the proof of Lemma 5(b) can be shortened by using stochastic calculus arguments in [15, Section IV.3] differing from those in this technical note.

REFERENCES


Global Robust Output Regulation by State Feedback for Strict Feedforward Systems
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Abstract—This note studies the global robust output regulation problem by state feedback for strict feedforward systems. By utilizing the general framework for tackling the output regulation problem [10], the output regulation problem is converted into a global robust stabilization problem for a class of feedforward systems that is subject to both time-varying static and dynamic uncertainties. Then the stabilization problem is solved by using a small gain based bottom-up recursive design procedure.

Index Terms—Nonlinear systems, output regulation, robust control.

I. INTRODUCTION

Output regulation problem of nonlinear systems has been one of the central control problems for nearly two decades [3], [6]–[16], [20], [21], [23], [24]. The research was first focused on the local version of the problem where all the initial conditions and uncertain parameters are assumed to be sufficiently small [3], [8], [11], [12], [14], [20]. The research on the nonlocal version of the problem started in the late 1990s [6], [7], [10], [13], [15], [16], [21], [23], [24]. It is now well known (see, e.g., [10]) that the robust output regulation problem can be approached in two steps. In the first step, the problem is converted into a robust stabilization problem of a so-called augmented system which consists of the original plant and a suitably defined dynamic system called an internal model candidate, and in the second step, the robust stabilization problem of the augmented system is further pursued. The success of the first step depends on whether or not an internal model candidate exists which can usually be ascertained by the property of the solution of the regulator equations. Even though the first step can be accomplished, the success of the second step is by no means guaranteed due to at least two obstacles. First, the stabilizability of the augmented system is dictated not only by the given plant but also by the particular internal model candidate employed. An internal model candidate can be chosen from an infinite set of dynamic systems and a suitable internal model candidate is usually obtained from the past experience and some trial and error. Second, the structure of the augmented system may be much more complex than that of the original plant. Therefore, even though the stabilization of the original plant with the exogenous signal set to 0 is solvable, the stabilization of the augmented system may still be intractable. Perhaps, it is because of these difficulties, so far almost all papers on semi-global or global robust output regulation problem are focused on the lower triangular systems [6], [10], [13], [24], feedback linearizable systems [15], [16], and output feedback systems [7], [23].

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In this note, we study the global robust output regulation problem by state feedback for the following strict feedforward systems
\[
\dot{x}_i = f_i(x_{-i}, \ldots, x_1, u, v, w), \quad i = n, \ldots, 2
\]
\[
\dot{x}_1 = cu + f_1(v, w)
\]
\[
e = x_1 - q_\delta(v, w)
\]
where \(x = (x_1, \ldots, x_n) \in \mathbb{R}^n\) is the state, \(u \in \mathbb{R}\) the control, \(e \in \mathbb{R}\) the tracking error, \(w \in \mathbb{R}^w\) the uncertain constant parameter, \(v \in \mathbb{R}^v\) the state of the exosystem
\[
v = Sv
\]
(2)
where all eigenvalues of the matrix \(S\) are simple with zero real parts, \(c\) is a known nonzero real number, and for \(i = 1, \ldots, n\), the functions \(f_i\) and \(q_\delta\) are smooth functions satisfying \(f_i(0, 0, \ldots, 0, w) = 0\) and \(q_\delta(0, w) = 0\) for all \(w \in \mathbb{R}^w\).

**Global Robust Output Regulation Problem (GRORP)**

For any compact set \(V_0 \subset \mathbb{R}^n\) with a known bound and any compact set \(W \subset \mathbb{R}^w\) with a known bound, design for system (1) a dynamic state feedback controller in the following form:
\[
u = \mathcal{K}(\eta, x, e), \quad \eta = \mathcal{F}(\eta, x, e)
\]
(3)
where \(\eta\) is the compensator state and \(\mathcal{K}, \mathcal{F}\) are locally Lipschitz functions vanishing at the origin, such that the closed-loop system composed of (1) and (3) has the following properties:

a) for all \(v(0) \in V_0, w \in W\) and for all initial state \(x(0), \eta(0)\), the trajectory of the closed-loop system exists and is bounded for all \(t \geq 0\);

b) the tracking error converges to zero as \(t\) tends to infinity, i.e.,
\[
lim_{t \to \infty} e(t) = 0.
\]

To our knowledge, the only papers that are relevant to the problem described above are [2] and [19]. An approximate and restricted tracking problem for a class of block feedforward systems was studied in [2] via dynamic output feedback control. The term approximate refers to the approximate regulation which is achieved by utilizing the \(k\)-fold internal model [8]. The term restricted refers to the fact that the state of the exosystem should be sufficiently small. In [19], the authors studied the input disturbance suppression problem (IDSP) via dynamic state feedback control for the following system:
\[
\dot{x}_i = w_{-i} x_{i-1} + g_i(x_{i-1}, \ldots, x_1, w), \quad i = n, \ldots, 2
\]
\[
\dot{x}_1 = u - g_1(v)
\]
(4)
where \(w = (w_1, \ldots, w_{n-1})\) is the uncertain constant parameter and the functions \(g_i, i = 2, \ldots, n, \) vanish at \(0, 0, \ldots, 0\). The goal of IDSP is to achieve property a) of GRORP and \(\lim_{t \to \infty} x(t) = 0\).

