



# Parameter estimation and bias correction for diffusion processes

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## ABSTRACT

This paper considers parameter estimation for continuous-time diffusion processes which are commonly used to model dynamics of financial securities including interest rates. To understand why the drift parameters are more difficult to estimate than the diffusion parameter, as observed in previous studies, we first develop expansions for the bias and variance of parameter estimators for two of the most employed interest rate processes, Vasicek and CIR processes. Then, we study the first order approximate maximum likelihood estimator for linear drift processes. A parametric bootstrap procedure is proposed to correct bias for general diffusion processes with a theoretical justification. Simulation studies confirm the theoretical findings and show that the bootstrap proposal can effectively reduce both the bias and the mean square error of parameter estimates, for both univariate and multivariate processes. The advantages of using more accurate parameter estimators when calculating various option prices in finance are demonstrated by an empirical study.

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## 1. Introduction

Diffusion processes have been commonly used in finance to model stochastic dynamics of financial securities following the works of Black and Scholes (1973) and Merton (1973) which established the foundation of option pricing theory in finance. There has been phenomenal growth in financial products and instruments powered by these processes, as documented in Sundaresan (2000). A  $d$ -dimensional parametric diffusion process  $\{X_t \in \mathcal{R}^d; t \geq 0\}$  is defined by the following stochastic differential equation

$$dX_t = \mu(X_t; \theta)dt + \sigma(X_t; \theta)dB_t, \quad (1.1)$$

where  $\theta$  is a  $q$ -dimensional parameter,  $\mu(\cdot; \theta) : \mathcal{R}^d \rightarrow \mathcal{R}^d$  and  $\sigma(\cdot; \theta) = (\sigma_{ij})_{d \times p} > 0 : \mathcal{R}^d \rightarrow \mathcal{R}^{d \times p}$  are drift and diffusion functions representing, respectively, the conditional mean and variance of the infinitesimal change of  $X_t$  at time  $t$ , and  $B_t$  is a  $p$ -dimensional Brownian motion. The existence and uniqueness of the process  $\{X_t; t \geq 0\}$  satisfying (1.1) and its probability properties are given in Stroock and Varadhan (1979).

A unique feature of statistical inference for diffusion processes is that, despite these processes being continuous-time stochastic models, their observations are made only at discrete time points, say at  $n$  equally spaced  $\{t\delta\}_{t=0}^n$ . Here  $\delta$  is the sampling interval and can be either fixed or very small corresponding to high-frequency data. See Lo (1988), Bibby and Sørensen (1995), Ait-Sahalia (2002), Ait-Sahalia and Mykland (2003), and Fan (2005) for discussions and overviews for estimation of diffusion processes, based on discrete observations.

Short-term interest rates are fundamental quantities in finance, as they define excess asset returns and risk premiums of other assets and their derivative prices. A family of diffusion processes for the interest rates dynamics, consists of the following linear drift processes

$$dX_t = \kappa(\alpha - X_t)dt + \sigma(X_t, \psi)dB_t, \quad (1.2)$$

where  $\alpha$ ,  $\kappa$  and  $\psi$  are unknown parameters. The linear drift prescribes a mean-reversion of  $X_t$  toward the long term mean  $\alpha$  at a speed  $\kappa$ . The diffusion function  $\sigma(X_t, \psi)$  can accommodate a range of patterns in volatility. A sub-family of the above linear drift processes is given by assigning  $\sigma(X_t, \psi) = \sigma X_t^\rho$  with  $\psi = (\sigma^2, \rho)$ . Important members of this sub-family are the Vasicek model (Vasicek, 1977) with  $\rho = 0$  and the CIR model (Cox et al., 1985) with  $\rho = 1/2$ . Both Vasicek and CIR models are commonly used in finance due to (i) both having simple and

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attractive financial interpretations; and (ii) both admitting close-form solutions. The latter facilitates explicit calculations of various option prices.

Despite the critical roles played by these interest rate processes, it is well known, empirically, that estimation of the drift parameters  $\kappa$  and  $\alpha$  can incur large bias and/or variability; see for instance Ball and Torous (1996) and Yu and Phillips (2001). Merton (1980) had earlier discovered the difficulty with the drift parameter estimation for the Black–Scholes diffusion model, where both the drift and diffusion functions are constant functions. The difficulty with the drift parameter estimation is encountered by virtually all the estimation approaches, including the maximum likelihood estimation. The problem increases when the process has a lack of dynamics, which happens when  $\kappa$  is small. Indeed, as reported in Phillips and Yu (2005) and our simulation study, the maximum likelihood estimator for  $\kappa$  can incur relative bias of more than 200% even the processes are observed monthly for more than 10 years. This is serious, as poor qualitative estimates can produce severely biased option prices.

The objectives of this paper are (i) to understand the above empirical phenomena by developing expansions to the bias and variance of estimators for the Vasicek, CIR and general linear drift diffusion processes; and (ii) to propose a bias correction approach that is applicable to general diffusion processes. Two asymptotic regimes are considered in our analysis. One has  $\delta$  (the sampling interval) fixed while the sample size  $n \rightarrow \infty$ . The other has  $\delta$  converges to zero as  $n \rightarrow \infty$ . The latter corresponds to high frequency data, and allows simplification of results as compared to results for the fixed- $\delta$  case.

The bias and variance expansions reveal that regardless of whether  $\delta$  is fixed or diminishing to zero, the bias of the  $\kappa$  estimators and the variances of the two drift parameters estimators are effectively at the order of  $(n\delta)^{-1}$ , depending on the amount of time over which the process is observed. Our analysis also reveals that the bias and variance of the estimators for the diffusion parameter  $\sigma^2$  basically enjoys much smaller orders at  $n^{-1}$ . These explain why estimation of  $\kappa$  incurs more bias than the other parameters, and why the drift parameter ( $\kappa$  and  $\alpha$ ) estimates are more variable than those of the diffusion parameter.

We then propose a parametric bootstrap procedure for bias correction for general diffusion processes. Both theoretical and empirical analysis show that the proposed bias correction effectively reduces the bias without inflating the variance. The proposed bootstrap procedure can be combined with a range of parameter estimators, including the approximate likelihood estimation of Ait-Sahalia (2002, 2008).

The paper is structured as follows. Section 2 outlines parameter estimators used in our analysis. Bias and variance expansions for the estimators of Vasicek, CIR and the linear drift processes are presented in Section 3. Section 4 discusses the bootstrap bias correction. Simulation results are reported in Section 5. Section 6 analyzes a dataset of Fed fund rates. All technical details are deferred to the Appendix.

## 2. Parameter estimation for diffusion processes

### 2.1. A general overview

Let  $X_0, X_\delta, \dots, X_{n\delta}$  be discrete observations from process (1.1) at equally spaced time points  $\{t\delta\}_{t=0}^n$  over a time interval  $[0, T]$  where  $T = n\delta$ . To simplify notation, we write these observations as  $\{X_t\}_{t=0}^n$  by hiding  $\delta$  whenever doing so does not lead to confusion. As a diffusion process is Markovian, the maximum likelihood estimation (MLE) is the natural choice for parameter estimation if its transitional density is known. However, for most diffusion processes, their transitional distributions are not explicitly known,

which prevents the use of the MLE. In these cases, several methods are available, which include the martingale estimating equation approach of Bibby and Sørensen (1995); the pseudo-Gaussian likelihood approach of Nowman (1997); the Generalized Method of Moments (GMM) estimator of Hansen and Scheinkman (1995); the Efficient Method of Moments of Gallant and Tauchen (1996) and the approximate likelihood approach of Ait-Sahalia (2002). Ait-Sahalia and Mykland (2003, 2004) consider likelihood and the GMM based estimation when  $\delta$  is random. Nonparametric estimators for the drift and diffusion functions have been also proposed; see Fan (2005) for reviews.

We carry out our analysis under two asymptotic regimes. It is assumed, in the first regime, that  $n \rightarrow \infty$  while  $\delta$  is a fixed constant; and in the second regime that

$$n \rightarrow \infty, \delta \rightarrow 0, T = n\delta \rightarrow \infty$$

$$\text{and for some } k > 2 \quad T\delta^{1/k} \rightarrow \infty. \quad (2.1)$$

In the second regime,  $\delta$  diminishes to zero while the total observational time goes to infinity as  $n \rightarrow \infty$ . The last part of (2.1) is used to bound remainder terms in moment expansions. We note that  $T \rightarrow \infty$  mimics the standard asymptotic of  $n \rightarrow \infty$  and, as shown in our analysis, is the main driving force in determining the bias and variance properties in the drift parameter estimation.

The motivations for assuming  $\delta \rightarrow 0$  besides  $n \rightarrow \infty$  are two-fold. One is that high frequency financial data are increasingly available. Another is to accommodate discretization based estimators, which normally requires  $\delta \rightarrow 0$  to make the discretization error diminish to zero fast enough so that the estimators are consistent.

### 2.2. Estimation for Vasicek process

The Vasicek process satisfies the univariate stochastic differential equation

$$dX_t = \kappa(\alpha - X_t)dt + \sigma dB_t. \quad (2.2)$$

It is the Ornstein–Uhlenbeck process and was proposed by Vasicek (1977) for interest rate dynamics. The conditional distribution of  $X_t$  given  $X_{t-1}$  is

$$X_t|X_{t-1} \sim N \left\{ X_{t-1}e^{-\kappa\delta} + \alpha(1 - e^{-\kappa\delta}), \frac{1}{2}\sigma^2\kappa^{-1}(1 - e^{-2\kappa\delta}) \right\}$$

and the stationary distribution is  $N(\alpha, \frac{1}{2}\sigma^2\kappa^{-1})$ . The conditional mean and variance of  $X_t$  given  $X_{t-1}$  are

$$E(X_t|X_{t-1}) = X_{t-1}e^{-\kappa\delta} + \alpha(1 - e^{-\kappa\delta}) =: \mu(X_{t-1}) \quad \text{and} \quad (2.3)$$

$$\text{Var}(X_t|X_{t-1}) = \frac{1}{2}\sigma^2\kappa^{-1}(1 - e^{-2\kappa\delta}). \quad (2.4)$$

Let  $\phi(x)$  be the density function of the standard normal distribution  $N(0, 1)$ . Then, the likelihood function of  $\theta = (\kappa, \alpha, \sigma^2)$  is

$$L(\theta) = \phi \left( \sigma^{-1}\sqrt{2\kappa}(X_0 - \alpha) \right) \\ \times \prod_{t=1}^n \phi \left( \sigma^{-1}\sqrt{2\kappa(1 - e^{-2\kappa\delta})}^{-1} \{X_t - \mu(X_{t-1})\} \right).$$

Ignoring the first component in  $L(\theta)$  involving  $X_0$ , the maximum likelihood estimators (MLE) can be obtained explicitly by

$$\hat{\kappa} = -\delta^{-1} \log(\hat{\beta}_1), \quad \hat{\alpha} = \hat{\beta}_2 \quad \text{and} \quad \hat{\sigma}^2 = 2\hat{\kappa}\hat{\beta}_3(1 - \hat{\beta}_1^2)^{-1} \quad (2.5)$$

where

$$\hat{\beta}_1 = \frac{n^{-1} \sum_{i=1}^n X_i X_{i-1} - n^{-2} \sum_{i=1}^n X_i \sum_{i=1}^n X_{i-1}}{n^{-1} \sum_{i=1}^n X_{i-1}^2 - n^{-2} \left( \sum_{i=1}^n X_{i-1} \right)^2},$$

$$\hat{\beta}_2 = \frac{n^{-1} \sum_{i=1}^n (X_i - \hat{\beta}_1 X_{i-1})}{1 - \hat{\beta}_1} \quad \text{and}$$

$$\hat{\beta}_3 = n^{-1} \sum_{i=1}^n \{X_i - \hat{\beta}_1 X_{i-1} - \hat{\beta}_2 (1 - \hat{\beta}_1)\}^2.$$

The conditional mean and variance (2.3) and (2.4) suggest that the discrete observations  $\{X_t\}_{t=0}^n$  follow an AR(1) process with  $\beta_1 = e^{-\kappa\delta}$  as the auto-regressive coefficient. As  $\beta_1 \rightarrow 1$  when  $\delta \rightarrow 0$ , we are observing a near unit root situation. Our analysis shows that

$$E(\hat{\beta}_1) = \beta_1 - \frac{4}{n} + \frac{3\kappa\delta}{n} + \frac{7}{n^2\kappa\delta} + o(n^{-2}\delta^{-1} + n^{-1}\delta). \quad (2.6)$$

Here the bias of  $\hat{\beta}_1$  is controlled by two forces of asymptotic:  $\delta$  and  $n$ , due to the continuous-time nature of the process. The expansion (2.6) echoes

$$E(\hat{\beta}_1) = \beta_1 - \frac{1 + 3\beta_1}{n} + O(n^{-2}) \quad (2.7)$$

given by [Marriott and Pope \(1954\)](#) and [Kendall \(1954\)](#) for the discrete-time AR(1) process.

