Elliptic Approximations of Propositional Formulae

Hans van Maaren

Abstract. A propositional formula can be approximated by a concave quadratic function. This approximation is obtained as a second order Taylor expansion of a convex smooth model. It is shown that in the 3- SAT case, the involved parameters can be set to such values that yield optimal discriminative properties. Two concentric (generally elliptic) quadratic convex regions are established, the inner one containing only satisfiable assignments and the outer one *excluding* the average non satisfiable assignment and *including* all satisfiable assignments.
Keywords. Satisfiability, convexity, quadratic approximations, quadratic valid cuts.
1991 Mathematics Subject Classification. Primary 68Q25,90C09, Secondary 90C10, 90C27.

1 Introduction.

Smooth convex and concave transforms of propositional formulae were introduced by van Maaren, Groote and Rozema in [4]. In [5] the eigenvalues of the associated Hessian matrices were used to design a branching variable heuristic which has (experimentally) been shown to result in relatively small search trees, even in case that no other additional node procedure than unit clause elimination was involved.
In this paper we investigate the second order Taylor expansion of the smooth concave model on its discriminative properties. It is shown that, in general, a parametric family of valid convex quadratic cuts can be derived and that, specifically in the 3-SAT case, parameter values can be established in such a way that these valid cuts separate the "average" non satisfiable truth assignment from all satisfiable ones. This is done by deriving a threshold value which is the solution to a parametric convex quadratic programming problem. In other words, a convex quadratic region is derived which contains all solutions to the SAT problem involved but which excludes most of the non solutions. Generally this region is shown to be an elliptic region.

The above threshold value depends only on the *global* characteristics of the CNF formula involved, being the numbers S_m, which indicate the number of m-literal clauses.

The aim of this research is to provide geometric insight in the satisfiability problem and to make a start with using the quadratic valid cuts in order to yield linear valid cuts of specific interest.

As to the first goal we include a discussion on balanced formula [1] and their geometric representations. As to the second we explicitly derive a formula yielding the desired threshold in the mixed 2,3-SAT case.

We pay special attention to pure 2-SAT formulae. This is not because we want to contribute to the solution procedure for this class, as they can be solved in linear time. The reason why we do so is because the expressions involved are much simpler and yet the methods used generalize naturally to the other more complex cases. Although the paper is essentially self contained, the reader is supposed to be familiar with the SAT terminology, the Integer Programming Approach to the SAT-Problem as well as to some basic facts concerning the SAT problem. We refer to [2] for the above. For a detailed discussion of smooth convex models we refer to [4] and [5].
Related studies on smoothing binary programming problems are found in [3] and [6].

We conclude with the remark that in spite of the less attractive expresssions that are created by differentiating the smooth model, the purely combinatorial entities which show up are partly familiar in the SAT area (when differentiating once) and partly define new (to the best of the author's knowledge) and interesting characteristics of CNF formulae.

2 The smooth concave model.

In [4] and [5] smooth convex and concave models are introduced and discussed at length. In order to keep this paper self contained we shall (only very briefly) recall the relevant formulae. Since we only deal with the concave model here we omit superscripts o and * as they appear in the previously quoted papers.
For $\varepsilon > 0$ we consider $\mathcal{A}_\varepsilon : (-\infty, 1] \to [0,1]$ given by

$$\mathcal{A}_\varepsilon(x) = \frac{x + \sqrt{x^2+\varepsilon}}{1+\sqrt{1+\varepsilon}} \tag{2.1}$$

and for $r < 0$, $\mathcal{A}_r : [0,1]^S \to [0,1]$ is defined by

$$\mathcal{A}_r(x_1, \ldots, x_S) = (\frac{1}{S}\sum_{i=1}^{S} x_i^r)^{\frac{1}{r}} \tag{2.2}$$

The above $\mathcal{A}_\varepsilon$ and $\mathcal{A}_r$ are used to smooth the Binary Integer Programming Problem associated to the SAT problem. In fact, for $\varepsilon \downarrow 0$ and $r \to -\infty$ the concave model below converges to this BIP presentation.

2.1 Clause dependent notations and conventions.

A clause $\mathbf{C_s}$ is represented by two index sets I_s and J_s is such away that

$$\mathbf{C_s} = \bigvee_{i \in I_s} p_i \vee \bigvee_{j \in J_s} \sim p_j \tag{2.3}$$

The length of a clause is $\#I_s + \#J_s$ and is denoted by ℓ_s. Throughout the paper we assume that $I_s \cap J_s = \emptyset$ and that $\ell_s \geq 2$, that is, no one literal clauses are considered. The concave transform of a clause $\mathbf{C_s}$ now is goven by

$$C_s(x) = 1 - \mathcal{A}_{\varepsilon_s}(1 - (\sum_{i \in I_s} x_i + \sum_{j \in J_s}(1 - x_j))) \tag{2.4}$$

where ε_s is a parameter depending on the length ℓ_s of the clause only. We shall identify freely $\varepsilon_s, \varepsilon_{\ell_s}$ or ε_m in cases where no confusion can appear and where m obviously refers to clause lengths. The value of $C_s(x)$ at the center $c = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ of the cube is

$$C_m = C_{\ell_s}(c) = C_s(c) = 1 - \mathcal{A}_{\varepsilon_m}(1 - \frac{1}{2}m) \tag{2.5}$$

2.2 CNF dependent notations and conventions

A CNF formula Φ of N variables is a conjunction of clauses $\mathbf{C_s}(s \in S)$ where each I_s and J_s are subsets of $\{1, \ldots, N\}$. Thus

$$\Phi = \bigwedge_{s \in S} \mathbf{C_s} \tag{2.6}$$

We let $S_m \subset S$ be the index set refering to the m-literal clauses of Φ. In numerical expressions we shall also use $S, S_2, S_3, \ldots$ to denote the numbers $\#S, \#S_2, \#S_3, \ldots$. M shall stand for the maximal clause length appearing in Φ. Thus $S = S_2 \cup S_3 \cup \ldots \cup S_M$ and $S = S_2 + S_3 + \ldots + S_M$ both will have a meaning, being the obvious one.
The concave transform of a CNF Φ now is

$$\Phi(x) = \mathcal{A}_r(C_1(x), \ldots, C_S(x)) \tag{2.7}$$

Notice that in the above one parameter $r < 0$ and $M - 1$ parameters $\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_M > 0$ are involved. In [5] it is shown that a threshold value c_Φ exists, depending only on the parameters and the global characteristics of Φ (the numbers $S_2, S_3, \ldots, S_M$) such that

$$\Phi(V) \geq c_\Phi \text{ if and only if } \Phi \text{ is true at assignment } V \tag{2.8}$$

In other words, the smooth convex region defined by the inequality

$$\Phi(x) \geq c_\Phi \tag{2.9}$$

separates "true" vertices from "false" vertices (assignments). Again, we mention that for $\varepsilon_s \downarrow 0$ and $r \to -\infty$, (2.9) converges to the BIP representation of the satisfiability problem for Φ.
In this paper we investigate a weaker, but more accessable inequality, namely the one obtained by replacing $\Phi(x)$ by its second order Taylor expansion at c, the center of the cube. Doing so, we shall establish a convex quadratic region in $\mathbb{R}^N$ which has weaker discriminative properties but is much more adapted for calculations and is suitable for deriving linear valid cuts with specific features. The value $\Phi(c)$ shall frequently occur in our expressions. Notice that

$$\Phi(c) = (\frac{1}{S}(\sum_{m \leq M} S_m C_m^r))^{\frac{1}{r}} \tag{2.10}$$

is a typical global characteristic of Φ.
In order to establish the second order Taylor expension of Φ we have to differentiate $\Phi(x)$ twice. This activity is in itself a trivial one, but it generates, as one might expect, less attractive coefficients. The combinatorial entities obtained however, are interesting and deserve extra attention. We shall list them here.

