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Some Relations Between Spectral Geometry and Number Theory

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In his paper [Mc], whose object was to show that the spectrum of the Laplacian of a Riemannian surface S determines the surface up to finitely many possibilities, Henry McKean proved the following result, which he called the "Riemann hypothesis for Riemann surfaces":

THEOREM (MCKEAN [MC]).
If S is a hyperbolic Riemann surface, then the first eigenvalue $\lambda_1(S)$ of the Laplacian on S satisfies

$$\lambda_1(S) \geq 14.$$

Here, the term "Riemann surface" denotes a compact, oriented surface with a constant curvature metric. The term "hyperbolic" means that the constant is equal to -1, so that S has genus > 1.

The number 14 arises because it is the bottom $\lambda_0(H^2)$ of the L^2-spectrum of the Laplacian of the hyperbolic plane. The content of McKean's theorem was thus to relate the first eigenvalue $\lambda_1(S)$ with the bottom of the spectrum of the universal cover $\widetilde{S} = H^2$.

Actually, the term "proved" is used somewhat loosely here, because McKean's proof, to which we will return shortly, contained a fatal mistake. Indeed, it was observed shortly afterwards by Burt Randol that the result was indeed wrong, in the following strong sense:

THEOREM (B. RANDOL [RA]). *Let S be a hyperbolic Riemann surface. Then there exist arbitrarily large i-fold coverings S_i of S such that $\lambda_1(S_i) \to 0$ as $i \to \infty$.*

The examples of Randol are fairly easy to describe: Let γ be a simple closed geodesic on S which does not divide S into two pieces. Then we may open S up by cutting it along γ, make i copies of S for some large i, and have them "link hands" to form a circle S_i of surfaces:

To see that $\lambda_1(S_i) \to 0$ as $i \to \infty$, we may construct test funcxtions on S_i in the following way: Assuming that i is even, divide S_i into two sets A and B of copies of S— for instance, the left half of S_i and the right half— so that A and B meet in two coipes of γ. Then let f_i1 be $+1$ on A and -1 on B, and change in some standard way from $+1$ to -1 in neighborhoods of the two copies of γ. Substituting f_i into the Rayleigh characterization of λ_1 given by

$$\lambda_1(S_i) = \inf_f \frac{\int_{S_i} ||grad(f)||^2}{\int_{S_i} f^2},$$

where the inf is taken over all f satisfying $\int_{S_i} f = 0$, we see that $\int_{S_i} ||grad(f_i)||^2$ is bounded independent of i, while $\int_{S_i} f_i^2$ grows linearly in i. It follows that $\lambda_1(S_i)$ tends to zero.

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Here is another example to convince one that McKean's theorem is quite false: Let γ now be a closed geodesic on S which does divide S into two pieces. Then pinching γ smaller and smaller produces a family of Riemann surfaces with a long thin neck:

As the neck gets longer and thinner, one may easily produce test functions f_t whose Rayleigh quotients get smaller and smaller, proving that $\lambda_1(S_t) \to 0$ as $t \to \infty$.

Thus, McKean's theorem is quite false even for surfaces of fixed genus.

Somewhat previous to this development, however, was a fascinating theorem of Selberg [Se], who showed the following:

Theorem ([Se]). *Let* $\Gamma = PSL(2, Z)$, *and*

$$\Gamma_n = \{(\,a\,bcd) \in \Gamma : (\,a\,bcd) \equiv \pm(\,1\,001) \pmod{n}\}\,.$$

Then, for all n,

$$\lambda_1(H^2/\Gamma_n) \geq 316.$$

The Γ_n's are called the congruence subgroups of Γ. Note that H^2/Γ_n is a finite area (non-compact) surface, which covers the orbifold H^2/Γ. The number 316 enters in a kind of curious way, but for our purposes here, note that it is less than and vaguely in the same neighborhood as 14. Indeed, Selberg raised the conjecture that one could improve 316 to be 14. Our interest in this theorem is that he obtains a lower bound for $\lambda_1(H^2/\Gamma_n)$ independent of n, even though the surfaces H^2/Γ_n get larger and larger.

Selberg's proof makes crucial use of the Weil theorem on zeta functions of curves over finite fields (the "Riemann hypothesis for curves over finite fields"), although a weaker bound could be obtained using an elementary technique due to Davenport ([**Dav**]).

