I. INTRODUCTION

The problem of constructing control algorithms that allow us to handle time delays is an important issue in system theory. It is a key issue in process control. In fact, time-delay systems can be used to model a large number of phenomena occurring, for example, in engineering [10], [11], [17], and biology [20].

Dead-time compensation methods are available in the literature for linear systems modeled by a transfer function. These methods have been motivated by the pioneering work in [18], in which the well-known Smith predictor is developed. The control of linear time-delay systems represented in state-space form has also been addressed since the early 1970’s by considering the input–output decoupling problem [9], [15], [19]. In the same manner, the disturbance decoupling problem (DDP) for time-delay systems has been treated using the geometric approach [16], [21]. At the same time, the stability and stabilization of linear time-delay systems has been studied using Lyapunov-like functions [5], [6], [12].

Only very recently has the control of time-delay nonlinear systems been considered. In [7] and [11], a Smith predictor is used to control nonlinear chemical processes with an input time delay. Despite these contributions, a general theory for nonlinear time-delay systems remains to be developed.

In this paper, the DDP for a class of single-input–single-output (SISO) nonlinear systems with multiple delays in the input and the state is tackled. The contributions of the paper are as follows:

- pioneering mathematical frame for time-delay nonlinear systems whose viability is shown for solving control problems, such as disturbance decoupling;
- classification of various causal compensators (static and dynamic);
- necessary and sufficient conditions for the solvability of the DDP by so-called bicausal compensators;
- introduction of geometric concepts that are claimed to embody the insight of the structure of time-delay nonlinear systems.

This paper is organized as follows. The notations and basic assumptions on the time-delay systems under investigation are stated in Section II as well as some preliminary definitions related to their inherent structure. Section III is devoted to the classification of various causal compensators. In Section IV, the DDP is stated. In Section V (respectively, Section VI), pure shift dynamic solutions (respectively, dynamic solutions) are given to the problem. Finally, some conclusions are presented in Section VII.

II. NOTATIONS AND PRELIMINARY DEFINITIONS

We consider SISO nonlinear time-delay systems described by

$$\begin{align*}
\dot{x}(t) &= f(x(t-\tau), \tau \in \mathbb{R}_+ \\
\Sigma: \quad &+ \sum_{i=0}^{n}\gamma_i (x(t-\tau_i), \tau \in \mathbb{R}_+) u(t-\tau_i) \\
y(t) &= h(x(t-\tau), \tau \in \mathbb{R}_+) \\
x(t_0 + \phi) &= \varphi(\phi), \quad u(t) = u_0, \quad \forall t \in [t_0 - \tau, t_0]
\end{align*}$$

where only a finite number of variables \(\tau, \tau_0, \tau_1, \ldots, \tau_m \in \mathbb{R}_+\), the fixed time delays, occur, with \(\tau_0 = 0\). The state \(x \in \mathbb{R}^r\), the input \(u\), and the output \(y \in \mathbb{R}\). The entries of \(f\) and \(\gamma_i\) are meromorphic functions of their arguments. The notation \(f(x(t-\tau), \tau \in \mathbb{R}_+)\) stands for the existence of a bicausal feedback that solves the DDP. Sufficient conditions on the time-delay systems under investigation are stated in Section VI.
for \( f(x(t), x(t - \tau_1), \ldots, x(t - \tau_m)) \), \( \phi \) is a continuous function of initial conditions, where \( \tau^* = \max \{\tau \in \mathbb{R}_+\} \) and \( -\tau \leq \phi \leq 0 \).

Let \( \mathcal{K} \) be the field of meromorphic functions of a finite number of variables in

\[
\{x(t - \tau), u^{(k)}(t - \tau), \tau \in \mathbb{R}_+, k \in \mathbb{N}\}.
\]

These variables are independent in the sense that they are not related by any equation, except differential/difference equations. Let also \( \mathcal{E} \) be the formal vector space over \( \mathcal{K} \) given by

\[
\mathcal{E} = \text{span}_{\mathcal{K}} \{d\xi | \xi \in \mathcal{K}\}.
\]

The notion of relative degree plays a key role in solving some control problems associated with delay-free nonlinear systems (e.g., [8] and [14]). For this kind of system, the relative degree is nothing, but the order of time differentiation that has to be applied to the output to have explicit dependence on the input. Time-delay systems are, on the other hand, subject to two operators, the time differentiation and the time-delay shift, which introduces the definition of a nonnegative integer, the relative degree, and a nonnegative real, the relative shift. These definitions are formalized as follows.