2.3 CNF dependent combinatorial entities.

$$POS(m, k) = \#\{s \in S_m | p_k \text{ is a literal in } \mathbf{C_s}\} \tag{2.11}$$

$$NEG(m, k) = \#\{s \in S_m | \sim p_k \text{ is a literal in } \mathbf{C_s}\} \tag{2.12}$$

$$DIF(m, k) = POS(m, k) - NEG(m, k) \tag{2.13}$$

$$DIF(m) \quad = (DIF(m, 1), DIF(m, 2), \ldots, DIF(m, N)) \tag{2.14}$$

viewed as a column vector.

The above entities arise, establishing the first order partial derivatives of $\Phi(x)$, at c. The ones below constitute the ingredients of the second order partial derivatives.

$$
\begin{array}{lll}
(2.15)\ EOR(m,i,j) & = & \#\{s \in S_m | p_i \text{ and } p_j \text{ are oriented equally in } \mathbf{C_s}\} \\
(2.16)\ UOR(m,i,j) & = & \#\{s \in S_m | p_i \text{ and } p_j \text{ are oriented differently in } \mathbf{C_s}\} \\
(2.17)\ DIF^2(m,i,j) & = & EOR(m,i,j) - UOR(m,i,j) \\
(2.18)\qquad DIF^2(m) & = & (DIF^2(m,i,j)) \text{ viewed as a symmetric squared } N \times N \text{ matrix.}
\end{array}
$$

In the above it is understood that for $i \neq j$ the propositional variables p_i and p_j both have to occur, and that for $i = j$ the occurrence of p_i counts as "p_i and p_j are oriented equally". Thus $EOR(m,i,i)$ is simply the number of occurencies of variable p_i in the m-literal clauses, and $UOR(m,i,i)$ is always zero.

We close this section by listing the frequently occurring coefficients, arising from the differentiation of $\Phi(x)$.

2.4 Occurring coefficients when differentiating Φ

(2.19) the parameters $r < 0$ and $\varepsilon_2, \varepsilon_3, \ldots, \varepsilon_M > 0$.

$$\alpha_m = \frac{1}{1 + \sqrt{1 + \varepsilon_m}}. \tag{2.20}$$

$$\rho_m = 1 + \frac{1 - \frac{1}{2}m}{\sqrt{(1 - \frac{1}{2}m)^2 + \varepsilon_m}}. \tag{2.21}$$

$$\sigma_m = \frac{\sqrt{1 + \varepsilon_m} - 1}{((1 - \frac{1}{2}m)^2 + \varepsilon_m)^{\frac{3}{2}}}. \tag{2.22}$$

$$C_m = 1 - \alpha_m((1 - \frac{1}{2}m) + \sqrt{(1 - \frac{1}{2}m)^2 + \varepsilon_m}\quad). \tag{2.23}$$

$$\Phi(c) = (\frac{1}{S} \sum_{m \leq M} S_m C_m^r)^{\frac{1}{r}}. \tag{2.24}$$

$$u_m = \frac{1}{S} \Phi(c)^{1-r} C_m^{r-1} \alpha_m \rho_m. \tag{2.25}$$

$$
\begin{array}{lrcl}
(2.26) & v_m & = & \frac{1}{S} C_m^{r-1} \alpha_m \rho_m \sqrt{(1-r)\Phi(c)^{1-2r}} = \\
(2.27) & & = & u_m \sqrt{\frac{1-r}{\Phi(c)}}.
\end{array}
$$

$$w_m = \frac{1}{S} \Phi(c)^{1-r} (C_m^{r-1} \sigma_m + (1-r) C_m^{r-2} \alpha_m^2 \rho_m^2). \tag{2.28}$$

To understand the main goals and reasoning of this paper the reader need not go necessarily in the sometimes tedious details of the calculations, involving unattractive expressions as in the above. It is enough to realize that these are just "real numbers" showing up because of using $\mathcal{A}_\varepsilon$ and $\mathcal{A}_r$ as given in (2.1) and (2.2). Essential is however, to keep in mind that they are completely determined by the parameters and the global characteristics of the CNF formula involved.

3 The quadratic approximation.

The second order Taylor expansion, taken at the center of the cube c, of our concave transform $\Phi(x)$, can be written as

$$\Phi(c+\xi) = \Phi(c) + \nabla\Phi(c).\xi + \frac{1}{2}\Delta\Phi(c)(\xi).\xi \tag{3.1}$$

where ∇ stands for gradient and Δ for Hessian matrix. As the notorious reader may verify we have

$$\begin{array}{ll} \nabla_i & = (\nabla\Phi(c))_i = \sum_{m\geq 1} u_m DIF(m,i) \\ u_m & = \frac{1}{S}\Phi(c)^{1-r} C_m^{r-1}\alpha_m\rho_m \end{array} \tag{3.2}$$

and

$$\begin{aligned} \Delta_{i,j} &= (\Delta\Phi(c))_{i,j} = \sum_{m\geq 1} v_m DIF(m,i) \sum_{m\geq 1} v_m DIF(m,j) \\ &\qquad - \sum_{m\geq 1} w_m DIF^2(m,i,j) \\ v_m &= \frac{1}{S} C_m^{r-1}\alpha_m\rho_m\sqrt{(1-r)\Phi(c)^{1-2r}} \\ w_m &= \frac{1}{S}\Phi(c)^{1-r}(C_m^{r-1}\sigma_m + (1-r)C_m^{r-2}\alpha_m^2\rho_m^2) \end{aligned} \tag{3.3}$$

The above can be written alternatively as

$$\varphi(\xi) = \Phi(c+\xi) - \Phi(c) = \sum_{s\in S} u_s C\ell_s(\xi) + \frac{1}{2}(\sum_{s\in S} v_s C\ell_s(\xi))^2 - \frac{1}{2}\sum_{s\in S} w_s C\ell_s(\xi)^2 \tag{3.4}$$

where subscript s in u_s, v_s and w_s should be read as ℓ_s, and

$$C\ell_s(\xi) = \sum_{i\in I_s}\xi_i - \sum_{j\in J_s}\xi_j. \tag{3.5}$$

The reader must be alert on the fact that in vertices V of the cube $V_i = \frac{1}{2} + \xi_i$ with $\xi_i = \pm\frac{1}{2}$. Thus, in the above, ξ-coordinates referring to truth assignments have value $\pm\frac{1}{2}$.

