6 = d_3 = d_4 = f_3 = f_4 = 0. \quad (12)$$

The solutions for the other parameters are given in the tables. The constants given by (10)-(12) show that the antipodes and the counits of all generators and the coproducts of raising/lowering operators are uniquely determined. We also found that for a free deformation parameter Q_1 at most one of the other deformation parameters, namely Q_2 or Q_3 , is nonzero. For $Q_2 \neq 0$, we have

$$aa^* - Q_1 a^* a = Q_2 q^{2N}. \quad (13)$$

Without loss of generality we can take $Q_2 = 1$ or by rescaling a and a^* the oscillator relation (13) can be reduced to

$$aa^* - Q_1 a^* a = q^{2N}. \quad (14)$$

We found that there are only six possible braided Hopf algebra solutions for this two parameter oscillator which are given in Table 1. Defining the free deformation parameter $Q_1 \equiv p^{-2}$ the algebra (14) can be rewritten as

$$aa^* - p^{-2} a^* a = q^{2N} \quad (15)$$

which is the more familiar form of the q, p oscillator algebra (Fibonacci oscillator) which we call the first-type Fibonacci oscillator. Using the actions (5) the eigenvalue of the operator $a^* a$ on the state $|n\rangle$ is found to be

$$a_n^* a_n = \frac{p^{-2n} - q^{2n}}{p^{-2} - q^2}. \quad (16)$$

Substituting $Q_1 = q^{-2}$ into (14) we obtain the Biedenharn-Macfarlane oscillator algebra which we call the first-type Biedenharn-Macfarlane oscillator defined by

$$aa^* - q^{-2} a^* a = q^{2N}. \quad (17)$$

The eigenvalue of the operator a^*a on the state $|n\rangle$ is found to be

$$a_n^*a_n = \frac{q^{2n} - q^{-2n}}{q^2 - q^{-2}}. \quad (18)$$

There are only six braided Hopf algebra solutions for the first-type Biedenharn Macfarlane oscillator (17) which can be obtained by substituting $Q_1 = q^{-2}$ into the solutions given in Table 1. Similarly, for $Q_3 \neq 0$ the oscillator relation

$$aa^* - Q_1a^*a = Q_3q^N \quad (19)$$

can again be reduced to

$$aa^* - Q_1a^*a = q^N \quad (20)$$

which we call the second-type Fibonacci oscillator. Setting the free deformation $Q_1 \equiv p^{-1}$ the eigenvalue of the operator a^*a on the state $|n\rangle$ is found to be

$$a_n^*a_n = \frac{p^{-n} - q^n}{p^{-1} - q}. \quad (21)$$

The second-type Fibonacci oscillator (20) has only two braided Hopf algebra solutions which are given in Table 2. The second-type Biedenharn-Macfarlane oscillator can be obtained simply by substituting $Q_1 = q^{-1}$ into (20) and the same substitution into Table 2 gives braided Hopf algebra solutions.

A wide variety of one parameter oscillators can be obtained by assigning $Q_1 = f(q)$ in the algebras (14) and (20). The braided Hopf algebra solutions for these oscillators can be obtained by substituting $Q_1 = f(q)$ into the Table 1 and Table 2. However, there are extra solutions for some values Q_1 which are given in Table 3.

The braided Hopf algebra structure of the quantum space (called braided space) defined by

$$x_i x_j = q x_j x_i \quad i > j \quad (22)$$

determines the structure of the braided integration, derivation and Fourier transform defined on that space⁹. For $Q_1 = q$, $Q_2 = 0$, $Q_3 = 0$, $Q_4 = 0$ the algebra (4) reduces to the three dimensional quantum space with the identifications

$$x_1 \equiv a^*, \quad x_2 \equiv q^N, \quad x_3 \equiv a. \quad (23)$$

The braided Hopf algebra solutions for the three dimensional braided space are given in Table 4. Setting the free parameter $g_1 = q^2$ in the sol5 of Table 4 gives the braided Hopf algebra given in the literature¹⁰ and references therein) as a special case.