Nonetheless, in light of Randol's examples, it makes sense to ask the question: From a geometric point of view, what is responsible for Selberg's theoremΓ In other words, what qualitative features of the congruence surfaces distinguish them from the kinds of surfaces where McKean's theorem is quite falseΓ

Our hope here is to capture for the geometer the essence of the number theory which is responsible for Selberg's 316 theorem. Roughly speaking, our answer is twofold:

(i) congruence surfaces are short and fat.

(ii) congruence surfaces have interesting symmetries.

We will be more precise about this below, but, regarding the first point, let us remark that Randol's examples are all long and thin. Therefore, the problem in (i) is to quantify the notion of "short and fat," and of course to verify that the congruence surfaces are indeed short and fat. Regarding the second point, we emphasize that it is not enough for there to be a lot of symmetries – rather, we demand that they should be interesting. We must also argue that conditions (i) and (ii) lead to good lower bounds for λ_1.

The plan of this paper is as follows: in §1, we discuss in general terms how one can understand the relationship between the spectrum of the Laplacian of a manifold M and the geometry of M, in the case where one has bounds on the geometry of M. In §2, we specialize to the case of constant curvature -1, where the machinery of spherical functions becomes available to us. In §3, we show how the same picture can be carried over to the context of the Laplacian acting on graphs. This is a powerful technique, since analysis on graphs is fairly easy to carry out, and in general it is not difficult to transfer results about graphs to results about hyperbolic manifolds. We then give an example of this line of thought in §4, where we take ideas which emerge naturally from the graph-theoretic picture and translate them into hyperbolic geometry to get our desired explanation of Selberg's theorem.

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§1 Spectral Geometry in the Presence of Bounded Geometry

A basic question is to understand the relationship between the spectrum of the Laplacian of a manifold M and the geometry of M. This is the question "Can one hear the shape of a drumΓ" raised by Mark Kac [Kc], understood in its broadest sense.

In this section, we will review this question in the following light: suppose that M is compact and has constant curvature, or at least has bounded curvature. How do the low eigenvalues of M affect, and how are they affected by, the geometry of MΓ

A very successful approach to understanding the first eigenvalue $\lambda_1(M)$ is contained in the following theorem, due to Jeff Cheeger:

Theorem (Cheeger's inequality [Ch]).

$$\lambda_1(M) \geq 14h^2 \ ,$$

where h is the Cheeger isoperimetric constant of M:

$$h(M) = \inf_S area(S) \min\left(vol(A), vol(B)\right),$$

where S runs over hypersurfaces of M dividing M into two parts A and B.

Here, if M is n-dimensional, we denote by vol(A) the n-dimensional volume of A, and by area(S) the $(n-1)$-dimensional volume of S.

Cheeger's inequality has an interesting converse, due to Peter Buser:

Theorem (Buser [Bu]).

$$\lambda_1(M) \leq c_1 h(M) + c_2 h^2(M),$$

where c_1 and c_2 are positive constants depending only on a lower bound for the Ricci curvature of M.

In particular, if we assume constant curvature -1, then λ_1 is bounded above and below by h, so that λ_1 will be near zero if and only if the same is true of h. We remark that the constants in Buser's theorem are explicit, but not very sharp.

In this way, we can understand all the examples of the introduction.

Cheeger's constant exerts powerful control over the geometry of M, given at least some local control over M. For instance, let us consider the diameter. Then one has:

THEOREM.

$$diam(M) \leq C_1(h,r) \log vol(M) + C_2(h,r)$$

where $C_1(h,r), C_2(h,r)$ are constants depending only on the isoperimetric constant h of M, and the injectivity radius r of M.

PROOF:— Let us pick a point x in M. Denote by $V(t,x)$ the volume of a metric ball of radius t about x.

If M has constant curvature -1 and r is the injectivity radius of M, then we may calculate $V(r,x)$ as the volume of a ball of radius r in hyperbolic space. If M has, say, negative curvature, then we may estimate $V(r,x)$ from below by the volume of a ball of radius r in Euclidean space. In the general case (no curvature assumptions at all!), a lower bound for $V(r/2,x)$ is given by Croke [Cr].