Although (2.3) and (2.4) suggest a link with AR(1) time series, a key difference between our current study and the AR(1) time series is that  $\delta$  may diminish to 0 in sampling a continuous-time process. Hence the existing theory on  $\beta_1$  from time series is not directly applicable when we consider the diminishing  $\delta$  asymptotic for Vasicek processes.

### 2.3. Estimation for CIR process

A CIR ([Cox et al., 1985](#)) diffusion process satisfies

$$dX_t = \kappa(\alpha - X_t)dt + \sigma\sqrt{X_t}dB_t, \quad (2.8)$$

with  $2\kappa\alpha/\sigma^2 > 1$ . Let  $c = 4\kappa\sigma^{-2}(1 - e^{-\kappa\delta})^{-1}$ , the transitional distribution of  $cX_t$  given  $X_{t-1}$  is non-central  $\chi^2_v(\lambda)$  with the degree of freedom  $v = 4\kappa\alpha\sigma^{-2}$  and the non-central component  $\lambda = cX_{t-1}e^{-\kappa\delta}$ .

The conditional mean is the same with (2.3) of the Vasicek process. However, due to the heteroscedasticity in the diffusion function, the conditional variance becomes

$$\text{Var}(X_t|X_{t-1}) = \frac{1}{2}\alpha\sigma^2\kappa^{-1}(1 - e^{-\kappa\delta})^2 + X_{t-1}\sigma^2\kappa^{-1}(e^{-\kappa\delta} - e^{-2\kappa\delta}). \quad (2.9)$$

Since the non-central  $\chi^2$ -density function is an infinite series involving central  $\chi^2$  densities, explicit expression of the MLEs for  $\theta = (\kappa, \alpha, \sigma^2)$  is not attainable. To gain insight into the parameter estimation, we consider pseudo-likelihood estimators proposed by [Nowman \(1997\)](#), as it can produce close-form expressions for the estimators. Nowman employed [Bergstrom \(1984\)](#)'s approximation

$$dX_t = \kappa(\alpha - X_t)dt + \sigma\sqrt{X_{m\delta}}dB(t) \quad \text{for } t \in [m\delta, (m+1)\delta] \quad (2.10)$$

which discretizes the diffusion function within each  $[m\delta, (m+1)\delta]$  by its value at the left end point of the interval, while keeping the drift unchanged.

Without confusion in the notation, let  $\{X_t\}_{t=0}^n$  be observations from process (2.10). Then they satisfy the following discrete time series model

$$X_t = e^{-\kappa\delta}X_{t-1} + \alpha(1 - e^{-\kappa\delta}) + \eta_t, \quad (2.11)$$

where  $E(\eta_t) = 0$ ,  $E(\eta_t\eta_s) = 0$  if  $t \neq s$  and  $E(\eta_t^2) = \frac{1}{2}\sigma^2\kappa^{-1}(1 - e^{-2\kappa\delta})X_{t-1} =: \xi(X_{t-1}, \theta)$ . However, unlike the Vasicek case, the discrete model (2.11) does not contain the same amount of information as the original process (2.8). Hence, results from discrete time series are not applicable.

By pretending  $\eta_t$  to be Gaussian distributed, a pseudo log-likelihood

$$\ell_n(\theta) = - \sum_{t=1}^n \left[ \frac{1}{2} \log\{\xi(X_{t-1}, \theta)\} + \frac{1}{2} \xi^{-1}(X_{t-1}, \theta) \times \{X_t - e^{-\kappa\delta}X_{t-1} - \alpha(1 - e^{-\kappa\delta})\}^2 \right] \quad (2.12)$$

is obtained which leads to a pseudo-MLEs

$$\hat{\kappa} = -\delta^{-1} \log(\hat{\beta}_1), \quad \hat{\alpha} = \hat{\beta}_2 \quad \text{and} \quad \hat{\sigma}^2 = \frac{2\hat{\kappa}\hat{\beta}_3}{1 - \hat{\beta}_1^2} \quad (2.13)$$

where

$$\hat{\beta}_1 = \frac{n^{-2} \sum_{i=1}^n X_i \sum_{i=1}^n X_{i-1}^{-1} - n^{-1} \sum_{i=1}^n X_i X_{i-1}^{-1}}{n^{-2} \sum_{i=1}^n X_{i-1} \sum_{i=1}^n X_{i-1}^{-1} - 1},$$

$$\hat{\beta}_2 = \frac{n^{-1} \sum_{i=1}^n X_i X_{i-1}^{-1} - \hat{\beta}_1}{(1 - \hat{\beta}_1)n^{-1} \sum_{i=1}^n X_{i-1}^{-1}} \quad \text{and}$$

$$\hat{\beta}_3 = n^{-1} \sum_{i=1}^n \{X_i - X_{i-1}\hat{\beta}_1 - \hat{\beta}_2(1 - \hat{\beta}_1)\}^2 X_{i-1}^{-1}.$$

We emphasize here that the discretized model (2.10) is used only to produce the estimators. It is the original CIR model (2.8) that is used when we analyze their properties.

### 2.4. Estimation for linear drift processes

As a generalization of the Vasicek and CIR processes, we consider the following linear drift diffusion process

$$dX_t = \kappa(\alpha - X_t)dt + \sigma(X_t; \psi)dB_t, \quad (2.14)$$

where  $\sigma(\cdot; \psi)$  is a general diffusion function with parameter  $\psi$ , and  $\theta = (\kappa, \alpha, \psi) \in R^d$  is the unknown parameter vector. The process includes, as special cases, Vasicek and CIR processes. The generality of the diffusion part means that the transitional density may not be analytically available. To facilitate maximum likelihood estimation, [Ait-Sahalia \(2002\)](#) has developed Edgeworth-type approximations to the transitional density of a univariate diffusion process, which can be used to derive approximate maximum likelihood estimates (AMLE) of  $\theta$ .

For a general univariate diffusion  $X_t$ , [Ait-Sahalia \(2002\)](#) first establishes an Edgeworth-type expansion to the transitional density  $f_Y(Y_t|Y_{t-1}, \theta)$  of a transformed process  $Y_t = \gamma(X_t; \psi) =: \int^{X_t} du/\sigma(u; \psi)$ . The  $J$ -term ( $J \geq 1$ ) expansion to  $f_Y(X_t|X_{t-1}; \theta)$  is

$$f_Y^{(J)}(Y_t|Y_{t-1}; \theta) = \delta^{-1/2} \phi\left(\frac{Y_t - Y_{t-1}}{\delta^{1/2}}\right) \exp\left(\int_{Y_{t-1}}^{Y_t} \mu_Y(u; \theta) du\right) \times \sum_{j=0}^J c_j(Y_t|Y_{t-1}; \theta) \frac{\delta^j}{j!},$$

where  $\phi(\cdot)$  is the standard normal density,  $c_0(Y_t|Y_{t-1}; \theta) \equiv 1$  and for  $j \geq 1$ ,

$$c_j(Y_t|Y_{t-1}; \theta) = j(Y_t - Y_{t-1})^{-j} \int_{Y_{t-1}}^{Y_t} (u - Y_{t-1})^{j-1} \times \left\{ \lambda_Y(u, \theta) c_{j-1}(u|Y_{t-1}; \theta) + \frac{1}{2} \frac{\partial^2 c_{j-1}(u|Y_{t-1})}{\partial u^2} \right\} du,$$

$$\lambda_Y(y; \theta) = - \{ \mu_Y^2(y; \theta) + \partial \mu_Y(y; \theta) / \partial y \} / 2 \quad \text{and}$$

$$\mu_Y(y; \theta) = \frac{\mu(\gamma^{-1}(y); \theta)}{\sigma(\gamma^{-1}(y); \psi)} - \frac{1}{2} \frac{\partial \sigma(\gamma^{-1}(y); \psi)}{\partial x}.$$

Transforming back the above expansion, the  $J$ -term expansion to  $f(X_t|X_{t-1}; \theta)$  is

$$f^{(J)}(X_t|X_{t-1}; \theta) = \sigma^{-1}(X_t; \psi) f_Y^{(J)}(\gamma(X_t)|\gamma(X_{t-1}); \theta). \quad (2.15)$$

The  $J$ -term approximate MLE (AMEL), say  $\hat{\theta}^{(J)}$ , is obtained by solving

$$\sum_{i=1}^n \frac{\partial [\log\{f^{(J)}(X_i|X_{i-1}; \theta)\}]}{\partial \theta} = 0.$$

Ait-Sahalia (2002) establishes the convergence of  $\hat{\theta}^{(J)}$  to the MLE as  $J \rightarrow \infty$  for each fixed sample size. An extension of the approximation to multivariate diffusion process is given in Ait-Sahalia (2008).

### 3. Main results

In Sections 3.1 and 3.2, we report results from both fixed  $\delta$  and diminishing  $\delta$  asymptotic analysis for Vasicek and CIR processes. In Section 3.3, we report results on the approximated MLE for the general linear drift diffusion. We will concentrate there on  $\hat{\theta}^{(1)}$ , based on the one term Edgeworth expansion. As  $J = 1$  is fixed, we will work under the diminishing  $\delta$  only.