4 Conclusion

The braidings imply the relations between independent copies of algebras. For example, the implication of the braiding $\psi(a \otimes a)$ can be found by using the identifications $a_1 \equiv a \otimes 1$ and $a_2 \equiv 1 \otimes a$ such that

$$a_2 a_1 = (1 \otimes a)(a \otimes 1) = \psi(a \otimes a) = z a \otimes a = z a_1 a_2.$$

Thus the braiding relations imply a 2-body system of oscillators which can be extended to n -body oscillators using the n -fold braided tensor product as done by Baskerville and Majid in the context of the braided version of the q -Heisenberg algebra¹¹. The Fock space representations of the n -fold braided tensor product of oscillators can also be found. This requires the braidings of the generators of the algebra with the states. All these constructions depend on the braiding relations of the generators of the algebra. Since each solution for the braiding gives a different system of oscillators, the solutions we found may provide a general framework for the interacting oscillators and hence for the statistical mechanical quantities calculated by using these oscillators.

Because of the connection between symmetry and statistics it is interesting to investigate the un-

derlying symmetry transformations of the braided oscillators which gives rise to the braiding relations given in the tables. It may also be interesting to find the unbraiding transformations¹² or the decoupling of the braided oscillators. The generalization of supersymmetry to fractional supersymmetry requires the deformation parameter to be a root of unity¹³ and this case deserves to be discussed on a separate study.

The braided Hopf algebra solutions we present for the oscillators and for the three dimensional braided space may provide a general frame on which other structures can be defined.

References

- ¹L. C. Biedenharn, J. Phys. A: Math. Gen. **22**, L873 (1989), A. J. Macfarlane, J. Phys. A: Math. Gen. **22**, 4581 (1989)
- ²M. Arik and A. Yildiz, J. Phys. A: Math. Gen. **30**, L255 (1997)
- ³M. Arik, E. Demircan, T. Turgut, L. Ekinçi and M. Mungan, Z. Phys. C **55**, 89 (1992), R. Chakrabarti and R. Jagannathan J. Phys. A: Math. Gen. **24**, L711 (1991), M. Daoud and M. Kibler, Phys. Lett. A **206**, 13 (1995)
- ⁴S. Majid, Beyond supersymmetry and quantum symmetry (an introduction to braided groups and braided matrices) *Quantum Groups, Integrable Statistical Models and Knot Theory* edited by M. L. Ge and H. J. Vega (Singapore: World Scientific), pp.231-282 (1993)
- ⁵R. S. Dunne, A.J. Macfarlane, J.A. Azcarraga and J.C. Bueno, Phys. Lett. B **387**, 294 (1996)
- ⁶S. Majid, Algebras and Hopf algebras in braided categories *Lecture Notes in Pure and Applied Mathematics* vol 158 (New York: Marcel Dekker), pp. 55-105 (1994)
- ⁷M. Durdevich, Israel Journal of Mathematics **98**, 329 (1997)
- ⁸Majid S *Foundations of Quantum Group Theory* (Cambridge: Cambridge University Press), (1995)
- ⁹A. Kempf and S. Majid, J. Math. Phys. **35**, 6802 (1994)
- ¹⁰G. Carnovale, J. Math. Phys. **40**, 5972 (1999)
- ¹¹W. K. Baskerville and S. Majid, J. Math. Phys. **34**, 3588 (1993)
- ¹²G. Fiore, H. Steinacker and J. Wess, Mod. Phys. Lett. A **16**, 261 (2001)
- ¹³R.S. Dunne, J. Math. Phys. **40**, 1180 (1999), H. Ahmedov and O. F. Dayi, Mod. Phys. Lett. A **15**, 1801 (2000), H. Ahmedov, A. Yildiz and Y. Ucan, J. Phys. A **34**, 6413 (2001)