More compact matrix notations for our entities are

$$\begin{array}{l} \nabla = \sum_{m\geq 1} u_m DIF(m) \\ \Delta = \frac{1-r}{\Phi(c)}\nabla\nabla^t - \sum_{m\geq 1} w_m DIF^2(m) \end{array} \tag{3.6}$$

Example: For $\Phi = (p\vee q)\wedge(\sim p\vee \sim r)\wedge(p\vee q\vee s)\wedge(\sim p\vee r\vee\sim s)$ we have

$$\nabla = u_2\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} + u_3\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\Delta = \frac{1-r}{\Phi(c)}\nabla\nabla^t - (w_2\begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + w_3\begin{pmatrix} 2 & 1 & -1 & 2 \\ 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & -1 \\ 2 & 1 & -1 & 2 \end{pmatrix})$$

and

$$\begin{aligned}\varphi(\xi) &= (u_2(\xi_q - \xi_r) + u_3(\xi_q + \xi_r)) + \frac{1}{2}(v_2(\xi_q - \xi_r) + v_3(\xi_q + \xi_r))^2 \\ &-\frac{1}{2}w_2(2\xi_p^2 + \xi_q^2 + \xi_r^2 + 2\xi_p\xi_q + 2\xi_p\xi_r) \\ &-\frac{1}{2}w_3(2\xi_p^2 + \xi_q^2 + \xi_r^2 + 2\xi_s^2 + 2\xi_p\xi_q - 2\xi_p\xi_r + 4\xi_p\xi_s + 2\xi_q\xi_s - 2\xi_r\xi_s)\end{aligned}$$

The aim of this paper is to establish numbers m_{sat} and M_{sat}, depending only on the parameters r and $\varepsilon_2, \ldots, \varepsilon_M$ and the global characteristics of the CNF formula involved (the numbers $S_2, S_3, \ldots, S_M$) in such a way that

$$\text{(3.7)} \qquad \begin{array}{ll} \{\xi | \varphi(\xi) < m_{\text{sat}}\} & \text{contains no satisfiable assignments} \\ \{\xi | \varphi(\xi) > M_{\text{sat}}\} & \text{contains only satisfiable assignments} \end{array}$$

Thus, by knowing m_{sat}, the quadratic convex cut

$$\text{(3.8)} \qquad \varphi(\xi) \geq m_{\text{sat}}$$

shall be a valid cut.

Also, specific values of the parameters shall be given which make these cuts as discriminative possible.

Allowing m_{sat} and M_{sat} to depend on other characteristics (such as average numbers of occurrencies, number of pure clauses, average length of clauses, ...) as well, certainly will increase the discriminative properties of the cuts, however, the analysis will be likewise more involved, and we shall not proceed along these lines. Still, our methods to establish these values leave some space for more specifity, as the reader may notice in the sequal.

4 The 2-SAT case.

In this section we suppose Φ to consist of only 2-literal clauses. This assumption has a considerable simplifying effect on our formulae. The reader may check that in this case

$$\text{(4.1)} \qquad \begin{array}{ll} \text{(a)} & \Phi(c)^{1-r} = C_2^{1-r} \\ \text{(b)} & \rho_2 = 1 \\ \text{(c)} & u_2 = \dfrac{1}{S}\alpha_2 \\ \text{(d)} & v_2 = \dfrac{1}{S}\alpha_2\sqrt{\dfrac{1-r}{C_2}} \\ \text{(e)} & w_2 = \dfrac{1}{S}(\sigma_2 + (1-r)\dfrac{\alpha_2^2}{C_2}) \end{array}$$

and moreover,

$$\text{(4.2)} \qquad \varphi(\xi) = u_2 \sum_{s\in S} C\ell_s(\xi) + \frac{1}{2}v_2^2\left(\sum_{s\in S} C\ell_s(\xi)\right)^2 - \frac{1}{2}w_2 \sum_{s\in S} C\ell_s(\xi)^2.$$

As motivated earlier, we are now interested in approximating the values

$$(4.3) \qquad \max_{\begin{cases} c+\xi \text{ binary} \\ \Phi(c+\xi) \text{ false} \end{cases}} \varphi(\xi) \qquad \text{and} \qquad \min_{\begin{cases} c+\xi \text{ binary} \\ \Phi(c+\xi) \text{ true} \end{cases}} \varphi(\xi)$$

only in terms of the parameters and global characteristics of Φ (in this case only S since $S_3 = S_4 = \ldots = 0$ and $S_2 = S$).

Now, by reasons of symmetry, we may restrict ourselves to the values $\varphi(c)$, or in other words, we shall investigate

$$(4.4) \qquad M_{\text{sat}} = \max_{\begin{cases} \Phi(e) \text{ false} \\ \Phi \text{ is 2-CNF} \end{cases}} \varphi(c) \qquad \text{and} \qquad m_{\text{sat}} = \min_{\begin{cases} \Phi(e) \text{ true} \\ \Phi \text{ is 2-CNF} \end{cases}} \varphi(c)$$

Doing so we shall obtain an upperbound (a lower bound respectively) for the values of 4.3.

For a 2-literal clause $\boldsymbol{C}_s$ we have

$$(4.5) \qquad C\ell_s(c) \in \{1, 0, -1\}$$

depending whether $\boldsymbol{C}_s$ contains two positive literals, two literals of opposite sign, or two negative ones. That is, if we assume that Φ consists of

$$(4.6) \qquad \begin{cases} S_{2,2} & \text{clauses having 2 positive literals} \\ S_{2,1} & \text{clauses having 1 positive literal} \\ S_{2,0} & \text{clauses having 0 positive literals} \end{cases}$$

we obtain

$$(4.7) \qquad \varphi(c) = u_2(S_{2,2} - S_{2,0}) + \frac{1}{2} v_2^2 (S_{2,2} - S_{2,0})^2 - \frac{1}{2} w_2 (S_{2,2} + S_{2,0}).$$

The above expression appears to be a *convex* function of $S_{2,2}$ and $S_{2,0}$ and the two values of 4.4 can now be established by solving

$$(4.8) \qquad M_{\text{sat}} = \max_{\begin{cases} S_{2,2}, S_{2,0} \geq 0 \\ S_{2,2} + S_{2,0} \leq S_2 \\ S_{2,0} \geq 1 \end{cases}} \varphi(c) \qquad \text{and} \qquad m_{\text{sat}} = \min_{\begin{cases} S_{2,2}, S_{2,0} \geq 0 \\ S_{2,2} + S_{2,0} \leq S_2 \\ S_{2,0} = 0 \end{cases}} \varphi(c) \; .$$

Both problems are easily solved. The first by notifying that a convex function on a simplex attains its maximal value in a vertex. The latter because it is a one dimensional smooth quadratic convex minimization problem. It is understood here that we consider $S_{2,2}$ and $S_{2,0}$ to be *real* variables.

We proceed with rewriting the two problems using

$$x = \frac{S_{2,2}}{S} \; ; y = \frac{S_{2,0}}{S} \; ; \; \overline{u}_2 = S u_2; \; \overline{v}_2 = S v_2; \; \overline{w}_2 = S w_2.$$

Notice that $\overline{u}_2, \overline{v}_2$ and $\overline{w}_2$ only depend on ε_2 and r.