There is distinct difference between IDSP and GRORP. Roughly speaking (see Remark 3.3 for more specific comparison between IDSP and GRORP), for the IDSP, the internal model consists of only one dynamic system associated with the input \(u\). The IDSP of system (4) can be converted into a global robust stabilization problem for a class of feedforward systems subject to input unmodeled dynamics. Several results about this robust stabilization problem have been reported, see e.g., [11], [17], [18], [22]. In contrast, for the GRORP, the internal model in general consists of \(n\) dynamic systems associated with \(x_2, \ldots, x_n\) and the input \(u\), respectively. The GRORP of system (1) can be converted into a global robust stabilization problem for a class of feedforward systems subject to both time-varying static and dynamic uncertainties described by (5) below. The stabilization problem of a system of the form (5) is itself interesting and is worth an independent study.

In order to solve the GRORP of strict feedforward system (1), we have to overcome the difficulties mentioned above. First, we need to identify the structural properties of the functions \(f_i\) and \(g\) in (1) so that an internal model candidate exists. Then, by looking for a suitable internal model and performing appropriate transformations on the augmented system consisting of (1) and the internal model, we can convert the GRORP of system (1) to a global robust stabilization problem of the system
\[
\dot{x}_i = f_i(x_1, x_{i-1}, \ldots, x_1, \xi_1, \bar{d})
\]
\[
\dot{\xi}_i = g_i(x_1, x_{i-1}, \ldots, x_1, \xi_1, \bar{d}), \quad i = n, \ldots, 2
\]
\[
\dot{\xi}_1 = f_1(x_1, \bar{d})
\]
\[
\dot{\bar{d}} = \bar{g}(\xi_1, \bar{d}, d)
\]
(5)
where for \(i = 1, \ldots, n, \dot{x}_i \in \mathbb{R}, \xi_i \in \mathbb{R}^{\xi_i}, d \in \mathbb{R}^d, \bar{d} \in \mathbb{R}, \bar{f}_i, \bar{g}_i\) are smooth functions vanishing at \(0, 0, \ldots, 0\), for all \(d \in D, n_{\xi_i} \) and \(n_d\) are dimensions of \(\xi_i\) and \(d\) respectively. System (5) contains two types of uncertainties, i.e., time-varying static uncertainty represented by the external disturbance \(d\) where \(d : [0, \infty) \to D\) is a continuous function with its range \(D\) a compact subset having a known bound, and dynamic uncertainty represented by dynamics governing \(\xi_1, \xi_2, \ldots, \xi_n\). The dynamics governing \(\xi_1, \xi_2, \ldots, \xi_n\) is called dynamic uncertainty because \(\xi_1, \xi_2, \ldots, \xi_n\) are not allowed for feedback. The global robust stabilization problem of system (5) had not been studied until recently [4] in which, a bottom-up recursive design procedure is presented to deal with the problem. Two types of the small gain theorem with restrictions adapted from [25] is applied to establish the local stability and global attractiveness of the closed-loop system at the origin respectively.

The rest of the note is organized as follows. Section II presents some definitions and preliminary results, and the result of the global robust stabilization problem for system (5). The main result of this note is contained in Section III. In Section IV, an illustrative example is elaborated. Finally, Section V concludes this note.

Like [25], we let \((x_1, x_2)\) with \(x_i \in \mathbb{R}^{x_i}, i = 1, 2, \) denote the vector \((x_1^T, x_2^T)^T \in \mathbb{R}^{x_1} \times \mathbb{R}^{x_2}\), and let \(L_c^\infty\) be the set of all piecewise continuous functions \(u : [0, \infty) \to \mathbb{R}^2\) with a finite supremum norm \(||u||_\infty = \sup_{t \geq 0} ||u(t)||\) and let \(||u||_c = \liminf_{t \to \infty} ||u(t)||\) denote the asymptotic \(L_c^\infty\) norm of \(u\), where \(||\cdot||\) denotes the standard Euclidean norm. A function \(\gamma : \mathbb{R}_+ \to \mathbb{R}_+\) is called a gain function if it is continuous, nondecreasing, and satisfies \(\gamma(0) = 0\). Finally, let \(I_n\) be \(n\) dimensional identity matrix.

**II. PRELIMINARY**

We first review some terminologies in [5], [25] for nonlinear systems of the following form:
\[
\dot{x} = f(x, u, d), \quad y = h(x, u, d)
\]
(6)
where \(x \in \mathbb{R}^n\) is the plant state, \(y \in \mathbb{R}^m\) the output, \(u \in \mathbb{R}^m\) the piecewise continuous input, \(f(x, u, d)\) and \(h(x, u, d)\) are locally Lipschitz functions vanishing at \(0, 0, \ldots, 0\) for all \(d \in D, d : [0, \infty) \to D\) is a continuous function with its range \(D\) a compact subset of \(\mathbb{R}^d\). Let \(x(t)\) denote the solution of system (6) with initial state \(x(0), \) input \(u,\) and \(d\).