**Definition 1:** The time-delay system \( \Sigma \) is said to have a relative degree \( \rho \) if a nonnegative integer \( \rho \) exists such that

\[
\rho = \min \left\{ k \in \mathbb{N} \; \exists \tau \in \mathbb{R}_+, \frac{\partial y^{(k)}(t)}{\partial u(t - \tau)} \neq 0 \right\}.
\]  

(1)

If, for all \((k, \tau) \in \mathbb{N} \times \mathbb{R}_+, \frac{\partial y^{(k)}(t)}{\partial u(t - \tau)} = 0\), we set \( \rho = \infty \).

The following result is a straightforward consequence of Definition 1.

**Proposition 2:** If system \( \Sigma \) has a finite relative degree \( \rho \), then

\[
\rho = \dim_{\mathcal{K}} \left( \text{span}_{\mathcal{K}} \{dy^{(k)}(t), k \in \mathbb{N}\} \cap \text{span}_{\mathcal{K}} \{dx(t - \tau), \tau \in \mathbb{R}_+\} \right).
\]

**Definition 3:** Assume that system \( \Sigma \) has a finite relative degree \( \rho \). Then, this time-delay system is said to have a relative shift \( \mu \) given by

\[
\mu = \min \left\{ \tau \in \mathbb{R}_+ | \frac{\partial y^{(\rho)}(t + \tau)}{\partial u(t)} \neq 0 \right\}.
\]

III. **Nonlinear Compensators for Time-Delay Systems**

A single operator acts on systems without delay, the differentiation with respect to time. Standard terminology distinguishes static-state feedbacks as a special case of general dynamic compensators. Systems with delays are described by differential-difference equations and subject to two operators, time differentiation and the time shift operator. General compensators for time-delay systems then split into several classes despite whether they have their own dynamics with respect to one or the other operator! One contribution in this section consists in such a classification of compensators and different state feedbacks will be considered in this work, namely, a static memoryless state feedback, a static-state feedback with delays, a pure shift dynamic compensator, and a dynamic compensator. These compensators are defined in Table I, where \((\cdot)\) stands for a generic time \((t - \tau)\) with \(z \in \mathbb{R}^r, \tau_i \in \mathbb{R}_+, \) for \(0 \leq i \leq m'\) and for some \(m' \in \mathbb{N}_0\) with \(\tau_0 = 0\) and \(\tau_i \in \mathbb{R}_+, M, N, M_n, N_n, M_i, N_i, \alpha, \beta\) are meromorphic functions of their arguments. The class of static memoryless state feedback has already been considered in the case of linear systems with delays [13], and the class of static feedback with delay can be seen as a generalization of the one used in [4] for the static case.

<table>
<thead>
<tr>
<th>Static memoryless feedback</th>
<th>Static feedback with delays</th>
</tr>
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<tbody>
<tr>
<td>( u = \alpha(x(t)) + \beta(x(t))v(t) )</td>
<td>( u = \alpha(x(t)) + \sum_{i=0}^{\rho} \beta_i(x(t))v(t - \tau_i) )</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>Pure shift dynamic compensator</th>
<th>Dynamic compensator</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z(t + \tau) = M(x(t),z(t)) + \sum_{i=0}^{\rho} N_i(x(t),z(t))v(t - \tau_i) )</td>
<td>( \dot{\eta}(t) = M_{\eta}(x(t),z(t),\eta(t)) + \sum_{i=0}^{\rho} N_{\eta i}(x(t),z(t),\eta(t))v(t - \tau_i) )</td>
</tr>
<tr>
<td>( u(t) = \alpha(x(t),z(t)) + \sum_{i=0}^{\rho} \beta_i(x(t),z(t))v(t - \tau_i) )</td>
<td>( \dot{z}(t) = M_{\eta}(x(t),z(t),\eta(t)) + \sum_{i=0}^{\rho} N_{\eta i}(x(t),z(t),\eta(t))v(t - \tau_i) )</td>
</tr>
</tbody>
</table>