Table 1: Braided Fibonacci oscillator of the first type

$aa^* - Q_1 a^* a = q^{2N}, aq^N = qq^N a, q^N a^* = qa^* q^N$						
	sol1	sol2	sol3	sol4	sol5	sol6
$k_1 =$	-1	-1	-1	-1	-1	-1
$b_1 =$	1	$1 - \frac{q}{\sqrt{Q_1}}$	$1 + \frac{q}{\sqrt{Q_1}}$	$\frac{Q_1}{q^2}$	0	0
$b_2 =$	$\frac{q^2 - Q_1}{Q_1}$	$\frac{(q^2 - Q_1)(\sqrt{Q_1} + q)}{qQ_1}$	$\frac{(Q_1 - q^2)(\sqrt{Q_1} - q)}{qQ_1}$	0	0	0
$b_3 =$	Q_1	Q_1	Q_1	$\frac{Q_1^2}{q^2}$	Q_1	Q_1
$z =$	$\frac{q^2}{Q_1}$	$\frac{q^2}{Q_1}$	$\frac{q^2}{Q_1}$	$\frac{Q_1}{q^2}$	$\frac{Q_1}{q^2}$	$\frac{Q_1}{q^2}$
$c_1 =$	$-\frac{q^2}{Q_1^2}$	0	0	$\frac{-1}{Q_1}$	$-\frac{q + \sqrt{Q_1}}{qQ_1}$	$\frac{\sqrt{Q_1} - q}{qQ_1}$
$c_2 =$	$\frac{q^2}{Q_1^2}$	$\frac{1}{Q_1}$	$\frac{1}{Q_1}$	$\frac{1}{Q_1}$	$\frac{1}{Q_1}$	$\frac{1}{Q_1}$
$c_3 =$	0	0	0	$\frac{Q_1 - q^2}{q^2}$	$\frac{(q^2 - Q_1)(q - \sqrt{Q_1})}{q^2 \sqrt{Q_1}}$	$\frac{(Q_1 - q^2)(q + \sqrt{Q_1})}{q^2 \sqrt{Q_1}}$
$d_1 =$	$\frac{q}{Q_1}$	$\frac{q}{Q_1}$	$\frac{q}{Q_1}$	q^{-1}	q^{-1}	q^{-1}
$d_2 =$	0	0	0	$\frac{Q_1 - q^2}{q^2}$	$\frac{Q_1 - q^2}{q^2}$	$\frac{Q_1 - q^2}{q^2}$
$f_1 =$	q	q	q	$\frac{q^2}{Q_1}$	$\frac{q^2}{Q_1}$	$\frac{q^2}{Q_1}$
$f_2 =$	$\frac{q^2 - Q_1}{Q_1}$	$\frac{q^2 - Q_1}{Q_1}$	$\frac{q^2 - Q_1}{Q_1}$	0	0	0
$g_1 =$	$\frac{q^2}{Q_1}$	$-\frac{q}{\sqrt{Q_1}}$	$\frac{q}{\sqrt{Q_1}}$	$\frac{Q_1}{q^2}$	$\frac{\sqrt{Q_1}}{q}$	$-\frac{\sqrt{Q_1}}{q}$
$g_2 =$	0	$\frac{(q^2 - Q_1)(\sqrt{Q_1} + q)}{qQ_1}$	$\frac{(Q_1 - q^2)(\sqrt{Q_1} - q)}{qQ_1}$	0	0	0
$g_3 =$	0	0	0	0	$-\frac{Q_1(Q_1 - q^2)^2}{q^2(Q_1 + q\sqrt{Q_1})}$	$\frac{Q_1(Q_1 - q^2)^2}{q^2(q\sqrt{Q_1} - Q_1)}$
$A_2 = A_3 = 0$ for all solutions						