**Definition 2.1:** [5] The output \(y\) of system (6) is said to satisfy a robust \(L_c^\infty\) stability bound (RLBW) with restrictions \(X, \Delta, x(0), u\) and gains \(\gamma^0, \gamma\) respectively if there exist an open subset \(X \subset \mathbb{R}^n\) containing the origin, a positive real number \(\Delta\), gain functions \(\gamma^0, \gamma, \) all independent of \(d\), such that, for each \(x(0) \in X, d \in D, \) \(||u||_c < \Delta\), the solution of (6) exists for all \(t \geq 0\) and
\[
||u||_c \leq \max\{\gamma^0(||x(0)||), \gamma(||u||_c)\}
\]
(7)
Assumption 2.1: [25] The output $y$ of system (6) is said to satisfy a robust $\alpha$-$L_\infty$ stability bound (RAB) with restrictions $X, \Delta \subset \mathbb{R}^n$ on $u$ and gain $\gamma$, if there exist an open subset $X$ of $\mathbb{R}^n$ containing the origin, a non-negative real number $\Delta$, and a gain function $\gamma$, all independent of $d$, such that, for each $(x(0), d) \in X \times d \in \mathbb{D}$, $\|y\|_\infty < \Delta$.

$$\|y\|_\infty \leq \gamma(\|u\|_\infty).$$

(8)

The output $y$ of system (6) is said to satisfy a robust asymptotic bound (RAB) with restrictions $X, \Delta \subset \mathbb{R}^n$ on $u$ and gain $\gamma$, if there exist an open subset $X$ of $\mathbb{R}^n$ containing the origin, a non-negative real number $\Delta$, and a gain function $\gamma$, all independent of $d$, such that, for each $(x(0), d) \in X \times d \in \mathbb{D}$ and piecewise continuous $u$ satisfying $\|u\|_\infty \leq \Delta$, the solution of (6) exists for all $t \geq 0$ and

$$\|y\|_\infty \leq \gamma(\|u\|_\infty).$$

(9)

Remark 2.1: In both Definitions 2.1 and 2.2, the word "robust" is used to emphasize that the inequalities (7)-(9) hold regardless of the presence of the disturbance $d$ in (6). For convenience, we will simply use LB, AB and a-LB to mean RLB, RAB and Ra-LB respectively. The combination of LB and AB can be used to study the asymptotic stability of system (6) with $u = 0$. More specifically, if the state $x$ of system (6) satisfies LB and AB with restrictions $X_\alpha$ and $x_\alpha$ on $x(0)$ respectively, then the equilibrium point $x = 0$ of system (6) with $u = 0$ is locally asymptotically stable and if, in addition, $X = \mathbb{R}^n$, then it is globally asymptotically stable. As for the relationship between a-LB and the combination of LB and AB, we refer the reader to [5].

For simplicity, if the output $y$ of system (6) satisfies LB with restriction on $x(0)$, restriction $\Delta$ on $u$ and gain $\gamma$, and satisfies AB with no restriction on $x(0)$, restriction $\Delta$ on $u$ and gain $\gamma$, then we will say $y$ satisfies LB with restriction AB with no restriction on $x(0)$, both with restriction $\Delta$ on $u$ and gain $\gamma$.

Like [1], [25], our approach will utilize saturation functions characterized as follows.

Definition 2.3: A locally Lipschitz function $\sigma(\cdot): \mathbb{R} \rightarrow [-\lambda, \lambda]$ is said to be a saturation function with saturation level $\lambda > 0$, if $\sigma(s) = s$ when $|s| \leq \lambda/2$, and $\lambda/2 \leq \text{sgn}(s)\sigma(s) \leq \min\{|s|, \lambda\}$ when $|s| \geq \lambda/2$.

In the rest of this section, we restate the stabilization result obtained in [4] for system (5). For this purpose, we first make two assumptions on system (5).

Assumption 2.1: For $i = 1, \ldots, n$, $D_i$ is a constant matrix and $\mu_i = c_i - D_i^2A^2_ii$ is a positive (or alternatively negative) constant.

Assumption 2.2: $\xi_i$ satisfies LB and AB with no restriction on $\xi_i(0)$, both with restriction $\Delta_i$ on $\bar{u}$ and gain $\bar{N}_i$, and for $i = 2, \ldots, n$, $\xi_i$ satisfies LB and AB with no restriction on $\xi_i(0)$, both with restriction $\Delta_i$ on $\bar{x}_{i-1}$, $\bar{x}_i$, $\bar{\xi}_{i-1}$, $\bar{\xi}_i$, $\bar{\xi}_i$, $\bar{d}$.

$$\begin{align*}
\ddot{x}_i &= D_i\xi_i + c_i\bar{x}_{i-1} + f_i^e(\xi_i, \bar{x}_{i-1}, \ldots, \bar{x}_i, \xi_i, \bar{u}, d) \\
\ddot{\xi}_i &= A_i\xi_i + B_i\bar{x}_{i-1} + \bar{f}_i^e(\xi_i, \bar{x}_{i-1}, \ldots, \bar{x}_i, \xi_i, \bar{u}, d), \\
i &= n, \ldots, 2
\end{align*}$$

(10)

where $f_i^e, \bar{f}_i^e$ are suitably defined smooth functions.