A pure shift dynamic compensator is said to be regular if \( i, 0 \leq i \leq m' \) and \( \tau \in \mathbb{R}_+ \) exist such that \( (\partial u(t + \tau))/(\partial v(t - \tau_i)) \neq 0 \). The following result follows for Pure shift dynamic compensators with \( \lambda_0 \neq 0 \), called bicausal compensators.

**Proposition 4:** The relative degree \( \rho \) and the relative shift \( \mu \) are invariant under bicausal compensation.

**Proof:** First, the relative degree and the relative shift do not decrease under any compensation. Let \( \mathcal{C} \) denote a bicausal compensator and \( C^{-1} \) its inverse. So, \( C^{-1} \circ C \) is the identity compensator and \( \rho(C^{-1} \circ C) = \rho \geq \rho(C) \geq \rho \), where \( \rho() \) denotes the relative degree of the compensated system. These inequalities imply the invariance of \( \rho \). A similar argument can be used to show the invariance of the relative shift \( \mu \).

IV. **STATEMENT OF THE DISTURBANCE DECOUPLING PROBLEM (DDP)**

Let us consider SISO nonlinear time-delays system of the form

\[
\begin{align*}
\dot{x}(t) &= f(x(t)) + \sum_{i=0}^{m} g_i(x(t))u(t - \tau_i) \\
\Sigma_{d'}: \quad y(t) &= h(x(t)), \\
\Sigma_{d}: \quad y(t) &= h(x(t)), u(t) = u_0, \quad \forall t \in [t_0 - \tau_i, t_0]
\end{align*}
\]

where \((\cdot)\) stands for a generic time \((t - \tau)\) with \(x \in \mathbb{R}^r, \tau \in \mathbb{R}_+, \) for \(0 \leq i \leq m'\) and for some \(m' \in \mathbb{N}_0\) with \(\tau_0 = 0\) and \(\tau_i \in \mathbb{R}_+, M, N, M_n, N_n, M_i, N_i, \alpha, \beta\) are meromorphic functions of their arguments. \(\varphi\) is the function of initial conditions.

The notion of relative degree discussed in Section II can be extended to system \(\Sigma_{d'}\). Formally, the relative degree \( \rho \) of system \(\Sigma_{d'}\) with respect to the control input \( u(\cdot) \) is defined as

\[
\rho = \dim_{\mathcal{K}} \left( \text{span}_{\mathcal{K}} \{dy^{(\rho)}(t), k \in \mathbb{N}\} \cap \text{span}_{\mathcal{K}} \{dx(t - \tau), dy^{(\rho)}(t - \tau), \tau \in \mathbb{R}_+, k \in \mathbb{N}\} \right)
\]
where $\mathcal{K}$ stands for the field of meromorphic functions of a finite number of variables in
\[
\left\{ x(t-\tau), u^{(k)}(t-\tau), q^{(k)}(t-\tau), \tau \in \mathbb{R}_+, k \in \mathbb{N} \right\}.
\]

The DDP under consideration is now stated.

**The Disturbance Decoupling Problem:** Consider a SISO nonlinear time-delay system $\Sigma_\tau$. Find, if possible, a compensator such that the output is not affected by the disturbance $d(\cdot)$, and more precisely, such that for the closed-loop system, we have
\[
dy^{(k)}(t) \in \text{span}_{\mathcal{K}^v} \left\{ dx(\cdot), dv^{(i)}(\cdot), 0 \leq i \leq k - 1 \right\}
\]
for any $k \geq 0$. ($\cdot$) stands, as before, for a generic time $(t-\tau)$ with $\tau \in \mathbb{R}_+$ and $\mathcal{K}^v$ denotes the field of meromorphic functions of a finite number of variables in $\left\{ x(t-\tau), z(t-\tau), \eta(t-\tau), v^{(k)}(t-\tau), q^{(k)}(t-\tau), \tau \in \mathbb{R}_+, k \in \mathbb{N} \right\}$.