The second problem now reads as

$$m_{\text{sat}} = \min_{0 \leq x \leq 1} \overline{u}_2 x + \frac{1}{2}\overline{v}_2^2 x^2 - \frac{1}{2}\overline{w}_2 x \tag{4.9}$$

which has a solution

$$m_{\text{sat}} = \begin{cases} 0 & \text{if} \quad \overline{w}_2 \leq 2\overline{u}_2 \\ -\frac{(\overline{u}_2 - \frac{1}{2}\overline{w}_2)^2}{2\overline{v}_2^2} & \text{if} \quad \overline{w}_2 \geq 2\overline{u}_2. \end{cases} \tag{4.10}$$

In order to solve the first problem we notice that we are only interested in getting estimates for larger formulae. This means that the constraint $y \geq \frac{1}{S}$ in the first problem may be relaxed to $y \geq 0$, thus obtaining an upper bound for the actual value.
Now we simply have to compare the values of $\varphi(c)$ at (0,0), (1,0) and (0,1), resulting in

$$\begin{aligned} M_{\text{sat}} &= \text{maximum of} \begin{cases} 0 \\ \overline{u}_2 + \frac{1}{2}\overline{v}_2^2 - \frac{1}{2}\overline{w}_2 \\ -\overline{u}_2 + \frac{1}{2}\overline{v}_2^2 - \frac{1}{2}\overline{w}_2 \end{cases} \\ &= \max(0, \overline{u}_2 + \frac{1}{2}\overline{v}_2^2 - \frac{1}{2}\overline{w}_2). \end{aligned} \tag{4.11}$$

Sofar we conclude that we obtained the estimates desired and that we have established a parametrized family of quadratic convex cuts of the form

$$\begin{cases} \Phi(c+\xi) \text{ true} & \Rightarrow \quad \varphi(\xi) \geq m_{\text{sat}}(r, \varepsilon_2) \\ \varphi(\xi) \geq M_{\text{sat}}(r, \varepsilon_2) & \Rightarrow \quad \Phi(c+\xi) \text{ true} \end{cases} \tag{4.12}$$

where the latter implication can be strengthened (see 4.17).

A glance at M_{sat} and m_{sat} immediately reveals that no parameter setting exists which makes a global separation of satisfiable vertices from non satisfiable vertices by means of these quadratic convex cuts possible ! That is, no ε_2 and r exist for which $M_{\text{sat}} \leq m_{\text{sat}}$ (unfortunately). This of course does not imply that such a separation is impossible if we allow our parameters to depend on more specific structure of Φ, rather than on the global characteristics. But we do not want them to do so.

The question whether some choices of ε_2 and r yield sharper cuts than others remains unanswered sofar. Now one of the obvious properties we want our first cut of 4.12 to satisfy is that it excludes as many non satisfiable assignments as possible. The expected numbers of 4.6, when dealing with *random* formulae, are

$$\begin{cases} S_{2,2} = \frac{1}{4}S_2 \\ S_{2,1} = \frac{1}{2}S_2 \\ S_{2,0} = \frac{1}{4}S_2 \end{cases} \tag{4.13}$$

that is, the expected value of $\varphi(\xi)$, at a vertex $c + \xi$, is

$$\varphi_{\text{exp}} = \overline{u}_2(\frac{1}{4} - \frac{1}{4}) + \frac{1}{2}\overline{v}_2^2(\frac{1}{4} - \frac{1}{4})^2 - \frac{1}{2}\overline{w}_2(\frac{1}{4} + \frac{1}{4}) = -\frac{1}{4}\overline{w}_2. \tag{4.14}$$

Thus parameter settings resulting in

$$-\frac{1}{4}\overline{w}_2 < m_{\text{sat}} \tag{4.15}$$

are favourable for our purposes.
A plausible choice, which also simplifies the formulae involved, is to select ε_2 and r such that $\overline{w}_2 = 2\overline{u}_2$. This poses the question whether (see 4.1)

$$\sigma_2 + (1-r)\frac{\alpha_2^2}{C_2} = 2\alpha_2 \tag{4.16}$$

is solvable for $\varepsilon_2 > 0$ and $r < 0$. The reader is invited to confirm that in fact $\varepsilon_2 = 1$ and $r = 1 - \sqrt{2}$ constitute a solution to 4.16 and that in this case 4.12 simplifies to

$$\begin{cases} \Phi(c+\xi) \text{ true} & \Rightarrow \quad \frac{1}{S}\sum_{s\in S} C\ell_s(\xi) + \frac{1}{2S^2}(\sum_{s\in S} C\ell_s(\xi))^2 - \frac{1}{S}\sum_{s\in S} C\ell_s(\xi)^2 \geq 0 \\ \Phi(c+\xi) \text{ false} & \Rightarrow \quad \frac{1}{S}\sum_{s\in S} C\ell_s(\xi) + \frac{1}{2S^2}(\sum_{s\in S} C\ell_s(\xi))^2 - \frac{1}{S}\sum_{s\in S} C\ell_s(\xi)^2 \leq \frac{1}{2}. \end{cases} \tag{4.17}$$

When not relaxing for the constraint $y \geq \frac{1}{S}$ in 4.11 a sharper result (valid for $S \geq 3$) is obtained: the right hand side of the second inequality may be taken as $\frac{1}{2} - \frac{4}{S} + \frac{2}{S^2}$ instead of $\frac{1}{2}$.

We see that putting $\varepsilon_2 = 1$ and $r = 1 - \sqrt{2}$ results in rather attractive formulae, a fact which invites us to present some examples.

First consider $\Phi = (p \vee q) \wedge (p \vee \sim q) \wedge (\sim p \vee q) \wedge (\sim p \vee \sim q)$. Here $\sum_{s\in S} C\ell_s(\xi) = 0$ and $\sum_{s\in S} C\ell_s(\xi)^2 = 4\xi_p^2 + 4\xi_q^2$. Hence 4.17 reads as

$$\begin{aligned} \Phi(c+\xi) \text{ true} \quad &\Rightarrow \quad \xi_p^2 + \xi_q^2 \leq 0 \\ \Phi(c+\xi) \text{ false} \quad &\Rightarrow \quad \xi_p^2 + \xi_q^2 \geq -\frac{1}{2}. \end{aligned}$$

The first implication yields unsolvability since $\xi_p^2 + \xi_q^2 = \frac{1}{2}$ at any vertex $c + \xi$.

More generally, if Φ is *perfectly balanced in signs* (which exactly means that $\sum_{s\in S} C\ell_s(\xi) = 0$) our first cut reads as

$$\Phi(c+\xi) \text{ true } \Rightarrow C\ell_s(\xi) = 0, \text{ for any } s \in S. \tag{4.18}$$

The above means that such formulae are extremely simple to solve, because the disjunction in any clause should be interpreted as an exclusive or.

The next example reveals some of the geometry associated with our quadratic convex cuts.
N.B. N does not denote the number of *variables* in this example.

Let Φ be the conjuction of

$$\begin{array}{l} p \vee q \\ p \vee \sim q \\ \left.\begin{array}{l} \sim p \vee q \\ \sim p \vee q \\ \vdots \\ \sim p \vee q \end{array}\right\} N \text{ times} \end{array}$$

We first calculate $\varphi(\xi)$:

$$\varphi(\xi) = \left(\frac{(N-2)^2}{2(N+2)^2} - 1\right)\xi_p^2 + \left(\frac{N^2}{2(N+2)^2} - 1\right)\xi_q^2 + \\ + \left(\frac{2N}{N+2} - \frac{2N(N-2)}{2(N+2)^2}\right)\xi_p\xi_q + \frac{2-N}{N+2}\xi_p + \frac{N}{N+2}\xi_q. \tag{4.19}$$

Figures 1 and 2 show the regions $\Phi(\xi) = 0$ for $N = 1, 10, 50$ and $N = 6, 7, 8$ respectively.