#### 3.1. Fixed $\delta$ analysis

The fixed  $\delta$  results for the maximum likelihood estimators of the Vasicek process are given in the following two theorems. Let

$$B_1(\theta, \delta) = (5/2 + e^{\kappa\delta} + e^{2\kappa\delta}/2),$$

$$B_2(\theta, \delta) = -\sigma^2 \delta^{-1} \left[ \kappa^{-1} \left\{ 2 - \kappa\delta - \frac{1}{2} e^{2\kappa\delta} (1 - e^{-\kappa\delta}) \right\} - 4\delta (1 - e^{-2\kappa\delta})^{-1} e^{-2\kappa\delta} \right],$$

$$V_1(\theta, \delta) = \delta^{-1} (e^{2\kappa\delta} - 1),$$

$$V_2(\theta, \delta) = \sigma^2 (2\kappa)^{-1} \delta (e^{\kappa\delta} - 1)^{-1} (e^{\kappa\delta} + 1) \quad \text{and}$$

$$V_3(\theta, \delta) = \sigma^4 (\kappa\delta)^{-2} \left\{ 2(\kappa\delta)^2 + (e^{\kappa\delta} - e^{-\kappa\delta}) \left( 1 - \frac{2\kappa\delta e^{-2\kappa\delta}}{1 - e^{-2\kappa\delta}} \right) \right\}.$$

**Theorem 3.1.1.** For a stationary Vasicek process, as  $n \rightarrow \infty$  while  $\delta$  is fixed,

$$E(\hat{\kappa}) = \kappa + (n\delta)^{-1} B_1(\theta, \delta) + O(n^{-2}),$$

$$\text{Var}(\hat{\kappa}) = (n\delta)^{-1} V_1(\theta, \delta) + O(n^{-2}),$$

$$E(\hat{\alpha}) = \alpha + O(n^{-2}), \quad \text{Var}(\hat{\alpha}) = (n\delta)^{-1} V_2(\theta, \delta) + O(n^{-2}),$$

$$E(\hat{\sigma}^2) = \sigma^2 + n^{-1} B_2(\theta, \delta) + O(n^{-2}) \quad \text{and}$$

$$\text{Var}(\hat{\sigma}^2) = n^{-1} V_3(\theta, \delta) + O(n^{-2}).$$

Theorem 3.1.1 indicates that the estimators for all three parameters have both their bias and variances at the order of  $n^{-1}$ . However, a closer inspection indicates that the variance of  $\hat{\kappa}$  and  $\hat{\alpha}$ , and the bias of  $\hat{\kappa}$  are effectively at the order of  $T^{-1}$ . The bias and variance of  $\hat{\sigma}^2$  is  $O(n^{-1})$ , and hence converge to zero much faster. It is a little surprising to see that the bias of  $\hat{\alpha}$  is at  $n^{-2}$ ,

which is smaller than that of  $\hat{\kappa}$ . We note that  $V_1(\theta, \delta)$  and  $V_2(\theta, \delta)$  are decreasing functions of  $\delta$ . This means that the variances of the drift parameter estimates increase as  $\delta$  gets smaller. On the other hand, when  $\kappa$  gets smaller for a fixed  $n$  and  $\delta$ , the ratio  $B_1(\theta, \delta)/\kappa$  becomes larger.

**Theorem 3.1.2.** For a stationary Vasicek process, let  $\hat{\theta} = (\hat{\kappa}, \hat{\alpha}, \hat{\sigma}^2)^T$  and  $\theta = (\kappa, \alpha, \sigma^2)^T$ . Then, as  $n \rightarrow \infty$

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, \Omega_1),$$

where  $\Omega_1 = \text{diag} \{ \delta^{-1} V_1(\theta, \delta), \delta^{-1} V_2(\theta, \delta), V_3(\theta, \delta) \}$ .

Theorem 3.1.2 illustrates that each component of  $\hat{\theta}$  converges at the same rate of  $n^{1/2}$  and different estimators are asymptotically uncorrelated.

We need some notations before presenting our analysis for the CIR process. Let

$$F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{z^k}{k!} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)}$$

be the hypergeometric function;  $\theta_\alpha = 2\kappa\alpha/\sigma^2$  and  $\theta_\beta = 2\kappa/\sigma^2$  which are parameters of the stationary Gamma distribution of the CIR process. Let  $\delta_{ij} = \delta|j - i|$ ,

$$S_{1,ij} = \sum_{l=1}^{\infty} \frac{(e^{-\kappa\delta_{ij}} e^{-\kappa\delta})^l \Gamma(l+1) \Gamma(\theta_\alpha - 2)}{\Gamma(\theta_\alpha - 1 + l)},$$

$$S_{2,ij} = \sum_{l=1}^{\infty} \frac{(e^{-\kappa\delta_{ij}} e^{-\kappa\delta})^l \Gamma(l+2) \Gamma(\theta_\alpha - 2)}{\Gamma(\theta_\alpha + l)},$$

$$C_{1,\theta} = (\theta_\alpha - 1) \left\{ 2e^{-\kappa\delta} (\theta_\alpha + 1) - \theta_\alpha (1 + e^{-\kappa\delta}) - \frac{\theta_\alpha}{\theta_\alpha - 1} + \frac{\theta_\alpha^2 + \theta_\alpha}{(\theta_\alpha - 1)} (1 - e^{-\kappa\delta}) \right\} \quad \text{and}$$

$$C_{2,\theta} = 2e^{-\kappa\delta} (\theta_\alpha - 1) - \theta_\alpha^2 (1 - e^{-\kappa\delta}) + \frac{(\theta_\alpha - 1)(\theta_\alpha^2 + \theta_\alpha)}{(\theta_\alpha - 2)} (1 - e^{-\kappa\delta}).$$

Furthermore, to quantify the bias and variance, we define

$$V_4(\theta, \delta)$$

$$= \frac{e^{2\kappa\delta}}{n\delta} \left( (1 - e^{-\kappa\delta}) \left\{ nC_{2\theta} - 2\theta_\alpha(\theta_\alpha - 1)(\theta_\alpha - 2)(1 - e^{-\kappa\delta}) \right. \right.$$

$$\times \sum_{i=1}^n \sum_{j=i+1}^n (e^{\kappa\delta} S_{1,ij} - S_{2,ij}) \left. \right\} + (1 - e^{-\kappa\delta})^2$$

$$\times \left[ \sum_{i=1}^n \sum_{j=1}^n \{ \theta_\alpha^2 (F(1, 1, \theta_\alpha; e^{-\kappa\delta_{ij}}) - 1) - \theta_\alpha e^{-\kappa\delta_{ij}} \}$$

$$- 2 \sum_{j=1}^n \sum_{i=j+1}^n \{ -\theta_\alpha e^{\kappa\delta} e^{-\kappa\delta_{ij}} + \theta_\alpha^2 (F(1, 1, \theta_\alpha; e^{-\kappa\delta_{(i-1)j}}) - 1) \}$$

$$- 2 \sum_{i=1}^n \sum_{j=i}^n \{ C_{1,\theta} e^{-\kappa\delta_{ij}} - \theta_\alpha(\theta_\alpha - 1)(\theta_\alpha - 2)$$

$$\times (e^{\kappa\delta} S_{1,ij} - S_{2,ij}) \left. \right] \Bigg),$$

$$B_3(\theta, \delta)$$

$$= -n^{-1} e^{\kappa\delta} (1 - e^{-\kappa\delta}) \left( \sum_{i=1}^n \sum_{j=1}^n [-e^{-\kappa\delta_{ij}} - \theta_\alpha^2 (F(1, 1, \theta_\alpha; e^{-\kappa\delta_{ij}}) - 1) + \theta_\alpha e^{-\kappa\delta_{ij}}] \right)$$

$$\begin{aligned}
 &+ \sum_{j=1}^n \sum_{i=j+1}^n \left[ -\theta_\alpha e^{\kappa\delta} e^{-\kappa\delta_{ij}} + \theta_\alpha^2 \{F(1, 1, \theta_\alpha; e^{-\kappa\delta(i-1)j}) - 1\} \right] \\
 &+ \sum_{i=1}^n \sum_{j=i}^n \left\{ C_{1,\theta} e^{-\kappa\delta_{ij}} - \theta_\alpha (\theta_\alpha - 1)(\theta_\alpha - 2)(e^{\kappa\delta} S_{1,ij} - S_{2,ij}) \right\} \\
 &+ \frac{\delta}{2} V_4(\theta, \delta),
 \end{aligned}$$

$$B_4(\theta, \delta) = \frac{1}{2}(\theta_\alpha - 1)^{-1}(1 - e^{-\kappa\delta})\sigma^2,$$

$$V_5(\theta, \delta) = \theta_\beta^{-2}\theta_\alpha \{2(e^{\kappa\delta} - 1)^{-1} + 1\},$$

$$V_6(\theta, \delta) = A_1(\theta, \delta)^2 Z_1(\theta, \delta) + A_2(\theta, \delta)^2 Z_2(\theta, \delta) \quad \text{where}$$

$$A_1(\theta, \delta) = \frac{\sigma^2\delta^{-1}}{\kappa e^{-\kappa\delta}} - \frac{2\delta\sigma^2}{(1 - e^{-2\kappa\delta})}, \quad A_2(\theta, \delta) = \frac{-2\delta^{-2}\kappa}{1 - e^{-2\kappa\delta}},$$

$$Z_1(\theta, \delta) = \frac{\theta_\alpha - 1}{\theta_\alpha}(1 - e^{-\kappa\delta}),$$

$$\begin{aligned}
 Z_2(\theta, \delta) = &\frac{1}{4\beta_3^2} \left[ 1 + \frac{1}{1 + e^{-\kappa\delta}} \left\{ 12e^{-2\kappa\delta} \right. \right. \\
 &+ (12\nu + 48)c(\theta, \delta)^{-1} \frac{e^{-\kappa\delta}\theta_\beta}{\theta_\alpha - 1} + (3\nu^2 + 12\nu) \\
 &\left. \left. \times c(\theta, \delta)^{-2} \frac{\theta_\beta^2}{(\theta_\alpha - 1)(\theta_\alpha - 2)} - \frac{2(\theta_\alpha + \theta_\alpha e^{-\kappa\delta} - 2e^{-\kappa\delta})}{(1 + e^{-\kappa\delta})(\theta_\alpha - 1)} \right\} \right],
 \end{aligned}$$

$$c(\theta, \delta) = 2\theta_\beta(1 - e^{-\kappa\delta})^{-1} \quad \text{and} \quad \nu = 2\theta_\alpha.$$

**Theorem 3.1.3.** For a stationary CIR process with  $\theta_\alpha \geq 2$ , as  $n \rightarrow \infty$  while  $\delta$  is fixed,

$$E(\hat{\kappa}) = \kappa + (n\delta)^{-1}B_3(\theta, \delta) + O(n^{-2}),$$

$$\text{Var}(\hat{\kappa}) = (n\delta)^{-1}V_4(\theta, \delta) + O(n^{-2}),$$

$$E(\hat{\alpha}) = \alpha + O(n^{-2}),$$

$$\text{Var}(\hat{\alpha}) = n^{-1}V_5(\theta, \delta) + O(n^{-2}),$$

$$E(\hat{\sigma}^2) = \sigma^2 + B_4(\theta, \delta) + O(n^{-1}) \quad \text{and}$$

$$\text{Var}(\hat{\sigma}^2) = n^{-1}V_6(\theta, \delta) + O(n^{-2}).$$

Theorem 3.1.3 conveys similar features to those in Theorem 3.1.1 for the Vasicek process, with respect to the orders of the bias and variance of the estimators. The coefficients of the bias and variance become more involved for CIR process. A major difference appears in the bias of  $\hat{\sigma}^2$ , where  $B_4(\theta)$  is a fixed constant that does not converge to 0. Hence  $\hat{\sigma}^2$  is not a consistent estimator of  $\sigma^2$ . This is due to the discretization of the diffusion function used in (2.10). The inconsistency is limited to the diffusion parameter, as the drift parameter estimators are asymptotically unbiased and consistent. This is because the discretization of the process is confined to the diffusion part only. In general, any estimation based on discretization is likely to encounter this kind of systematic bias (Lo, 1988), and  $\delta \rightarrow 0$  is required for estimators being consistent.