Table 2: Braided Fibonacci oscillator of the second type

$aa^* - Q_1 a^* a = q^N, aq^N = qq^N a, q^N a^* = qa^* q^N$		
	sol1	sol2
$A_2 =$	$\frac{q - Q_1}{q}$	$\frac{Q_1 - q}{Q_1}$
$A_3 =$	$\frac{q}{Q_1(Q_1 - q)}$	$q - Q_1$
$k_1 =$	$-\frac{Q_1}{q}$	$-\frac{q}{Q_1}$
$b_2 =$	0	$\frac{Q_1^2 - q^2}{qQ_1}$
$b_3 =$	q	q
$z =$	$\frac{q}{Q_1}$	$\frac{Q_1}{q}$
$c_2 =$	q^{-1}	q^{-1}
$c_3 =$	$\frac{q^2 - Q_1^2}{qQ_1}$	0
$d_1 =$	$\frac{1}{Q_1}$	$\frac{Q_1}{q^2}$
$f_1 =$	$\frac{q}{Q_1}$	Q_1
$g_1 =$	$\frac{q^2}{Q_1^2}$	$\frac{Q_1^2}{q^2}$
$b_1 = c_1 = d_2 = f_2 = g_2 = g_3 = 0$ for both solutions		

Table 3: Other braided oscillator solutions

$aa^* - Q_1 a^* a = Q_2 q^{2N} + Q_3 q^N, aq^N = qq^N a, q^N a^* = qa^* q^N$						
	sol1	sol2	sol3	sol4	sol5	
$Q_1 =$	q^2	$-q^2$	$-q^2$	q^2	$-q$	
$Q_2 =$	1	1	1	1	0	
$Q_3 =$	0	0	0	0	1	
$k_1 =$	-1	-1	-1	-1	-1	
$b_1 =$	0	0	$2 - q^2 c_1$	$2 + q^2 c_1$	0	
$b_3 =$	q^2	$-q^2$	$-q^2$	q^2	$-q$	
$z =$	1	-1	-1	1	-1	
$c_1 =$	0	0	c_1	c_1	0	
$c_2 =$	q^{-2}	$-q^{-2}$	$-q^{-2}$	q^{-2}	$-q^{-1}$	
$d_1 =$	q^{-1}	q^{-1}	q^{-1}	q^{-1}	q^{-1}	
$f_1 =$	q	q	q	q	q	
$g_1 =$	-1	-1	1	1	1	
$A_2 = A_3 = b_2 = c_3 = d_2 = f_2 = g_2 = g_3 = 0$ for all solutions						

Table 4: Three dimensional braided space

$aa^* = qa^* a, aq^N = qq^N a, q^N a^* = qa^* q^N$							
	sol1	sol2	sol3	sol4	sol5	sol6	sol7
$A_2 =$	A_2	A_2	0	0	0	0	0
$A_3 =$	$-qA_2$	qA_2	0	0	0	0	0
$k_1 =$	-1	1	-1	-1	-1	-1	-1
$b_2 =$	0	0	0	0	$-1 + g_1$	$-1 + qc_2$	0
$b_3 =$	q	q	qg_1	q	q	q	qz
$z =$	1	1	g_1	z	g_1	qc_2	z
$c_2 =$	q^{-1}	q^{-1}	q^{-1}	q^{-1}	$q^{-1}g_1$	c_2	q^{-1}
$c_3 =$	0	0	$g_1 - 1$	0	0	0	$z - 1$
$d_1 =$	q^{-1}	q^{-1}	q^{-1}	q^{-1}	$q^{-1}g_1$	q^{-1}	q^{-1}
$d_2 =$	0	0	$g_1 - 1$	0	0	0	0
$f_1 =$	q	q	qg_1	q	q	q	q
$f_2 =$	0	0	0	0	$-1 + g_1$	0	0
$g_1 =$	1	1	g_1	g_1	g_1	g_1	g_1
$b_1 = c_1 = g_2 = g_3 = 0$ for all solutions							