Analyzing the $N = 1$ case we see that linear cuts $\xi_p > -\frac{1}{2}$ and $\xi_q > -\frac{1}{2}$ are obtained by minimizing ξ_p and ξ_q respectively over the region $\varphi(\xi) \geq 0$. These cuts precisely correspond to the resolvents p and q, when applying resolution to Φ. Deriving linear cuts perpendicular to the axes of the ellipsoid $\varphi(\xi) \geq 0$ we obtain $\xi_p - \xi_q > -1$ and $\xi_p - \xi_q < 1$ (perpendicular to the short axis) and $\xi_p + \xi_q > 0$ (perpendicular to the longer axis, at (0,0)). Now the first two imply the clauses $p\vee \sim q$ and $\sim p \vee q$ respectively (clauses which were already present here). The latter in fact implies $p = q = 1$. Notice that our quadratic cut in fact separates satisfiable vertices from non satisfiable ones.

When N increases the discriminative properties of the cut gradually vanish, as the pictures show. However, the centre of the ellipsoid (given by the coordinates $\frac{N+2}{N^2+7N+2}(1, N)$) is always in the right quadrant!

Although the setting $\varepsilon_2 = 1$; $r = 1 - \sqrt{2}$ gives attractive formulae and meets the requirement that the cuts $\varphi(\xi) \geq m_{\text{sat}}$ exclude the "average" non satisfiable assignment, other settings exist which also meet this last requirement. We present another possible choice which gives more freedom when considering mixed $2,3-$ CNF formulae and which was in fact used in our experiments of [5].

By posing the condition that we want $M_{\text{sat}} = 0$ the equation

$$\overline{u}_2 + \frac{1}{2}\overline{v}_2^2 - \frac{1}{2}\overline{w}_2 = 0 \tag{4.20}$$

is investigated and turns out to simplify to

$$\alpha_2 = \frac{1}{2}\sigma_2 \tag{4.21}$$

which is a requirement *not* depending on parameter r, and is satisfied precisely if $\varepsilon_2 = \frac{1}{4}$. This setting meets 4.15 as well and results in the parametrized family of cuts

$$\begin{cases} \frac{1}{S}\sum_{s\in S} C\ell_s(\xi) + \frac{1}{2S^2}\tau_r(\sum_{s\in S} C\ell_s(\xi))^2 - \frac{1}{2S}(2+\tau_r)\sum_{s\in S} C\ell_s(\xi)^2 \geq -\frac{1}{8}\tau_r \\ \tau_r = \frac{1}{2}(1-r)(\sqrt{5}-1); r < 0. \end{cases} \tag{4.22}$$

For $r \to -\infty$ the above results in another nice cut:

$$\frac{1}{2S^2}(\sum_{s\in S} C\ell_s(\xi))^2 - \frac{1}{2S}\sum C\ell_s(\xi)^2 \geq -\frac{1}{8}. \tag{4.23}$$

Notice that in the above the effect of the first order term has vanished!

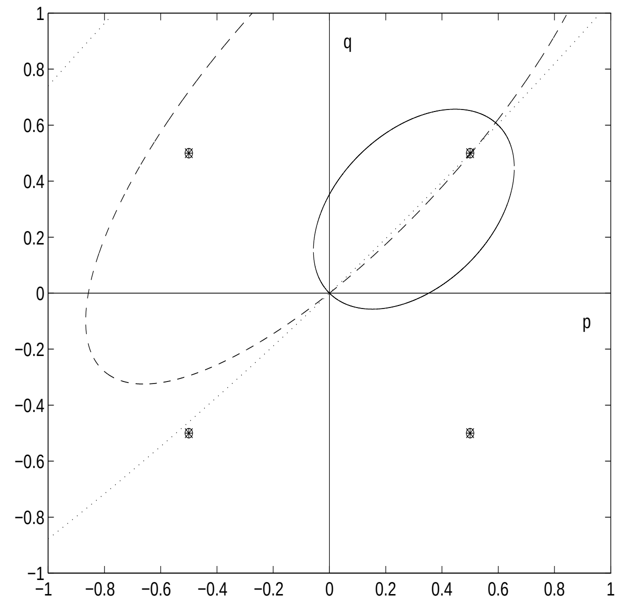

Figure 1: N=1 (solid), N=10 (dashed), N=50 (dotted)

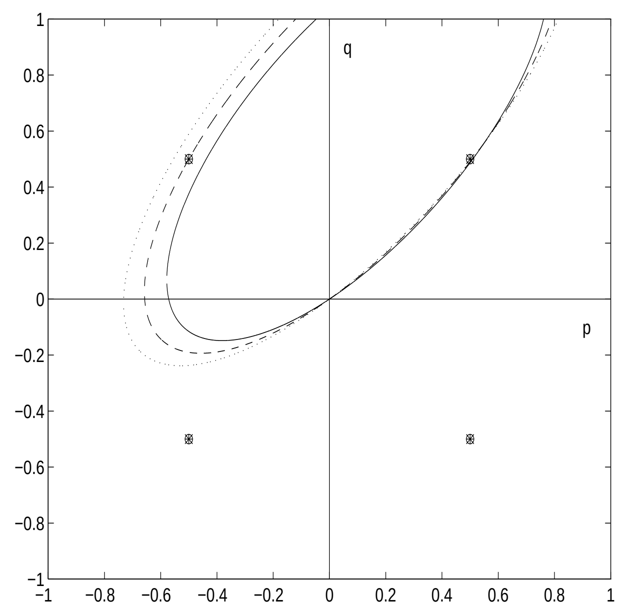

Figure 2: N=6 (solid), N=7 (dashed), N=8 (dotted)

5 The 3-SAT-case.

In this section we shall deal with pure 3-CNF formulae. We shall ask ourselves the same questions as for the 2-SAT case. It will appear that things become slightly more involved and that there is considerably less freedom in selecting appropriate parameters. We start with simplifying our formulae.

$$(5.1)\qquad \begin{array}{lcl} \Phi(c)^{1-r} & = & C_3^{1-r} \\ u_3 & = & \frac{1}{S}\alpha_3\rho_3 \\ v_3 & = & \frac{1}{S}\alpha_3\rho_3\sqrt{\frac{1-r}{C_3}} \\ w_3 & = & \frac{1}{S}(\sigma_3 + (1-r)\frac{\alpha_3^2\rho_3^2}{C_3}) \end{array}$$

and

$$(5.2)\qquad \varphi(\xi) = u_3 \sum_{s\in S} C\ell_s(\xi) + \frac{1}{2}v_3^2(\sum_{s\in S} C\ell_s(\xi))^2 - \frac{1}{2}w_3 \sum_{s\in S} C\ell_s(\xi)^2.$$

Again we put

$$(5.3)\qquad M_{\text{sat}} = \max_{\left\{\begin{array}{l}\Phi(e)\text{false}\\ \Phi\text{ is 3-CNF}\end{array}\right.} \varphi(c) \quad \text{and} \quad m_{\text{sat}} = \min_{\left\{\begin{array}{l}\Phi(e)\text{true}\\ \Phi\text{ is 3-CNF}\end{array}\right.} \varphi(c).$$

Now for a 3-literal clause $\boldsymbol{C}_s$ we have

$$(5.4)\qquad C\ell_s(\xi) \in \{\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\}.$$