**Theorem 3.1.4.** For a stationary CIR process with  $\theta_\alpha \geq 2$ , let  $\hat{\theta} = (\hat{\kappa}, \hat{\alpha}, \hat{\sigma}^2)^T$  and  $\tilde{\theta} = (\kappa, \alpha, \sigma^2 - B_4(\theta, \delta))^T$ . Then see Box I.

To take into account the inconsistency of  $\hat{\sigma}^2$ , we have adjusted  $\theta$  to  $\tilde{\theta}$  in the above asymptotic normality. A difference between Theorems 3.1.4 and 3.1.2 is that the pseudo-likelihood estimators of CIR process are no longer asymptotically uncorrelated.

### 3.2. Diminishing $\delta$ analysis

We now let  $\delta \rightarrow 0$ , which will simplify the bias and variance expansions of the estimators.

The following two theorems are counterparts of Theorems 3.1.1 and 3.1.2 respectively.

**Theorem 3.2.1.** For a stationary Vasicek process and under Condition (2.1),

$$E(\hat{\kappa}) = \kappa + 4T^{-1} - \{4\kappa n^{-1} - 7/(\kappa T^2)\} + o(n^{-1} + T^{-2}),$$

$$\text{Var}(\hat{\kappa}) = 2\kappa T^{-1} + o(T^{-1}),$$

$$E(\hat{\alpha}) = \alpha + o(T^{-2}), \quad \text{Var}(\hat{\alpha}) = \sigma^2\kappa^{-2}/T + o(T^{-1}),$$

$$E(\hat{\sigma}^2) = \sigma^2 + O(n^{-1}) \quad \text{and} \quad \text{Var}(\hat{\sigma}^2) = 2\sigma^4 n^{-1} + o(n^{-1}).$$

**Theorem 3.2.2.** Let  $\hat{\theta} = (\hat{\kappa}, \hat{\alpha}, \hat{\sigma}^2)^T$  and  $\theta = (\kappa, \alpha, \sigma^2)^T$ , and under the same conditions of Theorem 3.2.1 as  $n \rightarrow \infty$

$$R_{n,\delta}(\hat{\theta} - \theta) \xrightarrow{d} N(0, \Omega_3),$$

where  $R_{n,\delta} = \text{diag}(T^{1/2}, T^{1/2}, n^{1/2})$  and  $\Omega_3 = \text{diag}(2\kappa, \sigma^2\kappa^{-2}, 2\sigma^4)$ .

Theorem 3.2.1 reveals, first of all, that the leading order bias of  $\hat{\kappa}$  is  $4/T$ , and the leading order relative bias is  $4/(\kappa T)$ , which gets larger as  $\kappa$  gets smaller (weaker mean-reverting). Secondly, the leading order variance of  $\hat{\kappa}$  and  $\hat{\alpha}$  are both of  $1/T$ , which are larger orders than that of  $\text{Var}(\hat{\sigma}^2)$ . Hence, estimation for the drift parameters are much more variable than  $\hat{\sigma}^2$ . The theorem also reveals that despite  $\hat{\alpha}$  having a larger order variance, it is almost unbiased. At the same time, estimation of  $\sigma^2$  enjoys both smaller bias and less variability, as has been observed in various empirical studies. Theorem 3.2.2 shows that  $\hat{\kappa}$  and  $\hat{\alpha}$  converge in a different rate ( $T^{-1/2}$ ) from that of  $\hat{\sigma}^2$  ( $n^{-1/2}$ ). And it also indicates that  $\hat{\kappa}$ ,  $\hat{\alpha}$  and  $\hat{\sigma}^2$  are asymptotically uncorrelated.

The following theorems are counterparts of Theorems 3.1.3 and 3.1.4.

**Theorem 3.2.3.** For a stationary CIR process, and under Condition (2.1) with  $\theta_\alpha \geq 2$ ,

$$E(\hat{\kappa}) = \kappa + 4T^{-1} + o(T^{-1}), \quad \text{Var}(\hat{\kappa}) = 2\kappa T^{-1} + o(T^{-1});$$

$$E(\hat{\alpha}) = \alpha + o(n^{-1}), \quad \text{Var}(\hat{\alpha}) = 2\alpha\theta_\beta^{-1}\kappa^{-1}T^{-1} + o(T^{-1});$$

$$E(\hat{\sigma}^2) = \sigma^2 - \frac{\sigma^2\kappa\delta}{2(\theta_\alpha - 1)} + O(n^{-1}),$$

$$\text{Var}(\hat{\sigma}^2) = \sigma^4 \left( 2 - \frac{1}{\theta_\alpha - 1} \right) n^{-1} + o(n^{-1}).$$

**Theorem 3.2.4.** Let  $\hat{\theta} = (\hat{\kappa}, \hat{\alpha}, \hat{\sigma}^2)^T$  and  $\theta = (\kappa, \alpha, \sigma^2)^T$ , under the same conditions of Theorem 3.2.3, as  $n \rightarrow \infty$ ,

$$R_{n,\delta}(\hat{\theta} - \theta) \xrightarrow{d} N(0, \Omega_4),$$

where  $R_{n,\delta} = \text{diag}(T^{1/2}, T^{1/2}, n^{1/2})$  and

$$\Omega_4 = \begin{pmatrix} 2\kappa & 2 & 0 \\ 2 & 2\alpha\theta_\beta^{-1}\kappa^{-1} & 0 \\ 0 & 0 & \sigma^4 \left( 2 - \frac{1}{\theta_\alpha - 1} \right) \end{pmatrix}.$$

Theorems 3.2.3 and 3.2.4 reveal similar features to Theorems 3.2.1 and 3.2.2 for the Vasicek process. These include (i) the leading order bias of  $\hat{\kappa}$  is still  $T^{-1}$ ; (ii) estimation of  $\kappa$  and  $\alpha$  still incurs a larger order variance as compared to the estimation of  $\sigma^2$ . Perhaps, it is a little surprising to see that the leading order bias and variance of  $\hat{\kappa}$  are identical to those of the Vasicek case. This is due to the simplification of the bias and variance coefficient functions when  $\delta \rightarrow 0$ . When  $\delta$  is fixed, the leading order bias and variance are different between Vasicek and CIR processes. Another effect of letting  $\delta \rightarrow 0$  is that the bias of  $\hat{\sigma}^2$  is  $\delta$ . Hence,  $\hat{\sigma}^2$  becomes consistent.

$$\sqrt{n}(\hat{\theta} - \tilde{\theta}) \xrightarrow{d} N(0, \Omega_2) \quad \text{where}$$

$$\Omega_2 = \begin{pmatrix} V_4(\theta, \delta) & \delta^{-1}(e^{\kappa\delta} + 1) & -A_1(\theta, \delta)\delta^{-1}(e^{\kappa\delta} - e^{-\kappa\delta}) \\ \delta^{-1}(e^{\kappa\delta} + 1) & V_5(\theta, \delta) & A_1(\theta, \delta)\delta^{-1}(e^{\kappa\delta} + 1) \\ -A_1(\theta, \delta)\delta^{-1}(e^{\kappa\delta} - e^{-\kappa\delta}) & A_1(\theta, \delta)\delta^{-1}(e^{\kappa\delta} + 1) & V_6(\theta, \delta) \end{pmatrix}$$

Box I.

3.3. Analysis on approximate likelihood approach

We report an analysis on the AMLE  $\hat{\theta}^{(1)}$  based on the one-term likelihood expansion under  $\delta \rightarrow 0$ . These results may allow us to contemplate for the general  $J$ -term AMLE  $\hat{\theta}^{(J)}$ .

Let  $h(X_i, X_{i-1}; \theta) =: \frac{\partial[\log f^{(1)}(X_i|X_{i-1}; \theta)]}{\partial \theta}$  and

$$h(X; \theta) = \sum_{i=1}^n h(X_i, X_{i-1}; \theta), \tag{3.1}$$

be the approximate likelihood score for  $J = 1$ . Write  $h(X; \theta) = (h_1(X; \theta), \dots, h_d(X; \theta))^T \in R^d$  where  $d$  is the dimension of  $\theta$ . The AMLE  $\hat{\theta}^{(1)}$  is the solution of

$$h(X; \theta) = 0. \tag{3.2}$$

Let  $h_{rs} = \frac{\partial h_r(X; \theta)}{\partial \theta_s}$  and  $h_{rst} = \frac{\partial^2 h_r(X; \theta)}{\partial \theta_s \partial \theta_t}$  for  $r, s, t \dots \in \{1, \dots, d\}$ , and

$$Z_r(\theta) = n^{-1/2}(h_r - nv_r), \quad Z_{rs}(\theta) = n^{-1/2}(h_{rs} - nv_{rs}),$$

$$Z_{rst}(\theta) = n^{-1/2}(h_{rst} - nv_{rst})$$

where  $nv_r = E(h_r)$ ,  $nv_{rs} = E(h_{rs})$  and  $nv_{rst} = E(h_{rst})$  are the expectations evaluated at the true parameter  $\theta$ .

Under  $\delta \rightarrow 0$ ,  $\hat{\theta}^{(1)}$  (in fact for all  $\hat{\theta}^{(J)}$ ) converges to  $\theta$  in probability (Ait-Sahalia, 2008) which allows inverting (3.2) near  $\theta$ . Here, without much confusion, we use  $\theta$  to denote the true parameter value as well.

Using tensor notations (McCullagh, 1987) and letting  $\hat{\beta} = n^{1/2}(\hat{\theta} - \theta) = (\hat{\beta}_1, \dots, \hat{\beta}_d)^T$ , (3.2) becomes, for  $r = 1, \dots, d$ ,

$$0 = n^{1/2}v_r + n^{1/2}Z_r(\theta) + \{nv_{rs} + n^{1/2}Z_{rs}(\theta)\}\hat{\beta}^s/n^{1/2}$$

$$+ \{nv_{rst} + n^{1/2}Z_{rst}(\theta)\}\hat{\beta}^s\hat{\beta}^t/(2n)\{1 + o_p(1)\}. \tag{3.3}$$

In (3.3), we use the tensor convention where any repeated subscripts and super-scripts are understood as summations.

Solving (3.3) iteratively, we have the following expansion for  $\hat{\beta}_r$ ,

$$\hat{\beta}_r = -v^{rs}v_s - v^{rs}Z_s + n^{-1/2}(v^{rs}v^{tu}Z_{st}Z_u$$

$$- v^{ri}v^{sj}v^{tk}v_{ijk}Z_sZ_t/2)\{1 + o_p(1)\}, \tag{3.4}$$

where  $(v^{ij})$  denotes the matrix inverse of  $(v_{ij})$ . Taking expectation on both sides of (3.4), and transforming back to  $\theta$ , the bias of  $\hat{\theta}$  is

$$E(\hat{\theta}_r - \theta_r) = -v^{rs}v_s + n^{-1}(v^{rs}v^{tu}v_{st,u}$$

$$- v^{ri}v^{sj}v^{tk}v_{ijk}v_{s,t}/2)\{1 + o(1)\}, \tag{3.5}$$

where  $v_{a,b} = E(Z_aZ_b)$  and  $v_{ab,c} = E(Z_{ab}Z_c)$ , for  $a, b, c \in \{1, \dots, d\}$ . We note that, unlike the exact likelihood score, the approximate likelihood score  $h_r$  may not have a zero mean. This means that there will be a contribution to the bias from  $v_s$ .