Theorem 2.1: [4] Consider system (5). Under Assumptions 2.1-2.2, there exist $\lambda_i > 0$ and nonzero $k_i$ with the same sign as $\theta_i$, where $\theta_i = \mu_i/\lambda_i - k_i$, $i = 2, \ldots, n$, such that, under the control

$$\ddot{u}_i = -\sigma_i(k_i\bar{x}_i + \sigma_i(k_i\bar{N}_i)),$$

(11)

where for $i = 1, \ldots, n$, $\sigma_i$ is a saturation function with level $\lambda_i$, the closed-loop system at $(0, \ldots, 0)$ is globally asymptotically stable for all $d \in \mathbb{D}$. III. MAIN RESULT

In this section, we will first give conditions under which the GRORP of system (1) can be converted into a global robust stabilization problem of a well defined augmented system (19) which takes the form (5) satisfying all assumptions of Theorem 2.1. Thus, we can further conclude the solvability of the GRORP of system (1) by Theorem 2.1. The first step of our approach is to find an appropriate internal model. For this purpose, we make the following assumptions.

Assumption 3.1: There exist smooth functions $\mathbf{x}(v, w) = (x_1(v, w), \ldots, x_n(v, w))$ and $u(v, w)$ with $x(0, 0) = 0$ and $u(0, 0) = 0$ satisfying for all $v \in \mathbb{R}^n, w \in \mathbb{R}_w$

$$x_i(v, w) = f_i(x_{i-1}(v, w), \ldots, x_1(v, w), u(v, w), v, w),$$

(12)

$$i = n, \ldots, 2$$

$$x_1(v, w) = cu(v, w) + f_1(v, w),$$

$$x_1(v, w) = q_1(v, w).$$

Assumption 3.2: Let $\pi_i(v, w) = u(v, w), \pi_i(v, w) = x_i(v, w), i = 2, \ldots, n$. For each $1 \leq i \leq n$ such that $\pi_i(v, w)$ is not identically zero, there exist sufficiently smooth functions $\tau_i: \mathbb{R}^n \times \mathbb{R}^w \rightarrow \mathbb{R}_i$, $i = 1, \ldots, n$, vanishing at $(0, 0)$, matrix $\Phi_i$, and column vector $\Psi_i$, such that

$$\pi_i(v, w) = \Phi_i\pi_i(v, w).$$

(13)

where the pair $(\Phi_i, \Psi_i)$ is observable and all eigenvalues of $\Phi_i$ are simple with zero real parts

Remark 3.1: Equation (12) is called regulator equations and the solvability of these equations is necessary but not sufficient for the solvability of the robust output regulation problem [3], [9], [10], [14]. Assumption 3.2 is made for the existence of appropriate linear internal models. Both Assumption 3.1 and 3.2 are quite standard in literature. In particular, if $x(v, w)$ and $u(v, w)$ are polynomial in $v$, Assumption 3.2 is satisfied automatically [9], [10]. Under Assumption 3.2, for each $1 \leq i \leq n$ such that $\pi_i(v, w)$ is not identically zero, given a pair of controllable matrices $(M_i, N_i)$ with $M_i \in \mathbb{R}^{i \times i}$ Hurwitz and $N_i \in \mathbb{R}^{i \times i}$, there exists a unique and nonsingular matrix $T_i \in \mathbb{R}^{i \times i}$ satisfying the Sylvester equation

$$T_i\Phi_i - M_iT_i = N_i\Psi_i,$$

(14)

since the spectra of $M_i$ and $\Phi_i$ are disjoint and the pair $(\Psi_i, \Phi_i)$ is observable.

We define the following system:

$$\begin{align*}
\dot{u}_i &= M_i\eta_i + N_iu - M_iN_i\Psi_i \\
\dot{u}_i &= M_i\eta_i + N_ix_i, \\
i &= 2, \ldots, n
\end{align*}$$

(15)

as the internal model of (1) with output $(u, x_2, \ldots, x_n)$. Note that the dimension of $\eta_i$ is understood to be zero if $\pi_i(v, w)$ is identically zero.