In any case, the problem is now to estimate $V(t,x)$ for $t > r$. Here, we have

$$V'(t,x)V(t,x) \geq h \ ,$$

as long as $V(t,x) < 12$ Vol (M), so that, after integrating, we obtain

$$V(t,x) \geq e^{h(t-r)}V(r,x),$$

up until $V(t,x) = 12$ vol (M). This will happen, therefore, by time t_0, where

$$t_0 = 1h(\log(vol(M)) - \log(2) - \log(V(r))) + r.$$

It follows that, for any two points x and y, the balls of radius t_0 about x and y must intersect. Therefore, diam $(M) \leq 2t_0$, and the theorem is proved. □

Somewhat curiously, this estimate can be improved in some ways in higher dimensions. One has the following theorems of Gromov and Burger – Schroeder:

THEOREM ([GR]). *Let M be a manifold of dimension ≥ 4 whose curvatures are pinched between 0 and -1. Then*

$$diam(M) \leq (const) vol(M)^3.$$

In particular, the dependence of *diam* on h can be eliminated, at the expense of replacing the logarithmic dependence on volume with polynomial dependence.

THEOREM ([BS]). *Let M be a rank 1 symmetric space of dimension ≥ 4. Then*

$$diam(M) \leq (const)\log(vol(M)) + (const)\lambda_1.$$

Here, the dependence of $diam$ on h is replaced by dependence on λ_1.

Here, (const) denotes a constant which depends only on the dimension of M.

Probably, the rank 1 assumption in the theorem of Burger and Schroeder can be weakened to curvature pinched between $-a^2$ and -1.

The dimension assumption $n \geq 4$ is necessary here. In dimension 2, one may shrink a non-separating curve to 0, increasing diameter but leaving h, and hence λ_1, bounded away from 0. In dimension 3, one may do hyperbolic Dehn surgery about a short closed geodesic, producing manifolds of small injectivity radius, and hence large diameter, but with volume bounded from above and λ_1 bounded from below.

An important step in the proofs of these theorems is to show that this cannot happen in higher dimensions. While there are certainly hyperbolic manifolds of higher dimensions which are non-compact but with finite volume, there do not exist hyperbolic manifolds in dimensions greater than 3 which have arbitrarily small geodesics but whose volumes remain bounded. Thus, for hyperbolic manifolds of dimensions greater than 3, an upper bound on volume contains implicitly a lower bound on the injectivity radius. To prove these theorems, one must make this implicit bound more explicit. See **[BS]** for details.

§2 Spherical Functions

Let H^n denote hyperbolic space of dimension n. If we fix a point $y \in H^n$, and if $f(r)$ is a function of one variable $r \geq 0$, we may regard f as a function on H^n by setting

$$f(x) = f(dist(x, y)).$$

We now claim:

THEOREM. *For each λ, there is a unique function $S_\lambda(r) = S^n_\lambda(r)$ on H^n such that*

(i) $\Delta(S_\lambda) = \lambda \cdot S_\lambda$

(ii) $S_\lambda(0) = 1, \qquad S'_\lambda(0) = 0.$

PROOF:— Let us write down the differential equation for S_λ. We have

$$\Delta(S_\lambda) = -div(grad(S_\lambda)).$$

Writing the hyperbolic metric in polar coordinates

$$ds^2 = dr^2 + (\sinh r)^2 d\theta^2$$

where $d\theta^2$ is the standard metric on the sphere S^{n-1}, we have

$$grad(S_\lambda) = S'_\lambda \partial \partial r$$

so that

$$div(grad(S_\lambda))dvol = d\left[(\sinh(r))^{n-1} dr \wedge (d\theta)\left(S'_\lambda \partial\partial r\right)\right] = \left(S''_\lambda + (n-1)\cosh(r)\sinh(r)S'_\lambda\right) dvol.$$

Hence,

$$div(grad(S_\lambda)) = S''_\lambda + (n-1)\coth(r)S'_\lambda,$$

and so S_λ is the unique solution to the equation

$$S''_\lambda + [(n-1)\coth(r)]S'_\lambda + \lambda S_\lambda = 0$$

with initial conditions $S_\lambda(0) = 1$, $S'_\lambda(0) = 0$, as desired. □

Noting that $\coth(r) \to 1$ as $r \to \infty$, standard comparison arguments show that the solution to this equation will look like the solution to the equation

$$f'' + (n-1)\cdot f' + \lambda \cdot f = 0, \quad f'(0) = 0, f(0) = 1$$

for large r. This is then easily seen to be

$$f(r) = e^{-(n-1)2r}[\cosh((\sqrt{(n-1)^2 - 4\lambda})r)$$

$$+(n-1)2\sqrt{(n-1)^2 - 4\lambda}\sinh((\sqrt{(n-1)^2 - 4\lambda})r)].$$

Notice that the solutions change character at $\lambda = (n-1)^2 4$. For $\lambda < (n-1)^2 4$, $S_\lambda(r)$ is a positive function which is decaying exponentially, while for $\lambda > (n-1)^2 4$, the solution oscillates with amplitude on the order of $e^{-(n-1)2r}$. This change in behavior happens because $(n-1)^2 4$ is the bottom $\lambda_0(H^n)$ of the spectrum of H^n, see, for instance, [Su] for a discussion.