Let  $\theta_1 = (\kappa, \alpha)^T$  be the drift parameter,  $A_{1,t}(\psi) = -\log\{\sigma(X_t; \psi)\} - \frac{1}{2\delta}[\gamma(X_t; \psi) - \gamma(X_{t-1}; \psi)]^2$ ,  $A_{2,t}(\theta) = \int_{X_{t-1}}^{X_t} \frac{\mu_Y(\gamma(u); \theta)}{\sigma(u; \psi)} du$  and  $A_{3,t}(\theta) = \log\{1 + c_1(\gamma(X_t; \theta)|\gamma(X_{t-1}; \theta))\delta\}$ .

Define

$$m_1(\theta) = \delta^{-2}E\left\{\frac{\partial A_{3,t}(\theta)}{\partial \theta_1}\right\},$$

$$m_2(\theta) = \delta^{-1}E\left[\frac{\partial\{A_{1,t}(\psi) + A_{3,t}(\theta)\}}{\partial \psi}\right],$$

$$M_1(\theta) = \left[E\left\{\frac{\partial^2 \lambda_Y(Y_{t-1}; \theta)}{\partial \theta_1 \partial \theta_1^T}\right\}\right]^{-1},$$

$$M_2(\psi_0) = \left[E\left\{\frac{\partial^2 A_{1,t}(\theta)}{\partial \psi \partial \psi^T}\right\}\right]^{-1}$$

and  $M_3(\theta) = -M_1(\theta)E\{\partial^2 \lambda_Y(Y_{t-1}; \theta)/\partial \theta_1 \partial \psi^T\}M_2(\theta)$  where

$$\lambda_Y(y; \theta) = -\{\mu_Y^2(y; \theta) + \partial \mu_Y(y; \theta)/\partial y\}/2.$$

Derivations in the Appendix show that  $m_i(\theta)$  and  $M_i(\theta)$ ,  $i = 1, 2, 3$  are  $O(1)$ . Furthermore, we define

$$I_1(\theta) = M_1(\theta)m_1(\theta) + M_3(\theta)m_2(\theta),$$

$$I_2(\theta) = I_{2,1}(\theta) + I_{2,2}(\theta),$$

$$I_3(\theta) = M_2(\theta)m_2(\theta),$$

$$I_4(\theta) = I_{4,1}(\theta) + I_{4,2}(\theta),$$

$$Q_1(\theta) = \delta(v^{is}v^{jt}v_{s,t}), \quad i, j \in \{1, 2\} \quad \text{and}$$

$$Q_2(\theta) = (v^{is}v^{jt}v_{s,t}), \quad i, j \in \{3, \dots, d\},$$

where  $I_{2,1}(\theta) = \delta(v^{1i}v^{sj}v^{tk}v_{ijk}v_{s,t}, v^{2i}v^{sj}v^{tk}v_{ijk}v_{s,t})^T/2$ ,  $I_{2,2}(\theta) = \delta(v^{1s}v^{tu}v_{st,u}, v^{2s}v^{tu}v_{st,u})^T$ ,  $I_{4,1}(\theta) = (v^{3i}v^{sj}v^{tk}v_{ijk}v_{s,t}, \dots, v^{di}v^{sj}v^{tk}v_{ijk}v_{s,t})^T/2$  and  $I_{4,2}(\theta) = (v^{3s}v^{tu}v_{st,u}, \dots, v^{ds}v^{tu}v_{st,u})^T$ . It is shown in the Appendix that  $I_i(\theta)$ ,  $i = 1, \dots, 4$ ,  $Q_1(\theta)$  and  $Q_2(\theta)$  are  $O(1)$ . Moreover, let  $f_Y(y; \theta)$  be the stationary density of the transformed process  $Y_t$  and  $\partial \mathcal{Y}$  is the boundary set of the sample space of  $Y_t$ .

The following theorem reveals the bias and variance of the AMLE  $\hat{\theta}^{(1)} = (\hat{\theta}_1^{(1)}, \hat{\psi}^{(1)})$ , where  $\hat{\theta}_1$  is the estimator for the drift parameter  $\theta_1 = (\kappa, \alpha)$ .

**Theorem 3.3.1.** For a stationary linear drift diffusion (2.14), assume conditions under which (2.15) holds, (2.1) and  $f_Y(y; \theta) = 0$  for any  $y \in \partial \mathcal{Y}$  at the true  $\theta$ , then

$$E(\hat{\theta}_1^{(1)} - \theta_1) = \delta I_1(\theta) + T^{-1}I_2(\theta) + O(\delta^2) + O(T^{-1}\delta),$$

$$\text{Var}(\hat{\theta}_1^{(1)}) = T^{-1}Q_1(\theta) + O(T^{-1}\delta);$$

$$E(\hat{\psi}^{(1)} - \psi) = \delta I_3(\theta) + n^{-1}I_4(\theta) + O(\delta^2) + O(n^{-1}\delta) \quad \text{and}$$

$$\text{Var}(\hat{\psi}^{(1)}) = n^{-1}Q_2(\theta) + O(n^{-1}\delta).$$

The implications of Theorem 3.3.1 are the following. First of all, a bias of order  $O(\delta)$  appears in both the drift and diffusion parameters due to the likelihood approximation. We expect that this bias would be  $O(\delta^J)$  for a general  $J$  when more terms are included in the expansion; and hence will become more accurate. The other bias terms, of order  $T^{-1}$  for the drift parameters and of  $n^{-1}$  for the diffusion parameters, are comparable to those we have derived for Vasicek and CIR processes. The variance of the AMLE  $\hat{\theta}^{(1)}$  are similar to those of the Vasicek and CIR processes, with different rates between the drift and diffusion parameters.

#### 4. Bootstrap bias correction

Given the explicit bias expansion in the previous section, a simple bias correction for  $\hat{\kappa}$  for the Vasicek and CIR processes is  $\hat{\kappa}_1 = \hat{\kappa} - 4/T$  when  $\delta$  is small. This would remove the leading order bias without altering the variance. The limitation of this approach is that it would not be applicable to other processes unless similar bias expansions are established. Even a bias expansion is available as, for the case of the linear drift processes, the coefficients of the bias terms can be very involved for explicit estimation.

Phillips and Yu (2005) propose a jackknife method to correct bias in parameter estimation of diffusion processes. The proposal was motivated by the bias expansions (2.7) established for discrete time series. It consists of first dividing the entire sample of  $n$  observations into  $m$  consecutive non-overlapping blocks of observations of size  $l$  such that  $n = ml$ ; and then construct parameter estimators based on each block of observations, say  $\hat{\theta}_i$  for the  $i$ -th block. The jackknife bias-corrected estimator is

$$\hat{\theta}_j = \frac{m}{m-1} \hat{\theta} - \frac{\sum_{i=1}^m \hat{\theta}_i}{m^2 - m}.$$

They suggested using  $m = 4$  which was shown numerically to produce the best trade-off between bias reduction and variance inflation.

In conventional statistical settings, it is understood (Shao and Tu, 1995) that the jackknife tends to inflate variance more than the bootstrap when both are used for bias correction. Indeed, as shown in our simulations, the variance of the jackknife estimator can be much larger than the original estimator. This may be due to the fact that dividing the data into shorter blocks reduces the observation time, which has been shown in Theorems 3.2.1 and 3.2.3 to be the key in influencing the variability in the estimation of the drift parameters.

We propose a parametric bootstrap procedure for bias correction. The bootstrap (Efron, 1979) has been shown to be an effective method for bias correction and variance estimation for both independent and dependent observations, as summarized in Hall (1992). Although our theoretical analysis has been confined to univariate processes, the proposed bootstrap bias correction is applicable to general multivariate diffusion processes.

Let  $\hat{\theta}$  be a mean square consistent estimator of  $\theta$ . The parametric bootstrap procedure consists of the following steps:

Step 1. Generate a bootstrap sample path  $\{X_t^*\}_{t=1}^n$  with the same sampling interval  $\delta$  from  $dX_t = \mu(X_t; \hat{\theta})dt + \sigma(X_t; \hat{\theta})dB_t$ ;

Step 2. Obtain a new estimator  $\hat{\theta}^*$  from the bootstrap sample path by applying the same estimation procedure as  $\hat{\theta}$ ;

Step 3. Repeat Steps 1 and 2  $N_B$  number of times and obtain a set of bootstrap parameter estimates  $\hat{\theta}^{*,1}, \dots, \hat{\theta}^{*,N_B}$ .

Let  $\hat{\theta}^* = N_B^{-1} \sum_{b=1}^{N_B} \hat{\theta}^{*,b}$ , the bootstrap bias-corrected estimator is  $\hat{\theta}_B = 2\hat{\theta} - \hat{\theta}^*$  and the bootstrap estimates for the variance of  $\hat{\theta}$  is

$$\widehat{\text{Var}}(\hat{\theta}) = N_B^{-1} \sum_{b=1}^{N_B} (\hat{\theta}^{*,b} - \hat{\theta}^*) (\hat{\theta}^{*,b} - \hat{\theta}^*)^T.$$

Here  $A^T$  denotes matrix transpose.

In Step 1, we generate an initial value  $X_0^*$  from the stationary distribution. For a univariate process, the stationary density is known to be

$$\pi_\theta(x) = \frac{\xi(\theta)}{\sigma^2(x, \theta)} \exp \left\{ \int_{x_0}^x \frac{2\mu(t, \theta)}{\sigma^2(t, \theta)} dt \right\}.$$

If the transitional distribution of  $X_{t\delta}$  given  $X_{(t-1)\delta}$  is known, we can generate  $X_{t\delta}^*$  given  $X_{(t-1)\delta}^*$  from that distribution. If the transitional

distribution is unknown, we can use the approximate transitional density of Ait-Sahalia(1999). We may also apply the Milstein scheme (Kloeden and Platen, 2000) which is more accurate than the first-order Euler scheme.

The bootstrap bias correction shares some key features of the jackknife method. For instance, it can be applied to a general diffusion process (univariate or multivariate), and for a range of estimators, including the MLE, the pseudo-MLE and discretization based estimators. The bootstrap bias correction is justified in the following theorem.

Let  $\theta = (\theta_1, \dots, \theta_p)^T$  be a vector of parameters of the general diffusion process (1.1), and  $\hat{\theta} = (\hat{\theta}_1, \dots, \hat{\theta}_p)^T$  be a consistent estimator of  $\theta$ . Write the bootstrap bias corrected estimator  $\hat{\theta}_B = (\hat{\theta}_{B1}, \dots, \hat{\theta}_{Bp})^T$ . For  $l = 1, \dots, p$ , let  $b_{nl}(\theta) = E(\hat{\theta}_{nl}) - \theta_l$  and  $v_{nl}(\theta) = \text{Var}(\hat{\theta}_{nl})$  be the bias and variance components of  $\hat{\theta}_l$  respectively, and

$$b_{nl}(\theta) = \beta_{nl} b_{nl}^{(0)}(\theta) \quad \text{and} \quad v_{nl}(\theta) = v_{nl} v_{nl}^{(0)}(\theta)$$

so that both  $|b_{nl}^{(0)}(\theta)|$  and  $|v_{nl}^{(0)}(\theta)|$  are uniformly bounded away from infinity and zero with respect to  $n$  and  $\delta$ . Hence, both  $\beta_{nl}$  and  $v_{nl}$  are the exact orders of magnitude for the bias and variance of  $\hat{\theta}_l$  respectively.