Next, we will convert the GRORP for system (1) into a global robust stabilization problem for the augmented system composed of the original plant (1) and the internal model (15). Performing the following coordinate and input transformation

$$\bar{\xi}_i = x_i - x_i(v, w) = \xi_i + \bar{N}_i\bar{\xi}_i,$$
\[
\tilde{x}_i = x_i - \Psi T_{i-1}^{-1} \eta_i, \quad i = 2, \ldots, n
\]
\[
\tilde{\eta}_i = \eta_i - T_i \tau_i, \quad i = 1, \ldots, n
\]
\[
\tilde{u} = u - \Psi T_{i-1}^{-1} \eta_i
\]  
(16)

on the augmented system gives
\[
\tilde{x}_i = -\Psi T_{i-1}^{-1} \left[ (M_i + N_i \Psi T_{i-1}^{-1}) \tilde{\eta}_i + N_i \tilde{x}_i \right] + f_i(\tilde{x}_{i-1}, \tilde{\eta}_{i-1}, \tilde{u}, v, w)
\]
\[
\tilde{\eta}_i = (M_i + N_i \Psi T_{i-1}^{-1}) \tilde{\eta}_i + N_i \tilde{x}_i, \quad i = n, \ldots, 2
\]
\[
\tilde{x}_1 = e \Psi T_{1-1}^{-1} \eta_1 + c \tilde{u}
\]
\[
\tilde{\eta}_1 = (M_1 + N_1 \Psi T_{1-1}^{-1}) \tilde{\eta}_1 + N_1 \tilde{x}_1 - \frac{M_1 N_1 e}{c}
\]  
(17)

where
\[
f_2(\tilde{x}_1, \tilde{\eta}_1, \tilde{u}, v, w) = -f_2(x_1, u, v, w) + f_2(\tilde{x}_1 + x_1, \tilde{u} + \Psi T_{1-1}^{-1} \tilde{\eta}_1 + u, v, w)
\]

and
\[
f_i(\tilde{x}_{i-1}, \tilde{\eta}_{i-1}, \ldots, \tilde{x}_1, \tilde{\eta}_1, \tilde{u}, v, w)
\]
\[
= -f_2(x_{i-1}, u, v, w) + f_2(\tilde{x}_{i-1} + x_{i-1}, \tilde{u} + \Psi T_{i-1}^{-1} \tilde{\eta}_{i-1} + u, v, w)
\]
\[
+ f_i(\tilde{x}_{i-1} + \Psi T_{i-1}^{-1} \tilde{x}_{i-1} + x_{i-1}, \ldots, \tilde{x}_1 + x_1, \tilde{u} + \Psi T_{1-1}^{-1} \tilde{\eta}_1 + u, v, w), \quad i = 3, \ldots, n
\]

It is known from [9] and [10] that the GRORP of system (1) will be solved if we can make the equilibrium point \((\tilde{x}, \tilde{\eta}) = (0, 0)\) of system (17) globally asymptotically stable for any trajectories \(v(t)\) starting from \(V_0\) and any \(w \in W\), where \(\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)\) and \(\tilde{\eta} = (\tilde{\eta}_1, \ldots, \tilde{\eta}_n)\). A system of the form (17) has never been encountered and there is no clue whether or not this system is stabilizable at the equilibrium point. Nevertheless, by performing some further coordinate and input transformations on (17), it is possible to convert (17) to the form of (5) with all desirable properties. For this purpose, we introduce two more assumptions.

Assumption 3.3: For each \(2 \leq i \leq n\) such that \(\pi_i(v, w)\) is not identically zero, \(\Psi_i\) is invertible.

Assumption 3.4: For \(i = 2, \ldots, n\)
\[
\frac{\partial f_i}{\partial x_{i-1}^{-1}} \left[ (x_{i-1}, \ldots, x_i) - (x_{i-1}, \ldots, x_i, u, v) \right]
\]

is a positive (or alternatively negative) constant.

Now define the following coordinate and input transformation:
\[
\xi_1 = c \tilde{\eta}_1 - N_1 \tilde{x}_1
\]
\[
\xi_i = (M_i + N_i \Psi T_{i-1}^{-1}) \tilde{\eta}_i + N_i \tilde{x}_i, \quad i = 2, \ldots, n
\]
\[
\tilde{u} = \tilde{u} + \frac{\Psi T_{1-1}^{-1} N_1 e}{c}
\]  
(18)

From (14), \(M_i + N_i \Psi T_{i-1}^{-1} = T_i \Phi_i T_{i-1}^{-1}\), and then from Assumption 3.3 and \(c \neq 0\), the transformation (18) is globally invertible. Performing the transformation (18) on (17) yields
\[
\tilde{x}_i = -\Psi T_{i-1}^{-1} \xi_i + f_i(\tilde{x}_{i-1}, \xi_{i-1}, \ldots, \tilde{x}_1, \xi_1, \tilde{u}, d)
\]
\[
\tilde{\eta}_i = M_i \xi_i + N_i \tilde{f}_i(\tilde{x}_{i-1}, \xi_{i-1}, \ldots, \tilde{x}_1, \xi_1, \tilde{u}, d), \quad i = n, \ldots, 2
\]
\[
\tilde{x}_1 = \Psi T_{1-1}^{-1} \xi_1 + c \tilde{u}
\]
\[
\xi_1 = M_1 \xi_1
\]  
(19)

where \(d = (v, w)\)
\[
f_2(\tilde{x}_1, \xi_1, \tilde{u}, d) = -f_2(x_1, u, d) + f_2(\tilde{x}_1 + x_1, \tilde{u} + \frac{\Psi T_{1-1}^{-1} \xi_1}{c} + u, d)
\]

and
\[
f_i(\tilde{x}_{i-1}, \xi_{i-1}, \ldots, \tilde{x}_1, \xi_1, \tilde{u}, d)
\]
Then noting the form of (21) and Assumption 3.4 shows that Assumption 2.1 is satisfied.