V. PURE SHIFT DYNAMIC SOLUTIONS TO THE DDP

We now give a solution for the DDP by means of a pure shift dynamic compensator. We first introduce some concepts and definitions that permit to obtain a geometric characterization of the solution. Such a characterization gives a hint for an eventual generalization of the one recently obtained for linear systems with delays and that may be viewed as a special case of systems over a ring [4].

Given a time-delay $\tau$, denote the shift operator $\nabla$, defined from $\mathcal{K}$ to $\mathcal{K}$ by $\nabla(\xi(t)) = \xi(t-\tau)$. The definition of $\nabla$, is extended to $\mathcal{K}^v$ by $\nabla(\omega(t)) = \omega(t-\tau)$.

Denote $\mathcal{K}^v$ the ring of polynomials of a finite number of operators $\nabla_1, \ldots, \nabla_{\tau}$. Define $\mathcal{M}$ as the formal module over the ring $\mathcal{K}^v$, more precisely
\[
\mathcal{M} = \text{span}_{\mathcal{K}^v} \left\{ d\xi(\cdot) | \xi \in \mathcal{K} \right\}.
\]

Let $\{\omega_1, \ldots, \omega_k\}$ be a set of vectors of $\mathcal{E}$, and then denote $\text{span}_{\mathcal{K}^v}[\omega_1, \ldots, \omega_k]$ as the submodule of $\mathcal{M}$ spanned by $\{\omega_1, \ldots, \omega_k\}$.

A vector $\omega \in \mathcal{E}$ is said to be an exact form if $\xi \in \mathcal{K}$ exists such that $\omega = d\xi$.

**Definition 5—Integrability of a One-Form:** A one-form $\omega \in \mathcal{E}$ is said to be integrable if $\alpha(\nabla) \in \mathcal{K}[\nabla]$ and $\varphi \in \mathcal{K}$ exist such that $\omega = \alpha(\nabla) d\varphi$.

This notion of integrability will be instrumental to derive a complete solution to the DDP in Theorem 11.

**Lemma 6:** Let $\omega$ be an integrable one-form, then an integrating factor $\alpha \in \mathcal{K}$ exists if and only if $d\omega \wedge \omega = 0$.

**Proof—Necessity:** Let $\alpha, \varphi \in \mathcal{K}$ such that $\omega = \alpha d\varphi$. Then, $d\omega \wedge \omega = (d\alpha \wedge d\varphi) \wedge d\varphi = 0$.

**Sufficiency:** Any one-form may be written as
\[
\omega = \sum_j \alpha_j d\varphi(t-\tau_j) \\
\in \text{span}_{\mathcal{K}^v} \left\{ dx(t-\tau_j), du(t-\tau_j) \right\}, \quad \alpha_j \in \mathcal{K}.
\]

Then, $\omega$ belongs to a finitely generated formal vector space over $\mathcal{K}$, and therefore, by applying the Frobenius Theorem [3, p. 238], the existence of an integrating factor $\alpha \in \mathcal{K}$ is stated.

**Remark 7:** The previous result is not more valid when the integrating factor $\alpha \in \mathcal{K}$. Consider $\omega = dx_1(t) + x_2(t) dx_1(t-1)$ that can be rewritten as $\omega = (1 + x_2(t)\nabla) dx_1(t)$ with an integrating factor $\alpha = (1 + x_2(t)\nabla) \in \mathcal{K}[\nabla]$. However, $d\omega \wedge \omega = dx_2(t) \wedge dx_1(t-1) \wedge dx_1(t) \neq 0$.

Similarly to what has been introduced in [4] for linear time-delay systems, let us define a closure of a submodule of $\mathcal{M}$.