**Theorem 4.1.** Suppose that for each  $l = 1, \dots, p$ , (i)  $\beta_{nl}^2 + v_{nl} \rightarrow 0$  as both  $n$  and  $T \rightarrow \infty$  and (ii)  $b_{nl}^{(0)}(\theta)$  and  $v_{nl}^{(0)}(\theta)$ , as functions of  $\theta$ , are twice continuously differentiable within a hypersphere  $S$  in  $R^p$  that contains the real parameter  $\theta$ ; and (iii)  $E\{b_{nl}^{(0)}(\hat{\theta})\}^2 = O(1)$ . Then,

$$E(\hat{\theta}_{Bl}) = \theta_l + o\{b_{nl}(\theta)\} \quad \text{and} \quad \text{Var}(\hat{\theta}_{Bl}) = v_{nl}(\theta) + o\{v_{nl}(\theta)\}. \quad (4.1)$$

The theorem shows that  $\hat{\theta}_B$  reduces the bias of the original estimator  $\hat{\theta}$  while having the same leading order variance as  $\hat{\theta}$ . It can be seen from (A.40) in the Appendix that the bias in the bootstrap bias corrected estimator is in fact  $O\{b_{nl}(v_{nl} + b_{nl}^2)^{1/2}\}$ , indicating a bias reduction. The processes to which the bootstrap can be applied are general processes in (1.1), that include both univariate and multivariate processes, and under both fixed and diminishing  $\delta$  situations. This indeed makes the method generally applicable. The effectiveness of the bootstrap bias correction depends on the nature of the bias function  $b_n(\theta)$ . Indeed, as seen from the proof of the theorem in the appendix, it depends on how well  $E\{b_n(\hat{\theta})\}$  approximates  $b_n(\theta)$ , which is critically dependent on the smoothness of  $b_n(\theta)$ . If this bias function is erratic, then it is harder for the bootstrap bias correction, provided the same sample size is kept the same. More sophisticated bootstrap, like the double bootstrap (Hall, 1992), may have to be employed.

#### 5. Simulation studies

We report results from simulation studies which were designed to (i) confirm the theoretical findings of Theorems 3.2.1 and 3.2.3, (ii) evaluate the performance of the proposed bootstrap bias correction, and (iii) compare the bootstrap proposal with the jackknife bias correction proposed by Phillips and Yu (2005) and the indirect estimation of diffusion processes (Gourieroux et al., 1993). Both univariate and bivariate processes were considered in the simulation. All the simulation results reported in this section were based on 5000 simulations and 1000 bootstrap resamples.

##### 5.1. Univariate processes

To confirm the theoretical results in Section 3, we simulated three sets of Vasicek and CIR processes. The parameter used for the Vasicek process were  $\theta = (\kappa, \alpha, \sigma^2) = (0.858, 0.0891, 0.00219)$

(Vasicek model 1), (0.215, 0.0891, 0.0005) (Vasicek model 2) and (0.140, 0.0891, 0.0003) (Vasicek model 3). For the CIR process,  $\theta = (\kappa, \alpha, \sigma^2) = (0.892, 0.09, 0.033)$  (CIR model 1), (0.223, 0.09, 0.008) (CIR model 2) and (0.148, 0.09, 0.005) (CIR model 3) respectively. Both Vasicek model 2 and CIR model 2 have only a quarter of the mean-reverting force of Vasicek model 1 and CIR 1 respectively. In models 3 of Vasicek and CIR, the autoregressive coefficient of the discrete time model is 0.99. Such settings are designed to check the performance of the parameter estimation in the situation of the near unit root case. We chose  $\delta = 1/12$  that corresponds to monthly observations in an annualized term. The sample size  $n$  was 120, 300, 500 and 2000. As the transitional distribution of these two processes are known, the simulated sample paths were generated from the known transitional distribution with the initial value  $X_0$  from their known stationary distributions, respectively.

Table 1 report the average bias, relative bias (R. Bias), standard deviation (SD) and root mean square error (RMSE) for CIR 1 and 2 models; and those for the Vasicek models and CIR 3 were similar and are not reported. We report, in parentheses, the asymptotic bias and standard deviation prescribed by expansions in Theorem 3.2.3. We observe that the severe bias in  $\kappa$  estimation was very clear, especially when the amount of the mean reverting was weak (Table 1(b)). At the same time, there was little bias in the estimation of  $\alpha$  and the overall quality in estimating  $\sigma^2$  was very high, even for a sample size as small as 120. These confirmed our theoretical findings. We find the difference between the simulated bias and SD and those predicted by the theoretical expansions, decreased as  $n$  and  $T$  were increased, and was very small at  $n = 2000$ .

We then applied the bootstrap bias correction to estimation of  $\kappa$  for the Vasicek and CIR models. The jackknife approach proposed by Phillips and Yu (2005) was also performed with  $m = 4$ . The simulation results are summarized in Table 2 for Vasicek and CIR models 1 and 2; and Table 3 for the near unit root case for Vasicek and CIR models 3. We see that the bootstrap bias correction effectively reduced the bias without increasing the variance of the estimation much. For the jackknife bias correction, there was some non-ignorable variance inflation. The bootstrap bias correction had less RMSE than the jackknife bias correction, as well as the original estimator. This extends to the near unit root case, as reported in Table 3. We see that when  $\kappa$  was small in both Vasicek and CIR models, the bootstrap bias correction delivers satisfactory performance by reducing the bias without inflating the variation.

Tables 2 and 3 also contain comparisons with the indirect estimation approach (Gourieroux et al., 1993; Phillips and Yu, forthcoming). The indirect approach is a simulation based method which estimates the parameter by minimizing

$$g(\vartheta) = \left\| \hat{\theta} - \frac{1}{K} \sum_{h=1}^K \hat{\theta}^{(h)}(\vartheta) \right\|$$

for  $\vartheta$  over the parameter space  $\Theta$ . Here  $\hat{\theta}$  is the original estimator,  $\hat{\theta}^{(h)}(\vartheta)$  is the estimator with the same construction as  $\hat{\theta}$  based on a sample generated from the parametric process with parameter  $\vartheta \in \Theta$ ,  $K$  is the number of simulations conducted for each  $\vartheta$  and  $\|\cdot\|$  is a norm. The indirect approach has advantages of correcting bias as well as stabilizing variance (Phillips and Yu, forthcoming). It is computationally more expensive than the bootstrap, as it simulates at many  $\vartheta$  until a minimum value  $\vartheta$  is found. The bootstrap method just uses one  $\vartheta$  at  $\hat{\theta}$ . In the simulation, we set  $K = 20$  and choose  $\|\cdot\|$  to be the  $L_2$ -norm. From the results displayed in Tables 2 and 3, we see that the proposed bootstrap approach had similar performance to the indirect approach. The indirect approach tends to have smaller variance but larger bias than the bootstrap method.

We also carried out estimation and bootstrap bias correction based on the approximate MLE of Ait-Sahalia (2002) for CIR model 2 with  $J = 2$ . This was designed to see if there were significant differences between the approximate MLEs and the pseudo-likelihood estimators of Nowman (1997). The results are reported in Table 4. We see that the use of the approximate likelihood did produce estimates which had slightly smaller bias and standard deviation. We also see that the bootstrap bias correction worked well for the approximate MLE in reducing both the bias and mean square error in  $\kappa$ -estimation.

## 5.2. Multivariate processes

To evaluate the applicability of the proposed bootstrap procedure for multivariate processes, we carry out simulations for the following bivariate processes:

$$dX_t = \kappa(\alpha - X_t)dt + \sigma(X_t)dB_t \quad (5.1)$$

where

$$X_t = \begin{pmatrix} X_{1t} \\ X_{2t} \end{pmatrix}, \quad \kappa = \begin{pmatrix} \kappa_{11} & 0 \\ \kappa_{21} & \kappa_{22} \end{pmatrix}, \quad \alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \quad \text{and} \\ \sigma(X_t) = \begin{pmatrix} \sigma_{11}X_{1t}^\rho & 0 \\ 0 & \sigma_{22}X_{2t}^\rho \end{pmatrix}$$

with  $\rho = 0$  and  $1/2$  respectively. Here  $\rho = 0$  corresponds to a bivariate Ornstein–Uhlenbeck process whose exact transitional density is known to be bivariate Gaussian, whereas  $\rho = 1/2$  corresponds to a bivariate extension of the Feller's process. We will report only simulation results for the bivariate Feller's process, as those for the bivariate Ornstein–Uhlenbeck were similar.

Unless  $\kappa_{21} = 0$ , the transitional density of the bivariate Feller's process does not admit an explicit form (Ait-Sahalia and Kimmel, 2008). We apply the approximate MLE for multivariate diffusion processes with  $J = 2$  to estimate the parameters.

We chose  $(\kappa_{11}, \kappa_{21}, \kappa_{22}, \alpha_1, \alpha_2, \sigma_{11}^2, \sigma_{22}^2) = (0.223, 0.4, 0.6, 0.09, 0.08, 0.008, 0.03)$ . The generation of the bivariate diffusion process was via the Milstein's scheme (Kloeden and Platen, 2000). We pre-run the process 1000 times before starting the real simulation, to make the simulated sample path stationary.

Table 5 summarizes the simulation performance of the AMLEs and the bootstrap bias corrected parameter estimation. We observe that, similar to the univariate case as reported in Tables 1 and 2, the estimators of the drift parameters in  $\kappa$  and  $\alpha$  had worse performance than the diffusion parameters in  $\sigma$ . It is encouraging to see that the bootstrap worked effectively in reducing both the bias and the mean square errors. We note that there was no substantial difference between the bootstrap bias corrected estimates for  $(\alpha_1, \alpha_2)$  and  $(\sigma_{11}, \sigma_{22})$  and the original estimates. This is consistent with the findings for univariate processes in Section 3.

## 6. A case study and option pricing

We analyze a Fed fund interest rate dataset consisting of 432 monthly observations from January 1963 to December 1998. This dataset has been analyzed in Ait-Sahalia (1999) to demonstrate his approximate likelihood estimation.

In addition to estimating the Vasicek and CIR processes, we computed two option prices driven by these two processes:  $P_{t,T}(\theta)$ , the price of a zero-coupon-bond at time  $t$  that pays \$ 1 at a maturity time  $T$ ; and  $C_{t,T,S,K}(\theta)$ , the price at time  $t$  of a European call option with maturity  $T$  and a strike price  $K$  on a zero-coupon bond maturing at  $S > T$ . Vasicek (1977) and Cox et al. (1985) provided the formula for the two options, as functions of parameters of the interest rate processes.