Standard comparison arguments also show that $S_\lambda(r)$ is a decreasing function of λ up to the first zero of $S_\lambda(r)$. In particular, $S_\lambda(r)$ is decreasing in λ for all r and all $\lambda < (n-1)^2 4$.

§3 Spectral Geometry of Graphs

In understanding the global analysis of manifolds with bounded geometry, it is often helpful and interesting to model the problem at hand by a graph. The reasons for doing so are two-fold.

First of all, if M_0 is a given manifold, then a rich class of manifolds which share the same bounded geometry assumption is given by the covering spaces $\{M_i\}$ of M_0. After fixing generators $g_1, \ldots, g_k$ for $\pi_1(M_0)$, the coarse global geometry of

M_i can be modeled on what might be called the Cayley graph Γ_i described as follows:

(i) the vertices of Γ_i are the cosets $\pi_1(M_0)/\pi_1(M_i)$.

(ii) Two vertices are joined by an edge if and only if they differ by left-multiplication of a generator g_i.

If one can successfully translate the analytic problem at hand in terms of a problem about graphs, then one has a powerful technique available to answer it.

Furthermore, if $\pi_1(M_0)$ is rich enough (for instance, if it maps onto a free group on two generators), then the covering spaces of M_0 form a family which is as rich as possible in terms of the graph theory.

The second reason for doing this is that analysis on graphs is usually fairly easy. If one can understand the problem there, one can usually work one's way back to the geometric case with a fair amount of insight.

We apply these considerations to the kinds of questions considered in §1. If Γ is a graph, finite or infinite, one may define the Laplacian of Γ as follows: defining $L^2(\Gamma)$ to be the space of L^2 functions on the vertices of Γ, we may define

$$\Delta(f)(x) = \sum_{y \sim x} [f(x) - f(y)].$$

where "$y \sim x$" means that the vertices x and y are joined by an edge. Then Δ is a self-adjoint positive semidefinite operator on $L^2(\Gamma)$, and therefore has a spectrum.

Here are some sample theorems illustrating the connection between the Laplacian on graphs and on manifolds:

THEOREM 1 ([**B2**]). *Let M' be an infinite covering of M, and let Γ' be the graph of $\pi_1(M)/\pi_1(M')$, relative to some choice of generators $g_1, \ldots, g_h$ of $\pi_1(M)$. Then*

$$\lambda_0(M') = 0 \qquad ifandonlyif \;\; \lambda_0(\Gamma') = 0.$$

In particular, if $M' = \widetilde{M}$ is the universal covering of M, then we have $\lambda_0(\widetilde{M}) = 0$ if and only if $\pi_1(M)$ is an amenable group.

THEOREM 2 ([**B5**], [**B1**]). *Let $\{M_i\}$ be a family of finite coverings of M, and Γ_i the corresponding graphs.*

Then $\lambda_1(M_i) \to 0$ if and only if $\lambda_1(\Gamma_i) \to 0$.

SKETCH OF PROOFS:— The idea of the proofs is to make use of Cheeger's inequality. Indeed, there is are analogues of the inequalities of Cheeger and Buser for graphs: if Γ is a finite graph, then set

$$h(\Gamma) = \inf_E \#(E)\mathrm{min}(\#(A), \#(B))$$

where E runs over collections of edges such that $\Gamma - E$ has two components A and B. Then one has that

$$(const)h^2 \leq \lambda_1(\Gamma) \leq (const)h,$$

where (const) depends on the maximal number of edges meeting at a vertex.

To see that $\lambda_1(M_i) \leq \text{(const)}h(\Gamma_i)$, we construct test functions f_i which are constant inside each fundamental domain, and which taper off in a standard way whenever a fundamental domain for A adjoins one for B. We then have

$$\int_{M_i} ||grad(f_i)||^2 \leq (const)\#(E),$$

and

$$\int_{M_i} f_i^2 \geq (const)\#(A).$$

This shows that $\lambda_0(\Gamma) = 0$ implies $\lambda_0(M') = 0$ (in Theorem 1) or that $\lambda_1(\Gamma_i) \to 0$ implies $\lambda_1(M_i) \to 0$ (in Theorem 2).