**Definition 8—Closure:** Let $\omega \in \mathcal{M}$, $\omega \neq 0$, the closure of $\text{span}_{\mathcal{K}^v}[\omega]$ is the largest one-dimensional submodule of $\mathcal{M}$, containing $\text{span}_{\mathcal{K}^v}[\omega]$.

**Lemma 9:** Let $\omega$ be a one-form such that $\text{span}_{\mathcal{K}^v}[\omega]$ is equal to its closure. If $\omega$ is integrable, then $\alpha \in \mathcal{K}$.

**Proof:** Because $\omega$ is integrable, then it can be written as $\omega = \alpha(\nabla) d\varphi$, where $\alpha(\nabla) \in \mathcal{K}[\nabla]$ and $\varphi \in \mathcal{K}$. Assume now that $\alpha(\nabla) \notin \mathcal{K}$: this implies that $\text{span}_{\mathcal{K}^v}[\omega] \supseteq \text{span}_{\mathcal{K}^v}[\alpha(\nabla) d\varphi]$, which is in contradiction with the closure of $\text{span}_{\mathcal{K}^v}[\omega]$. This proves the lemma.

From the previous results, it is possible to relate the notion of integrability of a one-form $\omega$ with its closure.

**Corollary 10:** $\omega \in \mathcal{M}$ is integrable if and only if $d\pi \wedge \pi = 0$, where $\pi$ is any basis of the closure of $\text{span}_{\mathcal{K}^v}[\omega]$.

Let us define the following submodule of $\mathcal{M}$:
\[
\Omega := \text{span}_{\mathcal{K}^v} \left\{ dx(t) \right\} \quad \exists \pi \in \mathcal{K}[\nabla], \quad P \omega^{(k)} \in \text{span}_{\mathcal{K}^v} \left\{ dx(t) \right\} + \text{span}_{\mathcal{K}^v} \left\{ dy^{(0)}(t), \ldots, dy^{(k+\pi-1)}(t) \right\}, \quad k \in \mathbb{N}
\]
which is the limit of the following algorithm:
\[
\Omega_1 = \text{span}_{\mathcal{K}^v} \left\{ dx(t) \right\} \\
\Omega_{k+1} = \left\{ \omega \in \Omega_k | \exists P \in \mathcal{K}[\nabla], P \omega \in \Omega_k + \text{span}_{\mathcal{K}^v} \left\{ dy^{(0)}(t) \right\} \right\}
\]
(3)

In the special case of linear or nonlinear systems without delays, $\Omega$ reduces to the standard notion of a controllability subspace or a controllability codistribution. In plain words, it represents the states not affected by any control or disturbance input, except through the output channels. The submodule $V^+((\text{Im}\mathcal{D})^k)$ has been introduced for linear systems with delays [1]. In this special case, it is proved that $\Omega = V^+((\text{Im}\mathcal{D})^k)$.

Let $\mathcal{K}^v$ be any complementary submodule to $\Omega$ so that $\text{span}_{\mathcal{K}^v} \left\{ dx(t), du(t) \right\} = \Omega \oplus \Omega_0$.

We are now ready to state our main result.

**Theorem II:** The DDP is solvable via bicausal compensation if and only if $\omega_0 \in \mathcal{E}$ exists such that
i) $\omega_0 \in \text{span}_{\mathcal{K}^v}[\{dx(t-\tau), du(t-\tau), \tau \geq \mu\}]$;
ii) $dy^{(0)}(t) - \omega_0 \in \Omega$;
iii) $\omega_0$ is integrable.