**Table 1**

Bias, Relative bias (R. bias), standard deviation (SD) and the root mean squared error (RMSE) of the pseudo-likelihood estimator for CIR models 1 and 2; figures inside the parentheses are those predicted by the theoretical expansions in Theorem 3.2.3.

n	(a) CIR model 1			
	True value	$\kappa$	$\alpha$	$\sigma^2$
120	Bias (A. Bias)	0.464 (0.400)	$2.4 \times 10^{-4}$ ( $4.3 \times 10^{-4}$ )	$8.7 \times 10^{-4}$ ( $3.5 \times 10^{-4}$ )
	R. Bias (%)	52.005	0.270	2.661
	SD (Asy. SD)	0.627 (0.431)	0.020 (0.021)	0.005 (0.004)
	RMSE	0.780	0.020	0.005
300	Bias (A. Bias)	0.179 (0.160)	$2.2 \times 10^{-4}$ ( $1.7 \times 10^{-4}$ )	$5.8 \times 10^{-4}$ ( $3.5 \times 10^{-4}$ )
	R. Bias (%)	20.107	0.250	1.778
	SD (Asy. SD)	0.334 (0.273)	0.012 (0.013)	0.003 (0.003)
	RMSE	0.380	0.012	0.003
500	Bias (A. Bias)	0.107 (0.096)	$6.4 \times 10^{-5}$ ( $1.0 \times 10^{-4}$ )	$4.9 \times 10^{-4}$ ( $3.5 \times 10^{-4}$ )
	R. Bias (%)	12.037	0.070	1.510
	SD (Asy. SD)	0.247 (0.211)	0.009 (0.01)	0.002 (0.002)
	RMSE	0.269	0.009	0.002
2000	Bias (A. Bias)	0.025 (0.024)	$3.5 \times 10^{-5}$ ( $3.0 \times 10^{-5}$ )	$3.5 \times 10^{-4}$ ( $3.5 \times 10^{-4}$ )
	R. Bias (%)	2.805	0.039	1.061
	SD (Asy. SD)	0.112 (0.106)	0.005 (0.005)	0.001 (0.001)
	RMSE	0.115	0.005	0.001
n	(b) CIR model 2			
	True value	$\kappa$	$\alpha$	$\sigma^2$
120	Bias (A. Bias)	0.509 (0.400)	$1.2 \times 10^{-3}$ ( $1.7 \times 10^{-3}$ )	$1.5 \times 10^{-4}$ ( $2.9 \times 10^{-5}$ )
	R. Bias (%)	228.251	1.343	1.796
	SD (Asy. SD)	0.507 (0.216)	0.036 (0.042)	0.001 (0.001)
	RMSE	0.719	0.036	0.001
300	Bias (A. Bias)	0.185 (0.160)	$9.2 \times 10^{-4}$ ( $7.0 \times 10^{-4}$ )	$8.7 \times 10^{-5}$ ( $2.9 \times 10^{-5}$ )
	R. Bias (%)	82.836	1.018	1.062
	SD (Asy. SD)	0.222 (0.136)	0.025 (0.026)	0.0007 (0.0006)
	RMSE	0.289	0.025	0.001
500	Bias (A. Bias)	0.108 (0.096)	$3.7 \times 10^{-4}$ ( $4.1 \times 10^{-4}$ )	$5.5 \times 10^{-5}$ ( $2.9 \times 10^{-5}$ )
	R. Bias (%)	48.612	0.408	0.669
	SD (Asy. SD)	0.148 (0.106)	0.019 (0.02)	0.0005 (0.0005)
	RMSE	0.183	0.019	0.001
2000	Bias (A. Bias)	0.025 (0.024)	$9.2 \times 10^{-5}$ ( $1.0 \times 10^{-4}$ )	$2.9 \times 10^{-5}$ ( $2.9 \times 10^{-5}$ )
	R. Bias (%)	11.145	0.102	0.346
	SD (Asy. SD)	0.058 (0.053)	0.009 (0.01)	$2.6 \times 10^{-4}$ ( $2.5 \times 10^{-4}$ )
	RMSE	0.063	0.009	$2.6 \times 10^{-4}$

We first estimated the parameters of the underlying diffusion processes (Vasicek and CIR) by the maximum likelihood method, and carried out the bootstrap bias correction. Then, we calculated the option prices  $P_{t,T}(\theta)$  and  $C_{t,T,S,K}(\theta)$  by plugging-in the parameter estimates of Vasicek or CIR process with  $t = 0, T = 1, S = 3$  and the initial interest rate at 5%. The face value of the European Call option on a three year discount bond was \$100 with a strike price  $K = \$90$ .

The bootstrap was used to estimate both the bias of the parameter estimates and the option prices, as well as their standard deviations, based on 1000 resamples. The bootstrap implementation for the option prices were readily made by extending Steps 2 and 3 in the procedure outlined in Section 4, to include computation of the option prices in each resample. The empirical results are reported in Table 6. It is observed that the bootstrap bias estimates (Estimated Bias) were quite large in both  $\hat{\kappa}$  and the option price  $\hat{C}(0, 1, 3, 90)$ . It was alarming to see a large under-estimation (more than 10%) in  $\hat{C}(0, 1, 3, 90)$ . Also, the bootstrap estimate of the standard deviation for both  $\kappa$  and  $C(0, 1, 3, 90)$  was quite large too. The large variability in the option price should be taken into consideration, and indicates the difficulties in producing accurate estimated prices. The empirical analysis also indicated that the European call option is more affected by the biased parameter estimates of the underlying interest rate process than the zero-coupon bond. We also supplied,

in parentheses, the estimated standard deviation based on the leading order variance terms prescribed by Theorems 3.2.1 and 3.2.3, which were comparable with the bootstrap estimates.

### 7. Discussion

The estimation of the drift parameters in diffusion processes has been known to be challenging when the process is lacking dynamics. The challenge was found as early as Merton (1980) who studied the estimation of the average return  $\mu$  in the Black–Scholes diffusion process  $dX_t/X_t = \mu dt + \sigma dB_t$  where  $X_t$  is the price process of a financial security. Our analysis reported in Theorems 3.2.1–3.2.4 quantifies the underlying sources of the challenge for two commonly used interest rate processes, and the analysis reported in Theorem 3.3.1 extends to the general linear drift diffusion processes using Ait-Sahalia (2002)’s AMLE as the estimator. The main message from these analyses is that it is  $T$ , the total observation time, rather than the sample size  $n$ , that governs the bias and/or variance in estimation of  $\kappa$  and  $\alpha$ . This is different from estimation for discrete time series where  $n$  defines the bias and variability of estimators. Our analysis is based on continuous-time diffusion processes, and improves the heuristic justification used in Phillips and Yu (2005) which was based on results like (2.7) from discrete time series. And most importantly, the results in these theorems nicely explain various empirical results reported in the literature.

**Table 2**  
Comparisons of bias corrections for Vasicek and CIR models 1 and 2,  $\hat{\kappa}_j$ ,  $\hat{\kappa}_B$  and  $\hat{\kappa}_I$  are, respectively, the jackknife, the bootstrap and the indirect bias corrected estimators for  $\kappa$ .

n		(a) Vasicek model 1				CIR model 1			
		$\hat{\kappa}$	$\hat{\kappa}_j$	$\hat{\kappa}_B$	$\hat{\kappa}_I$	$\hat{\kappa}$	$\hat{\kappa}_j$	$\hat{\kappa}_B$	$\hat{\kappa}_I$
120	Bias	0.481	-0.120	0.001	0.03	0.464	-0.122	0.002	0.022
	R. Bias (%)	56.039	14.941	0.118	4.219	52.005	13.650	0.178	2.677
	SD	0.659	0.767	0.623	0.611	0.627	0.730	0.651	0.602
	RMSE	0.816	0.778	0.623	0.612	0.780	0.739	0.651	0.603
300	Bias	0.181	-0.026	-0.003	-0.006	0.179	-0.027	-0.004	-0.0325
	R. Bias (%)	21.082	3.070	0.406	0.702	20.107	3.094	0.447	3.79
	SD	0.329	0.353	0.321	0.326	0.334	0.365	0.326	0.326
	RMSE	0.375	0.354	0.321	0.326	0.380	0.366	0.326	0.328
500	Bias	0.111	0.005	0.001	-0.006	0.107	-0.008	0.007	0.002
	R. Bias (%)	12.880	0.586	0.073	0.720	12.037	0.842	0.826	0.258
	SD	0.240	0.250	0.235	0.240	0.247	0.257	0.245	0.248
	RMSE	0.265	0.250	0.235	0.240	0.269	0.257	0.245	0.248
n		(b) Vasicek model 2				CIR model 2			
		$\hat{\kappa}$	$\hat{\kappa}_j$	$\hat{\kappa}_B$	$\hat{\kappa}_I$	$\hat{\kappa}$	$\hat{\kappa}_j$	$\hat{\kappa}_B$	$\hat{\kappa}_I$
120	Bias	0.507	-0.112	0.032	0.078	0.509	-0.088	0.030	0.097
	R. Bias (%)	236.344	51.974	14.774	36.416	228.251	39.283	13.579	43.497
	SD	0.519	0.645	0.510	0.456	0.507	0.623	0.501	0.486
	RMSE	0.726	0.655	0.511	0.461	0.719	0.630	0.502	0.495
300	Bias	0.191	-0.029	0.0018	0.0022	0.185	-0.032	0.008	-0.032
	R. Bias (%)	88.985	13.465	0.829	1.020	82.836	14.428	3.461	14.920
	SD	0.221	0.261	0.219	0.194	0.222	0.265	0.226	0.204
	RMSE	0.292	0.262	0.219	0.194	0.289	0.267	0.226	0.208
500	Bias	0.114	-0.011	0.002	-0.016	0.108	-0.0161	0.003	-0.015
	R. Bias (%)	53.033	5.230	0.861	7.612	48.612	7.209	1.325	6.728
	SD	0.150	0.170	0.147	0.139	0.148	0.167	0.150	0.139
	RMSE	0.189	0.171	0.147	0.140	0.183	0.168	0.150	0.140

**Table 3**  
Comparisons of bias corrections for Vasicek and CIR models 3, the near unit root case,  $\hat{\kappa}_j$ ,  $\hat{\kappa}_B$  and  $\hat{\kappa}_I$  are, respectively, the jackknife, the bootstrap and the indirect bias corrected estimators for  $\kappa$ .

n		Vasicek model 3				CIR model 3			
		$\hat{\kappa}$	$\hat{\kappa}_j$	$\hat{\kappa}_B$	$\hat{\kappa}_I$	$\hat{\kappa}$	$\hat{\kappa}_j$	$\hat{\kappa}_B$	$\hat{\kappa}_I$
120	Bias	0.519	-0.058	0.050	0.105	0.520	-0.075	0.059	0.041
	R. Bias (%)	362.596	40.573	34.655	73.458	349.932	50.520	39.597	19.17
	SD	0.498	0.658	0.501	0.460	0.507	0.613	0.504	0.483
	RMSE	0.719	0.660	0.503	0.466	0.719	0.618	0.507	0.484
300	Bias	0.190	-0.026	0.010	0.015	0.192	-0.029	0.007	0.025
	R. Bias (%)	133.046	17.819	6.713	10.876	129.455	19.700	4.459	17.670
	SD	0.206	0.250	0.213	0.196	0.222	0.251	0.214	0.207
	RMSE	0.280	0.251	0.213	0.196	0.289	0.253	0.214	0.209
500	Bias	0.110	-0.011	0.003	-0.015	0.111	-0.016	0.003	-0.012
	R. Bias (%)	76.593	7.700	2.000	10.624	74.514	10.900	1.830	8.450
	SD	0.130	0.159	0.147	0.115	0.135	0.156	0.133	0.121
	RMSE	0.170	0.159	0.147	0.116	0.135	0.157	0.133	0.122

**Table 4**  
Parameters estimation and bias correction for CIR model 2 based on the approximate likelihood method of Ait-Sahalia (1999, 2002).

n		CIR model 2				
		$\hat{\kappa}$	$\hat{\alpha}$	$\hat{\sigma}^2$	$\hat{\kappa}_j$	$\hat{\kappa}_B$
True value		0.223	0.09	0.008	0.223	0.223
120	Bias	0.494	0.004	$1 \times 10^{-4}$	-0.072	0.035
	R. Bias (%)	221.684	4.778	1.507	32.412	15.559
	SD	0.490	0.058	0.001	0.596	0.514
	RMSE	0.696	0.058	0.001	0.601	0.516
300	Bias	0.180	0.001	$6 \times 10^{-5}$	-0.035	0.013
	R. Bias (%)	80.559	1.349	0.700	15.803	5.618
	SD	0.223	0.0262	0.001	0.262	0.234
	RMSE	0.286	0.0262	0.001	0.265	0.234
500	Bias	0.1001	$7 \times 10^{-4}$	$4 \times 10^{-5}$	-0.022	-0.003
	R. Bias (%)	45.279	0.834	0.478	9.806	1.493
	SD	0.147	0.019	0.001	0.166	0.151
	RMSE	0.178	0.019	0.001	0.167	0.151

**Table 5**

Bias, relative bias (R. bias), standard deviation (SD) and the root mean squared error (RMSE) of the approximate likelihood estimator (Ait-Sahalia, 2008; Ait-Sahalia and Kimmel, 2008) for a Bivariate Feller's process; figures in parentheses are those for the bootstrap bias corrected estimators.