We define, as in 4.6,

$$(5.5)\qquad S_{3,i} = \#\{s \in S | \mathbf{C}_\text{s} \text{ contains exactly } i \text{ positive literals}\}$$

obtaining

$$(5.6)\qquad \begin{array}{lcl} \varphi(c) & = & u_3(\frac{3}{2}S_{3,3} + \frac{1}{2}S_{3,2} - \frac{1}{2}S_{3,1} - \frac{3}{2}S_{3,0}) + \\ & & \frac{1}{2}v_3^2(\frac{3}{2}S_{3,3} + \frac{1}{2}S_{3,2} - \frac{1}{2}S_{3,1} - \frac{3}{2}S_{3,0})^2 - \\ & & \frac{1}{2}w_3(\frac{9}{4}S_{3,3} + \frac{1}{4}S_{3,2} + \frac{1}{4}S_{3,1} + \frac{9}{4}S_{3,0}). \end{array}$$

Again, the above expression is a convex function of the $S_{3,i}$ and 5.3 can be solved similarly (but more tediously) as in the 2-CNF case. Following the same lines (but everything with a extra dimension) we obtain:

$$(5.7)\qquad m_{\text{sat}} = \left\{ \begin{array}{lll} \frac{1}{8}\bar{v}_3^2 - \frac{1}{8}\bar{w}_3 - \frac{1}{2}\bar{u}_3 & \text{if} & 2\bar{u}_3 - \bar{w}_3 \geq \bar{v}_3^2 \\ \frac{9}{8}\bar{v}_3^2 - \frac{9}{8}\bar{w}_3 + \frac{3}{2}\bar{u}_3 & \text{if} & 2\bar{u}_3 - \bar{w}_3 \leq -3\bar{v}_3^2 \\ -\frac{3}{8}\bar{w}_3 - \frac{1}{8\bar{v}_3^2}(2\bar{u}_3 - \bar{w}_3)^2 & \text{else} & \end{array} \right.$$

and, for $S \to \infty$

$$(5.8) \qquad M_{\text{sat}} = \text{maximum of} \begin{cases} \frac{3}{2}\bar{u}_3 + \frac{9}{8}\bar{v}_3^2 - \frac{9}{8}\bar{w}_3 \\ \frac{1}{2}\bar{u}_3 + \frac{1}{8}\bar{v}_3^2 - \frac{1}{8}\bar{w}_3 \\ -\frac{1}{2}\bar{u}_3 + \frac{1}{8}\bar{v}_3^2 - \frac{1}{8}\bar{w}_3 \\ -\frac{3}{2}\bar{u}_3 + \frac{9}{8}\bar{v}_3^2 - \frac{9}{8}\bar{w}_3. \end{cases}$$

Now we have completed the establishment of m_{sat} and M_{sat} as functions of r and ε_3 and hence we obtained a parametrized family of quadratic convex cuts for the (pure) 3-SAT problem.

Next we investigate whether it is possible to select r and ε_3 in such a way that the cut $\varphi(\xi) \geq m_{\text{sat}}$ excludes the "average" non satisfiable assignment. For a random 3-CNF formula the expected values of the $S_{3,i}$ are

$$(5.9) \qquad \begin{cases} S_{3,3} = \frac{1}{8}S_3 \\ S_{3,2} = \frac{3}{8}S_3 \\ S_{3,1} = \frac{3}{8}S_3 \\ S_{3,0} = \frac{1}{8}S_3 \end{cases}$$

whence

$$(5.10) \qquad \begin{aligned} \varphi_{\exp} &= \bar{u}_3(\frac{1}{8}.\frac{3}{2} + \frac{3}{8}.\frac{1}{2} + \frac{3}{8}. - \frac{1}{2} + \frac{1}{8}. - \frac{3}{2}) + \ldots - \frac{1}{2}\bar{w}_3(\frac{9}{4}.\frac{2}{8} + \frac{1}{4}.\frac{6}{8}) \\ &= -\frac{3}{8}\bar{w}_3. \end{aligned}$$

Investigating the requirement

$$(5.11) \qquad -\frac{3}{8}\bar{w}_3 < m_{\text{sat}}$$

excludes the possibility $2\bar{u}_3 - \bar{w}_3 \notin [-3\bar{v}_3, \bar{v}_3^2]$, as the reader may verify. The third possible case in m_{sat} also violates the above requirement, however, by putting the condition

$$(5.12) \qquad 2\bar{u}_3 = \bar{w}_3$$

it comes as close as possible. We conclude that we *cannot* find a parameter setting meeting our strict requirement of 5.11 but that, by putting $2\bar{u}_3 = \bar{w}_3$ the "average" non satisfiable assignment is on the boundary of the cut $\varphi(\xi) \geq m_{\text{sat}}$. We can be slightly more detailed here: In establishing m_{sat} in the $2\bar{u}_3 = \bar{w}_3$ setting it appears that the minimal value of the associated convex minimization problem is obtained exactly only if $\frac{3}{4}S$ is a natural number. This means that in fact the "average" non satisfiable assignment may appear on the boundary of $\varphi(\xi) = -\frac{3}{8}\bar{w}_3$ but that the satisfiable assignments are in the interior of the cut, that is, they meet the requirement $\varphi(\xi) > -\frac{3}{8}\bar{w}_3$ if S is *not* a multiple of 4.

The condition $2\bar{u}_3 = \bar{w}_3$ reads as

$$(5.13) \qquad (2\alpha_3\rho_3 - \sigma_3)C_3 = (1-r)\alpha_3^2\rho_3^2$$

which in fact defines a bounded solution set in $r < 0$ and restricts ε_3 to values exceeding approximately 0.55. Putting ε_3 such that

$$(5.14)\qquad \frac{(2\alpha_3\rho_3-\sigma_3)C_3}{\alpha_3^2\rho_3^2}(=1-r)>1$$

we obtain the quadratic convex cut

$$(5.15)\qquad \begin{cases} \frac{1}{S}\sum_{s\in S} C\ell_s(\xi)+\frac{1}{2S^2}(\kappa_\epsilon)(\sum_{s\in S} C\ell_s(\xi))^2-\frac{1}{S}\sum_{s\in S} C\ell_s(\xi)^2 \geq -\frac{3}{4} \\ \kappa_\varepsilon = 2-\frac{\sigma_3}{\alpha_3\rho_3} = 2-\frac{\varepsilon_3}{(\varepsilon_3+\frac{1}{4})(\sqrt{\varepsilon_3+\frac{1}{4}}-\frac{1}{2})} \end{cases}$$

where the inequality can be taken strict if S is not a multiple of 4.

The above simplifies when we put $\sigma_3=\alpha_3\rho_3=\overline{u}_3=\frac{1}{2}\overline{w}_3=\overline{v}_3^2$, which is satisfied if

$$(5.16)\qquad \begin{cases} \varepsilon_3=\frac{3}{4}+\frac{1}{2}\sqrt{3} \\ r=1-\frac{C_3}{\sigma_3}=-1.762551985. \end{cases}$$

Then 5.14 is met and $\kappa_\varepsilon=1$ in 5.15. For these parameter values $M_{\text{sat}}=\frac{3}{8}\bar{u}_3=\frac{3}{8}\alpha_3\rho_3$.
We resume:
For ε_3 and r as in 5.16 we obtain the following convex quadratic cuts:
(5.17)

$$\begin{cases} \Phi(c+\xi)\text{ true} \Rightarrow \frac{1}{S}\sum_{s\in S} C\ell_s(\xi)+\frac{1}{2S^2}(\sum_{s\in S} C\ell_s(\xi))^2-\frac{1}{S}\sum_{s\in S} C\ell_s(\xi)^2 \geq -\frac{3}{4} \\ \Phi(c+\xi)\text{false} \Rightarrow \frac{1}{S}\sum_{s\in S} C\ell_s(\xi)+\frac{1}{2S^2}(\sum_{s\in S} C\ell_s(\xi))^2-\frac{1}{S}\sum_{s\in S} C\ell_s(\xi)^2 \leq \frac{3}{8}-\frac{5}{S}+\frac{2}{S^2}. \end{cases}$$

Yet another possibility is $\varepsilon_3\to\infty$ in 5.15, resulting in $\kappa_\epsilon=2$. Corresponding r for this last setting is $r=-2$.