**Proof—Sufficiency:** From iii), an integrating factor $c(\nabla) \in \mathcal{K}[\nabla]$ and a function $\alpha \in \mathcal{K}$ so that $\omega_0 = c(\nabla) d\alpha(\cdot)$. If $\tau \neq 0$ exists such that $dy^{(0)}(t+\mu) / du(t-\tau) \neq 0$, we set
\[
\tau_c = \min \left\{ \tau \in \mathbb{R}_+ | \frac{dy^{(0)}(t+\mu)}{du(t-\tau)} \neq 0 \right\}
\]
Because system $\Sigma_\tau$ is affine in $u(\cdot)$, the function $\alpha(\cdot)$ is affine in $u(\cdot)$, and by i)
\[
\alpha(\cdot) = a(x(t-\tau), u(t-\tau'), \tau \in \mathbb{R}_+, \tau' \geq \tau_c) \\
b(x(t-\tau), \tau \in \mathbb{R}_+, u(t)
\]
and we define the pure shift dynamic compensator
\[
\left\{ \begin{array}{l}
\quad z(t+\tau_c) = - a(x(t-\tau), z(t+\tau_c-\tau') + v(t) \\
\quad u(t) = - a(x(t-\tau), z(t+\tau_c-\tau') + v(t)
\end{array} \right.
\]
(4)
where $\tau \in \mathbb{R}_+$ and $\tau' \geq \tau$. If $\tau_0 = 0$, this compensator reduces to a static feedback with delay

$$u(t) = \frac{-a(x(t - \tau), \tau \in \mathbb{R}_+) + v(t)}{b(x(t - \tau), \tau \in \mathbb{R}_+)}.$$ \hspace{1cm} (5)

Then, from ii) and the definition of $\Omega$, the disturbance decoupling problem is solved.

**Necessity:** Assume that the disturbance has been rejected by a bicausal compensator, say, $P(\nabla)u(t) = \alpha(x(t)) + Q(\nabla)v(t)$. Define a new bicausal compensator $Q(\nabla)v(t) = \omega(t)$. Consider now $\omega_0 \in \mathcal{K}^{\nabla}[\{dx\} \times \{dw\}]$ and $\omega \in \mathcal{K}^{\nabla}[\{dx\} \times \{dw\}]$ such that $dy^{(i)}(t) = \omega_0 + \omega$, and then in closed loop $\omega \in \mathcal{K}^{\nabla}[\{dx\} \times \{dw\}]$. Considering that $dy^{(i)}(t) = \sum \alpha_i dx_i + \beta_i dw_i$, then $P \in \mathcal{K}[\nabla]$ exists such that $P dw \in \mathcal{K}[\nabla][\{dx, dy^{(i)}\}]$ from where it is possible to obtain $P' \omega \in \mathcal{K}^{\nabla}[\{dx, dy^{(i)}\}]$ with $P' \in \mathcal{K}[\nabla]$. This proves ii). Condition i) follows from the definition of the proposed bicausal compensator.

The nonlinear compensators considered in Section III can finally be expressed in a more compact form and in accordance with the notation introduced here, as it is shown in Table II, where $P(\nabla)$ and $Q(\nabla) \in \mathcal{K}[\nabla]$.

**VI. DYNAMIC SOLUTIONS TO THE DDP**

The most general class of dynamic compensators that solve the DDP is now considered, giving some sufficient conditions for its solvability.

First, we extend the notion of relative degree introduced in Section IV. More precisely, the relative degree $\sigma$ of system $\Sigma_d$ with respect to the disturbance input $q(\cdot)$ is defined as

$$\sigma = \dim_{\mathbb{C}} \left( \text{span}_{\mathbb{K}} \left\{ dy^{(k)}(t), k \in \mathbb{N} \right\} \right) \cap \left( \text{span}_{\mathbb{K}} \left\{ dx(t - \tau), du^{(k)}(t - \tau), \tau \in \mathbb{R}_+, k \in \mathbb{N} \right\} \right).$$

We then have the following result.

**Theorem 12:** Consider system $\Sigma_d$, and assume it has finite relative degrees $\rho$ and $\sigma$, and a finite time-delay shift $\mu$. Let the differential form

$$\omega_0 := \sum_{\tau \geq \rho} \left[ \frac{\partial y^{(\sigma - i)}(t)}{\partial x(t - \tau)} dx(t - \tau) + \sum_{i=0}^{\sigma - 1} \frac{\partial y^{(\sigma - i)}(t)}{\partial u^{(i)}(t - \tau)} du^{(i)}(t - \tau) \right].$$

Then, a dynamic compensator exists that solves the disturbance decoupling problem if

i) $\omega_0$ is integrable;

ii) $dy^{(k)}(t) \in \mathcal{K}[\{dx(t - \tau), \omega^{(i)}(t - \tau'), \tau' \geq \mu, \tau \in \mathbb{R}_+, 0 \leq i \leq k - \rho\}]$ for all $k \geq \sigma - 1$.