Next, we show that system (19) also satisfies Assumption 2.2. For $i = 1$, the specific form of the last equation of (19) immediately implies that $\xi_1$ satisfies Assumption 2.2 with $\Delta_1 = 0$ and $\Delta_1 = \infty$. For $i = 2, \ldots, n$, let $\xi_i = \hat{f}_i(\hat{x}_{i-1}, \xi_{i-1}, \ldots, \hat{x}_1, \hat{\xi}_1, \hat{\xi}_2, d)$. Then $\xi_i$ subsystem in (19) is rewritten as $\xi_i = M_i \xi_i + N_i d$, where $M_i$ is Hurwitz, $\xi_i$ satisfies a-LB with no restriction on $\xi_i(0)$, no restriction on $\hat{a}_i$ and linear gain $\hat{a}_i$, and there exist constants $\bar{\xi}_i, \hat{\xi}_i$ independent of $d$ such that $||\hat{\xi}_i|| \leq \bar{\xi}_i ||(\hat{x}_{i-1}, \xi_{i-1}, \ldots, \hat{x}_1, \hat{\xi}_1, \hat{\xi}_2, d)|| \leq \hat{\xi}_i$ and $d \in \mathbb{D}$. Thus, Assumption 2.2 is satisfied with $\hat{N}_i = \bar{\xi}_i \hat{\xi}_i$, and $\Delta_i = \hat{\xi}_i, i = 2, \ldots, n$.

By Theorem 2.1, there exist $\lambda_i > 0$ and nonzero $k_i$ such that the following control
\[
\hat{a} = -\sigma_1 (k_1 \hat{x}_1 + \sigma_2 (k_2 \hat{x}_2 + \cdots + \sigma_n (k_n \hat{x}_n)))
\] (22)
can globally asymptotically stabilize the origin of system (19) for all $d \in \mathbb{D}$. Noting (15), (16), (18) and (22) yields the controller (20), which solves the GRORP of system (1).

For the class of strict feedforward systems which only involve polynomial non-linearities, Assumptions 3.1 to 3.3 can be easily testified. To this end, let $v^{[1]} = v = (v_1, v_2, \ldots, v_k) \in \mathbb{R}^k$ and for $\ell \geq 2$
\[
v^{[\ell]} = (v^{[1]}, v^{[-1]}_1, v^{[-2]}_1, v^{[-1]}_2, v^{[-2]}_2, \ldots, v^{[-1]}_{\ell-1}, v^{[-2]}_{\ell-1}),
\] (23)
Then there exists an odd polynomial $x(v, w)$ in $v$ such that $\hat{x}(v, w) = f(v, w)$ for all trajectories $v(t)$ of the exosystem and $w \in \mathbb{R}^w$. Moreover, there exist an integer $r$ and $\Phi \in \mathbb{R}^{r \times r}, \Psi \in \mathbb{R}^{r \times \mathbb{R}}$, where $\Phi$ is nonsingular with all its eigenvalues simple and on the imaginary axis and the pair $(\Phi, \Psi)$ is observable. Moreover, since the characteristic polynomial of $\Phi$ is the minimal polynomial of $S^{[2+i]}$, $\Phi$ is nonsingular, and all its eigenvalues are simple and on the imaginary axis. Thus, Assumption 3.1 to 3.3 are satisfied if $f_1(v, w)$ is an odd polynomial in $v$ and for $i = 2, \ldots, n, f_i(v, w)$ is an odd polynomial in $(x_{i-1}, \ldots, x_1, u, v, w)$. (24)

Remark 3.2: If $q(v, w)$ is an odd polynomial in $v$, then by Proposition 3.1, it can be concluded that, Assumptions 3.1 to 3.3 are satisfied if $f_1(v, w)$ is an odd polynomial in $v$ and for $i = 2, \ldots, n, f_i(v, w)$ is an odd polynomial in $(x_{i-1}, \ldots, x_1, u, v, w)$. (25)

Remark 3.3: When $q(v, w) = 0$ and $f_i, i = 2, \ldots, n$, in (1) are independent of $u$ and vanish at $(0, \ldots, 0, w), v), \Phi$ is a GRORP of system (1) reduces to the IDSP studied in [19]. For this special case, $u(v, w) = -f_1(v, w)/c(x(v, w)) = 0$ and thus Assumption 3.1 is satisfied automatically. Moreover, since $x(v, w) = 0$, there is no need to estimate $x(v, w)$. It suffices to use one single system $\hat{y}_1 = M_1 \hat{y}_1 + N_1 u - M_1 \hat{y}_1/c$ to define the internal model which essentially reduces to the same case as what has been done in [19]. Also, Assumption 3.3 is not needed anymore and thus Assumption 3.2 with $i = 1$ and Assumption 3.4 become the assumptions to the IDSP of system (1). The IDSP of system (1) can be converted into a global robust stabilization problem for a class of feedforward systems with input unmodeled dynamics. On the other hand, when $q(v, w) \neq 0, x(v, w) \neq 0$ in general. To estimate $u(v, w)$ and $x(v, w)$, we define the internal model (15). If Assumptions 3.1-3.4 are satisfied, then the GRORP of system (1) can be solved by converting it into a global robust stabilization problem for a class of feedforward systems with both time-varying static and dynamic uncertainties. Thus, there is distinct difference between IDSP and GRORP.