To obtain the opposite implication, suppose that $h(M_i) \to 0$. Then, one can show, using techniques of geometric measure theory, that there is a hypersurface S_i which realizes the isoperimetric ratio (in the case of Theorem 1, one must exercise a modicum of care to guarantee compactness). Standard variational arguments then show that the mean curvature of S_i is bounded by $h(M_i)$.

It then follows that a ball of fixed radius in S_i, which can therefore only pass through finitely many fundamental domains, of M in M_i, must have a fixed amount of area. Denoting by E_i the collection of fundamental domains through which S_i passes, and A_i, B_i the components of $M_i - E_i$ the inequalities

$$\#(E_i) \leq (const)area(S_i)\#(A_i) \geq (const)volume(A_i)$$

from which

$$h(\Gamma_i) \leq (const)h(M_i)$$

is immediate.

This concludes the proofs of Theorems 1 and 2, except for the assertion $h(\Gamma) = 0$ if and only if $G = \pi_1(M)$ is amenable for Γ the Cayley graph of the group G. When properly translated, this is just the classical characterization of amenability due to Folner [**F**]. □

We should remark that there is an approach to Theorems 1 and 2, due to Marc Burger ([**Bur1**] and [**Bur2**]), which actually allow one to go further. If M is a compact manifold and M' a covering of M, let f be an eigenfunction of L^2 norm 1 of the Laplacian on M'. Denote by F the corresponding function on the graph of the covering whose value at any vertex is given by taking the average value of f over the corresponding copy of the fundamental domain of M in M'. In this way, one can see that all the low eigenvalues of M, not just the first one, are determined by the low eigenvalues of the graph. In effect, if the L^2 norm of F is large, then its Rayleigh quotient is controlled by the eigenvalues of the graph. On the other hand, if its L^2 norm is small, then in some fundamental domain, f has average value near zero, and so its eigenvalue is bounded from below by some large fraction of the first eigenvalue with Neumann conditions of the fundamental domain. See [**Bur1**] and [**Bur2**] for details.

In the case where the M_i's are normal coverings of M, so that $\pi_i = \pi_1(M)/\pi_1(M_i')$ is a group, the spectral properties of the graph Γ_i can be analyzed further. Indeed, we may then identify $L^2(\Gamma_i)$ with $L^2(\pi_i)$, which may be further decomposed into its irreducible components. To that end, it is worth remarking that there is an analogue of the Laplacian defined for unitary representations, which we call the representational Laplacian, given by the formula

$$\Delta_H(X) = \sum_i (X - g_i(X))$$

for g_i fixed generators of a group, and X an element of H a unitary representation of Γ.

As before, Δ_H is a self-adjoint, positive semi-definite operator, and so has a spectrum. The lowest eigenvalue $\lambda_0(H)$ has a special interpretation as the "Kazhdan distance" from H to the trivial representation, see [**B1**] for a discussion. One then has:

DEFINITION. A group Γ has Kazhdan's Property T if there exists an $\varepsilon > 0$ with the following property:

For all unitary representations H which do not contain the trivial representation as a direct sum,

$$\lambda_0(H) > \varepsilon.$$

It is then a theorem of Kazhdan [Kz] that if Γ is a discrete subgroup of a Lie group G with cofinite volume, G then Γ has Property T if and only if the same is true for G (with Property T for Lie groups suitably defined).

Furthermore, the simple Lie groups with Property T are known – for instance, the Lie groups of non-compact type of rank > 1 have Property T, while the hyperbolic groups (e.g. PSL(2,R)) do not, as is demonstrated by Randol's counterexamples to McKean's Theorem.

Using these ideas, it is a simple matter to construct large Riemann surfaces S_i whose first eigenvalues $\lambda_1(S_i)$ are bounded away from 0 – for instance, let $\Gamma = PSL(n, Z)$ for any $n \geq 3$, and let $\phi : \pi_1(S) \to \Gamma$ be any surjective homomorphism. If $\{\Gamma_i\}$ is a family of subgroups of finite index of Γ, with $[\Gamma : \Gamma_i] \to \infty$ as $i \to \infty$, then setting S_i to be the covering of S with $\pi_1(S_i) = \phi^{-1}(\Gamma_i)$, Property T says that

$$\lambda_0(L^2(\Gamma_i) \ominus (constants)) > \epsilon$$

for some ϵ, where "$\ominus$" denotes the "direct minus" — that is,

$$[L^2(\Gamma_i) \ominus (constants)] \oplus constants \simeq L^2(\Gamma_i).$$

Theorem 2 then says that $\lambda_1(S_i) > C$ for some positive constant C as $i \to \infty$.