**Proof:** Assume that i) holds. Then, meromorphic functions $F(\cdot)$ and $\Psi(\cdot)$ exist such that

$$y^{(\sigma - i)}(t) = F(x(t - \tau), 0 \leq \tau < \mu), \Psi(x(t - \tau), u^{(i)}(t - \tau), \tau' \geq \mu, 0 \leq i \leq \sigma - \rho - 1).$$

Again, if a $\tau \neq 0$ exists for which $\partial y^{(\sigma - i)}(t + \mu)/\partial u^{(i)}(t - \tau) \neq 0$, for some $0 \leq i \leq \sigma - \rho - 1$, we set

$$\tau_1 = \min \left\{ \tau \in \mathbb{R}_+ \mid \partial y^{(\sigma - i)}(t + \mu)/\partial u^{(i)}(t - \tau) \neq 0 \right\}.$$

**TABLE II**

<table>
<thead>
<tr>
<th>Static memoryless feedback</th>
<th>$u = \alpha(x(t)) + \beta(x(t))v(t)$</th>
</tr>
</thead>
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<td>Static feedback with delays</td>
<td>$u = \alpha(x(t)) + Q(\nabla)v(t)$</td>
</tr>
<tr>
<td>Pure shift dynamic compensator</td>
<td>$P(\nabla)u = \alpha(x(t)) + Q(\nabla)v(t)$</td>
</tr>
<tr>
<td>with $P(0) \neq 0$</td>
<td>$P(\nabla)u = \alpha(x(t)) + Q(\nabla)v(t)$</td>
</tr>
</tbody>
</table>

for some $0 \leq i \leq \sigma - \rho - 1$, and, because system $\Sigma_d$ is affine in $u(t)$, $\Psi = \alpha(x(t)), u^{(i)}(t), 0 \leq i \leq \sigma - \rho - 1 + b(x(t), u^{(i)}(t))$, and $0 \leq i \leq \sigma - \rho - 2$ $u^{(\sigma - \rho - 1)}(t)$. Defining the dynamic compensator

$$\eta_1(t) = \eta_2(t)$$

$$\vdots$$

$$\eta_{\sigma - \rho - 1}(t)$$

$$\eta_i(t) = -a(x(t - \tau_i), \eta_{i+1}(t - \tau_i), z(t + \tau_i - \tau_i) + v(t)/b(x(t - \tau_i), \eta_{i+1}(t - \tau_i)) u^{(i)}(t - \tau_i).$$

$$u(t) = \eta_1(t)$$

where $\tau \in \mathbb{R}_+, 0 \leq i \leq \sigma - \rho - 1$, and, because system $\Sigma_d$ is affine in $u(t)$, $\Psi = \alpha(x(t)), u^{(i)}(t), 0 \leq i \leq \sigma - \rho - 1 + b(x(t), u^{(i)}(t))$, and $0 \leq i \leq \sigma - \rho - 2$ $u^{(\sigma - \rho - 1)}(t)$. Defining the dynamic compensator

$$\eta_1(t) = \eta_2(t)$$

$$\vdots$$

$$\eta_{\sigma - \rho - 1}(t)$$

$$\eta_i(t) = -a(x(t - \tau_i), \eta_{i+1}(t - \tau_i), z(t + \tau_i - \tau_i) + v(t)/b(x(t - \tau_i), \eta_{i+1}(t - \tau_i)) u^{(i)}(t - \tau_i).$$

Then, the DDP is solvable by means of a dynamic memoryless state compensator if

i) $\omega_0$ is integrable;

ii) $dy^{(k)}(t) \in \mathcal{K}[\{dx(t - \tau), \omega^{(i)}(t - \tau'), \tau' \geq \mu, \tau \in \mathbb{R}_+, 0 \leq i \leq k - \rho\}]$ for all $k \geq \sigma - 1$.