The first example we consider is to affirm our conclusions about the impossibility of strictly separating satisfiable assignments from unsatisfiable ones using our quadratic convex cuts.
Let Φ be the conjunction of

$$\begin{aligned} &p\vee q\vee r \\ &p\vee \sim q\vee \sim r \\ &\sim p\vee q\vee \sim r \\ &\sim p\vee \sim q\vee r. \end{aligned}$$

Then $\sum C\ell_s(\xi)=0$ and 5.2 becomes

$$\varphi(\xi)=-\frac{1}{2}w_3(4\xi_p^2+4\xi_q^2+4\xi_r^2)=-\frac{3}{2}w_3=-\frac{3}{8}\bar{w}_3$$

We see that $\varphi(\xi)$ is constant on the vertices, of which there are satisfiable and non satisfiable ones. Notice that our first cut of 5.17 is

$$\xi_p^2+\xi_q^2+\xi_r^2\leq\frac{3}{4}$$

which defines a sphere with centre 0 containing all vertices of the cube $-c+[0,1]^3$ in its boundary.

If Φ is the conjunction of all eight clauses $(\sim)p \vee (\sim)q \vee (\sim)r$ we obtain precisely the same cut. Thus "spheres" may represent satisfiable formulae as well as non satisfiable ones.

We shall discuss the next example at some length. It demonstrates what can be done with quadratic cuts, using the eigenvalue and eigenspace structure of $\triangle$.

Let Φ be the conjunction of

$p \vee q \vee r$
$p \vee q \vee \sim r$
$p \vee \sim q \vee r$
$p \vee \sim q \vee \sim r$
$\sim p \vee q \vee r$
$\sim p \vee q \vee \sim r$

Here, $\varphi(\xi) \geq -\frac{3}{4}$ can be written as

$$17\xi_p^2 + 17\xi_q^2 + 18\xi_r^2 - 14\xi_p\xi_q - 6\xi_p - 6\xi_q \leq \frac{27}{2}. \tag{5.18}$$

Eigenvalues are 10, 24 and 18 with corresponding eigenvectors (1,1,0),(-1,1,0) and (0,0,1). Centre of the ellipsoid is $(\frac{3}{10}, \frac{3}{10}, 0)$. The above cut is separating with respect to true and false assignments. Transforming $\xi-$space to η-space (where the eigenvectors are taken as a base) through

$\eta_1 = \frac{1}{2}\sqrt{2}(\xi_p + \xi_q)$
$\eta_2 = \frac{1}{2}\sqrt{2}(\xi_p - \xi_q)$
$\eta_3 = \xi_r$

transforms 5.18 into

$$10(\eta_1 - \frac{3}{10}\sqrt{2})^2 + 24\eta_2^2 + 18\eta_3^2 \leq \frac{153}{10}. \tag{5.19}$$

Now $\eta_3^2 = \frac{1}{4}$ in all vertices, whence 5.19 reduces to

$$10(\eta_1 - \frac{3}{10}\sqrt{2})^2 + 24n_2^2 \leq \frac{108}{10}. \tag{5.20}$$

We obtain $\eta_2^2 \leq \frac{108}{240}$ which means

$$(\xi_p - \xi_q)^2 \leq \frac{216}{240} < 1. \tag{5.21}$$

This implies $\xi_p = \xi_q$ in a satisfiable vertex. Now 5.20 reduces to

$$(\eta_1 - \frac{3}{10}\sqrt{2})^2 \leq \frac{108}{100} \tag{5.22}$$

which in turn means $\eta_1 \geq \frac{3}{10}\sqrt{2} - \frac{1}{10}\sqrt{108}$, yielding

$$2\xi_p = \xi_p + \xi_q > \frac{6}{10} - \frac{1}{10}\sqrt{216} > -\frac{9}{10} > -1 \tag{5.23}$$

which finally leads to $\xi_p = \frac{1}{2}$.

6 General and mixed 2,3-SAT case.

In this section we shall only be concerned with the convex quadratic cut $\varphi(\xi) \geq m_{\text{sat}}$.

Again we notice that an estimation m_{sat} is desired which only depends on $S_2, S_3, \ldots, S_M$ and that by reasons of symmetry we may restrict ourselves to solve the problem

$$(6.1) \qquad m_{\text{sat}} = \min_{\begin{cases} \Phi(e) \text{ true} \\ \Phi \text{ has } S_i \text{ clauses with } i \text{ literals.} \end{cases}} \varphi(c)$$

Now $C\ell_s(c) = \frac{1}{2}(I_s - I_s) \in [-l_s, l_s]$. Let

$$(6.2) \qquad \begin{cases} S_m = S_{m,0} + S_{m,1} + \ldots + S_{m,m} \quad (j = 2, \ldots M) \\ S_{m,i} \text{ is the number of clauses with } m \text{ literals of which } i \text{ are positive} \\ S_{m,i} \geq 0 \end{cases}$$

then

$$(6.3) \qquad \begin{aligned} \varphi(c) = \; & \textstyle\sum_{m,i} \frac{1}{2} u_m (2i - m) S_{m,i} + \frac{1}{2}(\sum_{m,i} \frac{1}{2} v_m (2i - m) S_{m,i})^2 \\ & -\textstyle\frac{1}{2}(\sum_{m,i} \frac{1}{4} w_m (2i - m)^2 S_{m,i}) \end{aligned}$$

and, again, the above expression is seen to be convex in the $S_{m,i}$ and must be minimized under the constraints given in 6.2 and the additional constraints $S_{m,0} = 0)$ $(m \leq M)$. The above problem is a $\frac{1}{2}M(M-1)$ dimensional problem and can be actually solved adequately using an accurate convex programming solver if M is not too large and r and $\varepsilon_m (m \leq M)$ are specified! We shall concentrate ourselves to $M = 3$, thus restricting ourselves to the mixed 2,3-SAT case. Also, we shall fix r, ε_2 and ε_3 to the values of 4.21 and 5.16. We are then left with a 3-dimensional convex programming problem which we have solved by hand. We shall not present these tedious calculations here and just give the results.