If we now set $\Gamma = PSL(2, Z)$, and Γ_p the congruence subgroups $\ker(PSL(2, Z) \to PSL(2, Z/p)$, Selberg's theorem (except for the constant 316) is equivalent to the following assertion:

ASSERTION. *There exists a constant $\epsilon > 0$ such that, for all p,*

$$\lambda_1(H_p) > \epsilon$$

for $H_p = L^2(PSL(2, Z/p)) \ominus$ (constants), with fixed generators on $PSL(2, Z)$.

The left-hand side may be computed to some extent. To that end, we observe that the irreducible representations of $PSL(2, Z/p)$ for p a prime, are known and actually fairly understandable [GGP]. They come in two families, the discrete and continuous series (so named by analogy with the Lie Group case).

The dimensions of these representations are all at least $(p-1)2$, which is approximately the cube root of the order of $PSL(2, Z/p)$. Since the size of an irreducible representation of a group G can be of dimension at most the square root of G, it is rather striking that we obtain only representations of dimension the cube root of G. We remark that the representations which enter into the Randol examples are all from the group Z/p, and hence are only one-dimensional.

The high dimension of the irreducible representations would then seem to offer a striking ccontrast between these two cases. It remains to be seen that this contrast is reflected in the behavior of λ_1— we will see that that is the case in the next section.

We remark that a detailed discussion of the Kazhdan constants for the irreducible representations of $PSL(2, Z/p)$ is contained in **[B1]**. A crucial role there is played there by the Koosterman sums

$$S_\chi(a, b, p) = \sum_{y \not\equiv 0 (mod\ p)} \zeta^{(ay + by^{-1})},$$

for $\zeta = e^{2\pi i/p}$ and χ an even multiplicative character $(mod)p$. These Kloosterman sums, which also enter into Selberg's proof of his theorem via the Weil estimate

$$|S_\chi(a, b, p)| < 2\sqrt{p},$$

enter here as matrix coefficients for the irreducible representation associated to χ.

§4 Congruence subgroups

We may now complete our discussion in the introduction in the following way: Suppose S is a Riemann surface which has a group Γ of "interesting symmetries." From §3, we now know how to interpret the word "interesting" – the non-trivial irreducible representations of Γ should have large dimension relative to the order of Γ.

We then have the following:

THEOREM ([**B3**]). *The k-th eigenvalue $\lambda_k(S)$ satisfies:*

$$\left(\sqrt{1-4\lambda_k}\right) r(S) \leq \log\left(vol(S)k+1\right)+C,$$

where $r(S)$ is the injectivity radius of S, C is a constant, and the inequality is vacuous if $\lambda_k \geq 14$.

SKETCH OF PROOF:— Let f be a function in the span of all eigenfunctions of eigenvalue $\leq \lambda < 14$. We may lift f to a function in the hyperbolic plane H^2 which we also denote by f.

If now y is an arbitrary point in H^2, we may average f about y to obtain the function f^{av} defined by:

$$f^{av}(r) = \frac{1}{length(S(r))}\int_{S(r)} f(x)\ dx$$

where $S(r) = \{x : dist(x,y) = r\}$ is the hyperbolic circle about y of radius r.

If $f(y) > 0$, then one obtains the estimate

$$f^{av}(x) \geq f(y)S_\lambda(dist(x,y)),$$

which follows from the fact that the S_λ's are positive and decreasing in λ, for $\lambda < 14$.

If r is less than the injectivity radius of S, then we can carry out this calculation on S rather than in H^2, so that we have:

$$(\dagger)\ \int_S |f|^2 \int_{B(r,y)} |f(x)|^2 \geq \int_{B(r,y)} |f^{av}(x)|^2 \geq |f^{av}(y)|^2 \int_0^r |S_\lambda^2(r)|\left(length(r)\right)\ dr,$$

where the second inequality comes from the fact that averaging decreases L^2-norm, and where length(r) denotes the length of a circle of radius r in the hyperbolic plane.