The results presented in this paper will be partially shown in the following example

**Example 1:** Consider the time-delay system

$$\dot{x}_1(t) = [x_2(t-1) + u(t-1)] + x_3(t)[x_2(t-2) + u(t-2)] + x_1(t-1)$$

$$\dot{x}_2(t) = q(t)$$

$$\dot{x}_3(t) = x_2(t) + u(t)$$

$$\dot{x}_4(t) = \sin(x_1(t-1))$$

$$y(t) = x_1(t)$$

for which $\rho = 1, \sigma = 2,$ and $\mu = 1$. By (3), $\Omega$ can be computed as $\Omega_1 = \mathcal{K}[\{dx(t)\}], \Omega_2 = \mathcal{K}[\{dx_1(t), dx_2(t), dx_4(t)\}] = \Omega$, producing $dy(t) = \omega + \omega_0$, where

$$\omega = [x_2(t-2) + u(t-2)]dx_3(t) + dx_1(t-1) \in \Omega$$

$$\omega_0 = (\nabla + x_3(t)\nabla^2)dx_2(t) + u(t).$$

The conditions of the Theorem 11 are fulfilled, and a static memoryless feedback that solves the problem is given by $u(t) = -x_2(t) + v(t)$, as provided by the proof of the sufficiency.
VII. CONCLUSION

In the present paper, an innovative mathematical frame has been introduced for solving the DDP associated with a class of time-delay nonlinear systems. The geometric concepts introduced give an insight into the system's structure. We claim, as well, that the methods used in this paper may be used for the analysis and synthesis of other control problems associated with time-delay nonlinear systems (e.g., the input–output linearization problem).

REFERENCES


Nonoptimality of Static-Priority Policies in Unreliable Two-Part-Type Manufacturing Systems

Francis De Vericourt and Yves Dallery

Abstract—In this note, we show that for the two-part–type one-machine continuous flow model, a static-priority policy cannot be optimal when the hedging point of the part of highest priority is positive. This result was previously conjectured by Veatch and Caramanis [6].

Index Terms—Multiple part type, stochastic scheduling, unreliable manufacturing systems.

I. INTRODUCTION

The exploration of the optimal control in the continuous flow model of a two-part-type one machine make-to-stock problem has received a great interest recently. In this problem, production is modeled as a continuous flow. Sivatsans and Dallery partially [5] characterize the optimal policy. Furthermore, Veatch and Caramanis [6] have found conditions under which no inventory is needed. The optimal policy can be viewed under these conditions, as a static-priority policy, with its hedging point at the origin. Static-priority policies have been defined by Peña-Perez and Zipkin [4] in the context of the discrete part model. These policies always produce the same part-type when it is below its hedging point. Their simplicity makes them very easy to study and to implement (see, for instance, Nemec and Gershwin [3]). Veatch and Caramanis [6], however, argue that a static-priority policy with a non-null hedging point for the first part type cannot be optimal (where the first part-type is the most expensive in terms of its normalized back order cost). In this note, we prove this conjecture.

II. THE MODEL

The model considered here is a manufacturing system consisting of a single, flexible, unreliable machine producing two part types. This model is the same as the one studied in [5] and [6]. We briefly recall the main assumptions of the model. More details and explanations can be found in [5] and [6].

The times between successive failures and successive repairs are exponentially distributed with rates μ and r, respectively. α(τ) denotes the machine state; α is one when the machine is up and zero when the machine is down. Part type i has a maximum production rate, μi, and a deterministic demand rate di. The decision variables are the instantaneous production rates ui(τ), xi(τ) denotes the difference between actual production and demand at time τ. We call xi(τ) the surplus (or backlog if the difference is negative) of part type i. Boldface letters, d, x, u are used to denote the vector forms of demand, surplus, and production rate, respectively. The dynamics of the surplus can be expressed as

\[ \dot{x}_i(\tau) = u_i(\tau) - d_i, \quad \text{for all } i. \]