Below we use the real numbers

$$(6.4) \qquad \begin{array}{rcll} \mu_2 & = & \alpha_2 C_2^{r-1} & (\approx 0.9835...) \\ \mu_3 & = & \sigma_3 C_3^{r-1} & (\approx 0.7349...) \\ \gamma & = & (1-r)\frac{\alpha_2}{C_2} & (\approx 1.7073...) \end{array}$$

and the global characteristic

$$(6.5) \qquad \omega = \omega(S_2, S_3) = \frac{S_2}{S} C_2^r + \frac{S_3}{S} C_3^r.$$

It turned out that m_{sat} depends on the value $\frac{S_2}{S_3}$ according to the following three cases:

$$(6.6) \qquad \begin{array}{lcl} \text{case 1} & : & \dfrac{S_2}{S_3} \leq \dfrac{\mu_3}{2\mu_2} \\ \text{case 2} & : & \dfrac{\mu_3}{2\mu_2} \leq \dfrac{S_2}{S_3} \leq (1+\gamma)\dfrac{\mu_3}{\mu_2} \\ \text{case 3} & : & \dfrac{S_2}{S_3} \geq (1+\gamma)\dfrac{\mu_3}{\mu_2} \end{array}$$

and has the corresponding values
(6.7)

$$m_{\text{sat}} = \begin{cases} -\frac{3}{4}\mu_3 \frac{S_3}{S}\omega^{\frac{1}{r}-1} - \frac{1}{2}\gamma(1-r)\mu_2\mu_3 \frac{S_2 S_3}{S^2}\omega^{\frac{1}{r}-2} - \frac{1}{2}(1-r)\mu_2^2 \frac{S_2^2}{S^2}\omega^{\frac{1}{r}-2} & \text{case 1} \\ -\frac{3}{4}\mu_3 \frac{S_3}{S}\omega^{\frac{1}{r}-1} + \frac{1}{8}(1-r)\mu_3^2 \frac{S_3^2}{S^2}\omega^{\frac{1}{r}-2} - \frac{1}{2}(1+\gamma)(1-r)\mu_2\mu_3 \frac{S_2 S_3}{S^2}\omega^{\frac{1}{r}-2} & \text{case 2} \\ -\frac{3}{4}\mu_3 \frac{S_3}{S}\omega^{\frac{1}{r}-1} - \frac{1}{8}\gamma(\gamma+2)(1-r)\mu_3^2 \frac{S_3^2}{S^2}\omega^{\frac{1}{r}-2} - \frac{1}{4}(1+\gamma)(1-r)\mu_2\mu_3 \frac{S_2 S_3}{S^2}\omega^{\frac{1}{r}-2} & \\ -\frac{1}{8}(1-r)\mu_2^2 \frac{S_2^2}{S^2}\omega^{\frac{1}{r}-2} & \text{case 3.} \end{cases}$$

The reader is invited to confirm that for $S_3 = 0$ and $S_2 = 0$, m_{sat} takes the values derived for the separated cases respectively.

Also in this case one can show (as in 4.14 and 5.10) that the "average" non satisfiable vertex $c + \xi$ does *not* satisfy $\varphi(\xi) > m_{\text{sat}}$. In fact, it does *not* satisfy $\varphi(\xi) \geq m_{\text{sat}}$ as soon as $S_2 \neq 0$. However, it is questionable whether this is a useful observation, since mixed 2,3-SAT formulae are typically appearing when solving a 3-SAT problem and as such generally are not random.

7 Geometric evidence for the hardness of balanced formulae.

Numerical experiments (Dubois [1]) have shown that random 3-SAT formulae in the hard region tend to become even harder in so called balanced cases. We believe that our quadratic convex cuts explain this feature quite well.
First, we notice that for random formulae Φ, the eigenvalues of $\Delta\Phi(c) = (\Delta_{ij})$ are generally nonzero and consequently negative. This means, that the inequality $\varphi(\xi) \geq m_{\text{sat}}$ defines an ellipsoid, the center C of which is given by the solution of

$$\Delta(C) = -\nabla \tag{7.1}$$

Now a formula Φ which is balanced in sign defines an approximately zero gradient ∇. Therefore, the *linear term in* $\varphi(\xi)$ *vanishes* and consequently $C = 0$. Thus the geometrical picture of a sign balanced formula is an *ellipsoid with centre 0.* In this case, first order heuristics based on the Taylor expansion of $\Phi(x)$ at the center yields no information and, in case of pure 3-SAT formulae, $\varphi(\xi) \geq m_{\text{sat}}$ simplifies to (we use the parameter setting of 5.16 and 5.17):

$$\sum_{s \in S} C\ell_s(\xi)^2 \leq \frac{3}{4}S \tag{7.2}$$

In the above case, however, the length of the axes may differ.

Next, the reader is invited to confirm that for pure 3-SAT formulae the diagonal terms of Δ are given by

$$\Delta_{i,i} = \overline{u}_3 \left(\left(\frac{POS(3,i) - NEG(3,i)}{S}\right)^2 - 2\left(\frac{POS(3,i) + NEG(3,i)}{S}\right) \right) \tag{7.3}$$

Noticing that in the above expression the quadratic term is much smaller than the linear term, we see that formulae which are balanced in occurrences of the variables the diagonal terms of Δ are approximately equal. Eigenvalues of Δ, therefore, are approximately equal too. Hence the geometrical picture of an occurrence balanced formula is a sphere.
However, in this case, its centre need not be zero necessarily and hence first order heuristics may yield profit.

Now a formula which is balanced for both features has as its geometrical picture a sphere with centre zero. The inequality of 7.2 now has all coefficients of the ξ_i^2 approximately equal. Still, the off diagonal terms may cause some slight difference in the length of the axes! Consider for instance the double balanced formula

$$\Phi = (p \vee q \vee r) \wedge (p \vee \sim q \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee \sim q \vee \sim r)$$

Here, $\varphi(\xi) \geq m_{\text{sat}}$ simplifies to

$$4\xi_p^2 + 4\xi_q^2 + 4\xi_r^2 + 8\xi_q\xi_r \leq 3$$

which defines an elliptic cylinder (there is a zero eigenvalue here, with eigenvector along $\xi_q = \xi_r$). The eigenvector of the largest eigenvalue (notice that the inequality sign has reversed) is along $\xi_q = -\xi_r$. In fact, the above inequality yields

$$\xi_q\xi_r \leq 0$$

immediately, implying that $q \leftrightarrow \sim r$ is a necessary condition for satisfiability. The above means that, from a geometric point of view, double balanced formulae do not represent typically the hardest possible cases. However, if the off diagonal terms tend to vanish too, that is, if Φ is moreover *pairwise balanced in sign*, meaning

$$DIF^2(3, i, j) \approx 0 \text{ for all } i \neq j \tag{7.4}$$

the geometrical picture is an almost perfect sphere with centre 0 and 7.2 simplifies to the non informative inequality

$$\sum_{i \leq N} \xi_i^2 \leq \frac{\frac{3}{4}S}{\binom{\text{nr. of occurrences}}{\text{of the variables}}} \approx \frac{\frac{3}{4}S}{\frac{3S}{N}} = \frac{1}{4}N \tag{7.5}$$

We encourage the experimentalists to test the hardness of the above type of random 3-SAT formulae.

References

[1] O. Dubois. Double balanced formulae. 1996. DIMACS SAT workshop. Preliminary Release Note.

[2] J. Gu, P.W. Purdom, J. Franco and B.W. Wah. Algorithms for the Satisfiability (SAT) Problem: A Survey. 1996. DIMACS SAT workshop. Preliminary Release Note.

[3] L.Lovasz and A. Schrijver. 1991. Cones of matrices and set-functions and 0-1 optimization. SIAM J. Optimization. Vol. 1, No 2, 166-190.

[4] H. van Maaren, J.F. Groote and M. Rozema. 1995. Verification of propositional formulae by means of convex and concave transforms. Techn. rep. TU Delft, nr. 95-74.

[5] H. van Maaren. 1996. On the use of second order derivatives for the Satisfiability Problem. Techn. rep. TU Delft, nr. 96-34.

[6] H.D. Sherali and W.P. Adams. 1990. A hierarchy of relaxations between the continuous and convex hull representations for zero-one programming problems. SIAM J. Disc. Math. Vol. 3, No. 3, 411-430.