We now have the following:

Lemma. *If F is a family of functions on S of dimension k, then there exists a function $f \in F$ satisfying:*

(i) $\int_S |f|^2 = 1$

(ii) *For some point $y \in S$,*

$$|f(y)|^2 \geq kvol(S) \ .$$

Proof:— If $f_1, \ldots, f_k$ is an orthonormal basis of F, then $\int f_1^2 + \cdots + f_k^2 = k$. Hence there is a point y such that $f_1^2(y) + \cdots + f_k^2(y) \geq kvol(S)$.

If we now set

$$h = f_1(y) \cdot f_1 + \cdots + f_k(y) \cdot f_k,$$

we may verify that $\int_S h^2 = h(y)$, so that

$$f(x) = [\frac{1}{\sqrt{h(y)}}]h(x)$$

then satisfies the conclusions of the lemma. □

The proof of the theorem now follows from the observation that $1 = \int_S |f|^2 \geq \int_{B(r,y)} |f|^2$ and the inequality (†). □

We now apply these considerations to the congruence subgroups. To that end, let Γ be a cocompact arithmetic subgroup of $PSL(2,R)$, and let Γ_p be the pth congruence subgroup. Then $\Gamma/\Gamma_p \cong PSL(2,Z/p)$, and furthermore $S_p = H^2/\Gamma_p$ is a finite covering of $S = H^2/\Gamma$.

It is to S_p that we will apply the above considerations.

First of all, we have that

$$vol(S_p) = vol(S) \cdot \#(PSL(2,Z/p)) \sim (const)p^3,$$

and that the eigenfunctions of Δ are of one of two types:

Either

(i) They are invariant under $PSL(2,Z/p)$, and so descend to functions on S (and hence are bounded below by $\lambda_1(S)$),

or

(ii) They lie in eigenspaces of dimension at least $p - 12$.

We may now apply the above theorem once we know an estimate for the injectivity radius of S_p. But the injectivity radius is just 12 the length of the shortest closed geodesic, and for $\gamma \in PSL(2,R)$, primitive in Γ, the length $\ell(\gamma)p$ of the geodesic fixed by γ is given by:

$$2\cosh(\ell(\gamma)2) = tr(\gamma).$$

To compute tr(γ), we reduce (mod p) to obtain

$$\gamma \equiv \pm (1\,001) \qquad (mod$$

p),$sothattr(\gamma) \equiv \pm 2 \pmod p$. Actually, it is an elementary computation that these condition force $\text{tr}(\gamma) \equiv \pm 2 \pmod{p^2}$. The case $\text{tr}(\gamma) = \pm 2$ is ruled out by the assumption that Γ_p is cocompact, since $\text{tr}(\gamma) = \pm 2$ implies that H^2/Γ_p has a cusp.

It follows that

$$\gamma(S_p) \sim \log(p^2).$$

Applying our estimate, we see that

$$(\sqrt{1 - 4\lambda_k}) 2 \log(p) \geq 3 \log(p) - \log(k) + (const),$$

which gives a lower bound for λ_k as soon as $k >$ (const) $p^{1+\epsilon}$. For instance, setting $\epsilon = 1$, we get the bound

$$\lambda_{(const)p^2}(S) \geq 316.$$

This falls a little short of our desired goal, which would be

$$\lambda_{(const)p} \geq (const),$$

because either $\lambda_1(S_p) = \lambda_1(S)$, or $\lambda_1(S_p) = \lambda_{p-12}(S_p)$.

This discussion was completed in an elegant manner by Sarnak and Xue [SX], see also Huxley [Hu]. We will here only give a rough sketch of what is involved.

First of all, we observe that the assertion of our theorem can be recast in terms of the Selberg trace formula. Let us recall the framework of this formula briefly.

If $k(r)$ is a rapidly decreasing function of r, we may regard k as a function on hyperbolic space by the formula

$$k(x, y) = k(dist(x, y)).$$

For Γ a cocompact subgroup of $PSL(2, R)$, set

$$K_\Gamma(x, y) = \sum_{\gamma \in \Gamma} k(\gamma x, y).$$

Then, for $S = H^2/\Gamma$,

$$\int_S K_\Gamma(x, x) dx = \sum_i \hat{k}(\gamma_j),$$

where $\hat{k}$ is the "Selberg transform" of k:

$$\hat{k}(\lambda) = \int_0^\infty k(r) S_\lambda(r) \; area(r) \; dr \; .$$

The considerations of our theorem now follow from taking

$$k(r) \equiv 1 \quad for \quad r < r(S), \equiv 0 \quad for \quad r > r(S).$$

We are now in a position to push beyond the injectivity radius of S! By a careful choice of k, one can essentially trade off having a few short geodesics with the growth of balls of a larger radius to obtain better bounds on λ_1. This together with elementary bounds for integer points in regions is sufficient to obtain a proof of Selberg's theorem for cocompact arithmetic groups, with a constant slightly worse than 316.