IV. AN ILLUSTRATIVE EXAMPLE

We study the global robust output regulation problem of the following system:
\[
\dot{x}_2 = (1 + 0.05 w v^3_1) x_2 + 0.05 x_1 u + w (v_1 - v_1^3)
\]
\[
\dot{x}_1 = 10 u + 7 w v^2_1, \quad \dot{v}_1 = -v_2, \quad v_2 = v_1 \quad e = x_1 - w v^3_1
\] (25)
where for illustration, we assume $|w| \leq 1$ is the uncertain constant parameter and $||v(0)|| \leq 1$.

System (25) is in the form of (1). Let us verify that (25) satisfies Assumptions 3.1 to 3.4. Firstly, the regulator equations of (25) have a global solution as follows:
\[
x_1(v, w) = w v^3_1, \quad x_2(v, w) = w v^2_1, \quad u(v, w) = -w v^2_1 \quad (26)
\]
which implies that Assumption 3.2 is satisfied. Simple calculation shows that
\[
\Phi_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \Phi_2 = \begin{bmatrix} 0 & 1 \\ -9 & -10 \end{bmatrix}
\]
\[
\Psi_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \Psi_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}
\] (27)
From (25) and $\partial f_2/\partial x_1 = (x_1, -w (x_2(v, w), u, v, w)) = 1$, Assumptions 3.3 and 3.4 are satisfied. Thus, Theorem 3.1 can be applied to solve the global robust output regulation problem of system (25).
To design the internal model, let

\[
M_1 = \begin{bmatrix}
-4 & 0 & 0 \\
0 & -3 & 0 \\
0 & 0 & -2 \\
0 & 0 & 0
\end{bmatrix}, \quad M_2 = \begin{bmatrix}
-2 & 0 \\
0 & -1
\end{bmatrix},
\]

\[
N_1 = [1 \ 1 \ 1]^T, \quad N_2 = [1 \ 1]^T.
\]

Solving the Sylvester equation (14) gives

\[
T_1 = \begin{bmatrix}
0.2447 & -0.0612 & 0.0094 \\
0.3167 & -0.1056 & 0.1617 \\
0.4308 & -0.2154 & 0.0308 \\
0.5500 & -0.5500 & 0.0500
\end{bmatrix},
\]

\[
T_2 = \begin{bmatrix}
0.4 & -0.2 \\
0.5 & -0.5
\end{bmatrix}.
\]

Then the internal model takes the following form:

\[
\dot{\eta}_1 = M_1 \eta_1 + N_1 u - 0.1 M_1 N_1 \epsilon, \quad \dot{\eta}_2 = M_2 \eta_2 + N_2 x_2.
\]

Using the coordinate and input transformations (16) and (18), the augmented system consisting of (25) and (28) is put into the following form

\[
\dot{x}_2 = -\Psi_2 T_2^{-1} \xi_2 + \dot{x}_2 + 0.05(\dot{x}_1 + w v^2)(\bar{\sigma} + 0.1 \Psi_1 T_1^{-1} \xi_1)
\]

\[
\dot{\xi}_2 = M_2 \xi_2 + N_2 \dot{x}_2 + 0.05 N_2 (\dot{x}_1 + w v^2)(\bar{\sigma} + 0.1 \Psi_1 T_1^{-1} \xi_1)
\]

\[
\dot{x}_1 = \Psi_1 T_1^{-1} \xi_1 + 10 \bar{\sigma}
\]

\[
\dot{\xi}_1 = M_1 \xi_1
\]

In the following, Theorem 2.1 will be used to design the stabilizing control \( \bar{\sigma} = -\sigma_1 (k_1 \dot{x}_1 + \sigma_2 k_2 \dot{x}_2) \) for system (29). The design procedure follows the proof of [4, Theor. 3.1].

Performing the coordinate transformation

\[
z_1 = \dot{x}_1 - \Psi_1 T_1^{-1} M_2^{-1} \xi_1, \quad z_2 = \dot{x}_2 + \Psi_2 T_2^{-1} M_2^{-1} \xi_2 + \frac{\theta_2}{\bar{\theta}_1} z_1
\]

on (29) gives

\[
z_2 = \theta_2 \bar{\sigma} + 0.025(\dot{x}_1 + w v^2)(\bar{\sigma} + 0.1 \Psi_1 T_1^{-1} \xi_1) + \theta_2 k_1 \dot{x}_1
\]

\[
z_2 = M_2 \xi_2 + N_2 \dot{x}_2 + 0.05 N_2 (\dot{x}_1 + w v^2)(\bar{\sigma} + 0.1 \Psi_1 T_1^{-1} \xi_1)
\]

\[
\dot{\xi}_1 = \bar{\sigma} - \dot{x}_1
\]