References

[B1] R. Brooks, "Combinatorial Problems in Spectral Geometry," in Curvature and Topology of Riemannian Manifolds, Springer Lecture Notes 1201 (1988), pp. 14-32.

[B2] R. Brooks, "The Fundamental Group and the Spectrum of the Laplacian," Comm. Math. Helv. 56 (1981), pp. 581-596.

[B3] R. Brooks, "Injectivity Radius and Low Eigenvalues of Hyperbolic Manifolds," I. Reine Ang. Math. 390 (1988), pp. 117-129.

[B4] R. Brooks, "On the Angles Between Certain Arithmetically Defined Subspaces of C^n," Ann. Inst. Fourier XXXVII (1987), pp. 175-185.

[B5] R. Brooks, "The Spectral Geometry of a Tower of Coverings," J. Diff. Geom. 23 (1986), pp. 97-107.

[B6] R. Brooks, "The Spectral Geometry of k-Regular Graphs," to appear.

[Bur1] M. Burger, "Petites Valeurs Propres du Laplacien et Topologie de Fell," These, Lausanne, 1986

[Bur2] M. Burger, "Spectre du Laplacien, Graphes, et Topologie de Fell," Comm. Math. Helv. 63 (1988), pp. 226-252

[BS] M. Burger and V. Schroeder, "Volume, Diameter, and the First Eigenvalue of Locally Symmetric Spaces of Rank One," J. Diff. Geom. 26 (1989), pp. 273-284.

[Bu] P. Buser, "A Note on the Isoperimetric Constant," Ann. Sci. Ec. Norm. Sup. 15 (1982), pp. 213-230.

[Ch] J. Cheeger, "A Lower Bound for the Smallest Eigenvalue of the Laplacian," in Gunning, *Problems in Analysis* (1970), pp. 195-199.

[Cr] C. Croke, "Some Isoperimetric Inequalities and Eigenvalue Estimates," Ann. Sci. Ec. Norm. Sup. 13 (1980), pp. 419-435.

[Dav] H. Davenport, "On Certain Exponential Sums," Jour. fur Math. 169 (1933), pp. 158-176.

[D] P. Diaconis, *Group Representations in Probability and Statistics*, Institute of Mathematical Statistics Lecture Notes (1988).

[F] E. Folner, "On Groups with Full Banach Mean Values," Math Scand. 3 (1955), pp. 243-254.

[GGP] Gelfand, Graev, and Pyatetskii-Shapiro, *Representation Theory and Automorphic Functions*, W.B. Saunders Ca, (1969).

[Gr] M. Gromov, "Manifolds of Negative Curvature," J. Diff. Geom. 13 (1978), pp. 223-230.

[Hu] M. N. Huxley, "Exceptional Eigenvalues and Congruence Subgroups," in Hejhal, Sarnak, and Terras, *The Selberg Trace Formula and Related Topics*, Contemp. Math 53 (1986), pp. 341-349.

[Kc] M. Kac, "Can One Hear the Shape of a DrumΓ" Amer. Math Monthly 73 (1966), pp. 1-23.

[Kz] D. Kazhdan, "Connections of the Dual Space of a Group with the Structure of its Closed Subgroups," Funct, Anal. App. (1968), pp. 63-65.

[Mc] H.P. McKean, "Selberg's Trace Formula as Applied to a Compact Riemann Surface," Comm. P. & Appl. Math 25 (1972), pp. 225-246.

[Ra] B. Randol, "Small Eigenvalues of the Laplace Operator on Compact Riemann Surfaces," Bull. Ams (1974), pp. 990-1000.

[SX] P. Sarnak and X. Xue, rough notes.

[Se] A. Selberg, "On the Estimation of Fourier Coefficients of Modular Forms," Proc. Symp. Pure Math, VIII (1965), pp. 1-15.

[Su] D. Sullivan, "Related Aspects of Positivity in Riemannian Geometry," J. Diff. Geom. 25 (1987), pp. 327-351.

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