\[
\dot{\xi}_1 = M_1 \xi_1
\]

which implies

\[
(\dot{x}_1 + w v^2)(\bar{\sigma} + 0.1 \Psi_1 T_1^{-1} \xi_1)
\]

\[
\leq 0.50005[|x_1| + 1.5|\dot{x}_1| + (50[\Psi_1 T_1^{-1}]|] + 0.1[|\Psi_1 T_1^{-1}]||\xi_1||
\]

so that for \(|\dot{x}_1| \leq 1, |\dot{\sigma}| \leq 1 \text{ and } |\xi_1| \leq 1 \text{ Then we have}

\[
|z_2| \leq 0.1260|\dot{x}_1| + 0.0755|\bar{\sigma}| + 35.622|\xi_1|
\]

for \(|\dot{x}_1| \leq 1, |\bar{\sigma}| \leq 1 \text{ and } |\xi_1| \leq 1 \text{ Thus, } \xi_2 \text{ satisfies LB with restriction and AB with no restriction on } \xi_2(0), \text{ both with restriction min}\{\lambda_1/3, k_1/6, 1/6\} \text{ on } u_1 \text{ and gain } N_{AB} s, \text{ where } N_{AB} = 2(1.026 \times 6/k_1 + 0.075 \times 6).

Now let \(\zeta_1 = (\xi_2, z_1, \xi_1) \). Then (30) can be written in the form

\[
z_2 = \theta_2 \dot{u}_1 + \tilde{F}_2(\zeta_1, u_1, d)
\]

\[
\zeta_1 = G_1(\zeta_1, u_1, d)
\]

where

\[
\tilde{F}_2(\zeta_1, u_1, d) = 0.025(\dot{x}_1 + w v^2)(\bar{\sigma} + 0.1 \Psi_1 T_1^{-1} \xi_1) + \theta_2 h(\dot{x}_1, u_1)
\]

\[
h(\dot{x}_1, u_1) = k_1 \dot{x}_1 - u_1 - \sigma_1 (k_1 \dot{x}_1 - u_1), \quad \text{and } G_1 \text{ is a suitably defined function.}
\]

Let \(u_1 = -\sigma_2 (k_2 \dot{x}_2 - k_2 \Psi_2 T_2^{-1} M_2^{-1} \xi_2 - k_2 \theta_2 \z_1/\theta_1) \). Clearly, \(\tilde{F}_1(\zeta_1, u_1) \text{ has no contribution to } \tilde{F}_2(\zeta_1, u_1, d) \) when \(\omega < \text{n}\{k_1/12, k_1/6, 1/6\} \), then from (31), we obtain

\[
\tilde{F}_2(\zeta_1, u_1, d) \leq 0.1 k_1
\]

\[
\left\{ \begin{array}{l}
6 \min\{k_2, \lambda_2\} + 0.50005 \times 36 \lambda_2^2 + 18 \min\{k_2, \lambda_2\}^2 \\
\end{array} \right.
\]

Solving max\{\(\tilde{F}_1(\zeta_1, u_1)\), \(\tilde{F}_2(\zeta_1, u_1, d)\)\} < \(s \) for \(s > 0 \), we choose \(k_1 = 0.2, k_2 = 0.00041, \lambda_1 = 10 \) and \(\lambda_2 = 0.0049 \).

As a result, the global robust output regulation problem of system (25) is solved by the following dynamic state feedback control:

\[
u = \Psi_1 T_1^{-1} \eta_1 - 0.1 \lambda_1 \epsilon
\]

\[
-\sigma_2 (0.2 \epsilon + \sigma_2 (0.00041 (x_2 - \Psi_2 T_2^{-1} \eta_2))
\]

\[
\eta_1 = M_1 \eta_1 + N_1 u - 0.1 M_1 \epsilon, \quad \eta_2 = M_2 \eta_2 + N_2 x_2 \quad \text{on state variables}
\]

\[
\eta_1 = M_1 \eta_1 + N_1 u - 0.1 M_1 \epsilon, \quad \eta_2 = M_2 \eta_2 + N_2 x_2
\]

where \(\sigma_1, \sigma_2 \) are saturation functions with level 10 and 0.0049, respectively.

For illustration, Figs. 1 and 2 show the simulation result of system (25) under the control (32) with initial state

\[(x_1(0), x_2(0), v(0), \eta_1(0), \eta_2(0))\]

\[(5, -0.7, (0.5, -0.6), (0.5, 1, 1.5, 1), (5, 5)) \]

and \(w = 0.5 \).

V. CONCLUSION

In this note, we have presented the solvability conditions of the GRORP by state feedback for strict feedback systems. The problem is approached in two steps. In the first step, the GRORP of the system is converted into a global robust stabilization problem of an augmented system. In the second step, the stabilization problem of the augmented system is further addressed. For the success of the first step, a suitable internal model and appropriate transformations have to be found so that the augmented system takes a suitable form and is stabilizable. For the success of the second step, we need to globally robustly stabilize a class of feedback systems subject to both time-varying static
and dynamic uncertainties, which is solved by using the bottom-up recursive design procedure recently developed in [4].

REFERENCES