Bivariate Feller process								
$n = 120$	$\kappa_{11}$	$\kappa_{21}$	$\kappa_{22}$	$\alpha_1$	$\alpha_2$	$\sigma_1^2$	$\sigma_2^2$	
True value	0.223	0.4	0.6	0.09	0.08	0.008	0.03	
Bias	0.406 (0.135)	0.435 (0.105)	0.446 (0.129)	0.008 (0.002)	0.004 (-0.001)	-0.0003 (-0.0001)	-0.0007 (-0.0005)	
R. bias(%)	182.260 (60.53)	108.832 (26.25)	74.319 (21.48)	9.845 (2.122)	5.263 (1.273)	3.69 (1.245)	2.313 (1.672)	
SD	0.464 (0.583)	0.683 (0.746)	0.514 (0.638)	0.031 (0.034)	0.030 (0.031)	0.001 (0.001)	0.0040 (0.004)	
RMSE	0.617 (0.598)	0.810 (0.753)	0.680 (0.651)	0.0326 (0.034)	0.030 (0.031)	0.001 (0.001)	0.004 (0.004)	
$n = 300$	$\kappa_{11}$	$\kappa_{21}$	$\kappa_{22}$	$\alpha_1$	$\alpha_2$	$\sigma_1^2$	$\sigma_2^2$	
Bias	0.122 (-0.008)	0.198 (0.088)	0.243 (0.028)	0.004 (0.003)	0.007 (0.004)	$1 \times 10^{-4}$ ( $-1 \times 10^{-5}$ )	$3 \times 10^{-4}$ (-0.0002)	
R. bias(%)	54.698 (3.587)	49.611 (2.191)	40.478 (4.750)	4.980 (4.020)	8.529 (5.990)	1.26 (0.164)	0.321 (0.776)	
SD	0.153 (0.187)	0.399 (0.438)	0.327 (0.331)	0.0188 (0.02)	0.0229 (0.023)	0.0006 (0.0007)	0.0027 (0.0027)	
RMSE	0.195 (0.187)	0.446 (0.446)	0.407 (0.332)	0.019 (0.020)	0.023 (0.023)	0.0007 (0.0007)	0.0027 (0.0027)	
$n = 500$	$\kappa_{11}$	$\kappa_{21}$	$\kappa_{22}$	$\alpha_1$	$\alpha_2$	$\sigma_1^2$	$\sigma_2^2$	
Bias	0.089 (0.008)	0.040 (-0.004)	0.097 (-0.01)	0.005 (-0.002)	0.004 (0.001)	$-5 \times 10^{-5}$ ( $-6 \times 10^{-6}$ )	0.0002 (-0.0001)	
R. bias(%)	40.191 (3.490)	25.17 (1.010)	16.170 (1.671)	5.560 (2.99)	5.010 (1.51)	0.656 (0.077)	0.64 (0.359)	
SD	0.098 (0.108)	0.269 (0.277)	0.240 (0.248)	0.015 (0.016)	0.017 (0.018)	0.0005 (0.0005)	0.0024 (0.0022)	
RMSE	0.132 (0.109)	0.271 (0.277)	0.259 (0.250)	0.0158 (0.017)	0.018 (0.018)	0.0005 (0.0005)	0.0024 (0.0025)	

**Table 6**

Results for a case study:  $\hat{P}$  and  $\hat{C}$  are the estimated prices for the discount bond and European call options respectively; estimated bias, bootstrap estimates and  $\hat{SD}$  are respectively the bootstrap estimate of the bias, the bootstrap bias corrected estimate and the bootstrap estimation of the standard deviation; figures in parentheses are the asymptotic standard deviation (Asy. SD) based on the leading order variance given Theorems 3.2.1 and 3.2.3.

	$\hat{\kappa}$	$\hat{\alpha}$	$\hat{\sigma}^2$	$\hat{P}$	$\hat{C}$
(a) Under Vasicek process					
Estimates	0.261	0.07	0.0005	0.846	3.03
Estimated bias	0.125	$2 \times 10^{-5}$	$2 \times 10^{-6}$	-0.004	-0.313
Bootstrap estimates	0.136	0.07	0.0005	0.852	3.67
$\hat{SD}$ (Asy. SD)	0.17 (0.12)	0.015 (0.014)	$3.5 \times 10^{-5}$ ( $3.4 \times 10^{-5}$ )	0.015	1.146
(b) Under CIR process					
Estimates	0.146	0.07	0.0043	0.852	2.64
Estimated bias	0.127	$8 \times 10^{-4}$	$3 \times 10^{-5}$	-0.004	-0.294
Bootstrap estimates	0.018	0.069	0.0043	0.860	3.39
$\hat{SD}$ (Asy. SD)	0.152 (0.11)	0.02 (0.02)	$3.0 \times 10^{-4}$ ( $3.0 \times 10^{-4}$ )	0.014	0.996

The proposed bootstrap bias correction is generally applicable. A reason for the proposed parametric bootstrap method working more effectively than the jackknife method is its re-creation of the full observation in each resampling, that fully utilizes the amount of data available and the model for the process. While we have gained understanding on parameter estimation for linear drift diffusion processes, there is a need to understand more on estimation for multivariate processes, in particular estimation of parameters that control the correlation between components of the process. Another important issue is to extend the analysis for the general AMEL  $\hat{\theta}^{(j)}$ . We hope future research will address these issues.

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**Appendix. Proofs**

We will only present the proofs of Theorems 3.1.3, 3.1.4 and 3.2.3 for the CIR process, and Theorems 3.3.1 and 4.1. The proofs of Theorems 3.1.1, 3.1.2, 3.2.1 and 3.2.2 for Vasicek processes are very similar to those for CIR processes, if not easier, as the observed sample path is multivariate normally distributed. The proof of Theorem 3.2.4 is closely related to that of Theorem 3.1.4. Detailed

proofs for all the theorems are given in a technical report (Tang and Chen, 2008) which is available from the authors.

**Proof of Theorem 3.1.3.** We first note two basic facts regarding a sample  $\{X_i\}_{i=1}^n$  from a stationary CIR process: (i) for any  $j > i$ ,  $c \cdot X_j|X_i \sim \chi^2_\nu(\lambda)$  where  $\nu = \frac{4\kappa\alpha}{\sigma^2}$ ,  $\lambda = cX_i e^{-(j-i)\kappa\delta}$  and  $c = \frac{4\kappa}{\sigma^2(1-e^{-(j-i)\kappa\delta})}$ ; and (ii)  $X_i \sim \Gamma(\theta_\alpha, \theta_\beta)$  where  $\theta_\alpha = \frac{2\kappa\alpha}{\sigma^2}$  and  $\theta_\beta = \frac{2\kappa}{\sigma^2}$ . Based on (i) and (ii), it can be shown that for  $j < i$ ,  $cX_j|X_i \sim \chi^2_\nu(\lambda)$  where  $\nu = \frac{4\kappa\alpha}{\sigma^2}$ ,  $\lambda = cX_i e^{-(i-j)\kappa\delta}$  and  $c = \frac{4\kappa}{\sigma^2(1-e^{-(i-j)\kappa\delta})}$ . In other words, the backward transitional distribution of a stationary CIR process follows the same non-central chi-square distribution. This implies that  $E(X_j|X_i) = c^{-1}(\nu + \lambda) = X_i e^{-\kappa\delta_{ij}} + \alpha(1 - e^{-\kappa\delta_{ij}})$ , where  $\delta_{ij} = \delta|j - i|$ .

We then establish an expansion for  $\hat{\beta}_1$ . Let  $t_{1i} = X_i - \mu_1$ ,  $t_{2i} = X_i^{-1} - \mu_2$  and  $t_{3i} = X_{i-1}^{-1}X_i - \mu_3$ , where  $\mu_1 = E(X_i) = \alpha$ ,  $\mu_2 = E(X_i^{-1}) = \frac{\theta_\beta}{\theta_\alpha - 1}$  and  $\mu_3 = E(X_{i-1}^{-1}X_i) = \frac{\theta_\alpha - e^{-\kappa\delta}}{\theta_\alpha - 1}$  by using the above two facts. Define  $t_a = n^{-1} \sum_{i=1}^n t_{ai}$  and  $\tilde{t}_a = n^{-1} \sum_{i=1}^n t_{a(i-1)} = t_a + n^{-1}(t_{a0} - t_{an})$  for  $a = 1, 2, 3$ , we have,

$$\begin{aligned} \hat{\beta}_1 &= \frac{\mu_1\mu_2 - \mu_3 + \mu_1\tilde{t}_2 + \mu_2\tilde{t}_1 + \mu_1\tilde{t}_2 - t_3}{\mu_1\mu_2 - 1 + \mu_1\tilde{t}_2 + \mu_2\tilde{t}_1 + \tilde{t}_1\tilde{t}_2} \\ &= \frac{\mu_u}{\mu_d} - \frac{1}{\mu_d}t_3 + \frac{\mu_1}{\mu_d} \left(1 - \frac{\mu_u}{\mu_d}\right) t_2 \\ &\quad + \frac{\mu_2}{\mu_d} \left(1 - \frac{\mu_u}{\mu_d}\right) t_1 + \frac{1}{\mu_d} \left(1 - \frac{\mu_u}{\mu_d}\right) t_1 t_2 \\ &\quad - \frac{1}{\mu_d^2} \left(1 - \frac{\mu_u}{\mu_d}\right) (\mu_1^2 t_2^2 + \mu_2^2 t_1^2 + 2\mu_1\mu_2 t_1 t_2) \\ &\quad + \frac{1}{\mu_d^2} (\mu_2 t_1 t_3 + \mu_1 t_2 t_3) + R_n, \end{aligned} \tag{A.1}$$

where  $\mu_u = \mu_1\mu_2 - \mu_3$  and  $\mu_d = \mu_1\mu_2 - 1$ . The  $R_n$  is  $O_p(n^{-3/2})$  if we consider  $\delta$  as fixed, and is  $O_p(T^{-3/2})$  if  $\delta$  diminishes. And the last equation is established by noting that  $\tilde{t}_a = t_a + O_p(n^{-1})$ . We note here that all the expansions in this paper, including the above (A.1), are made under the assumption that  $\kappa$  is a fixed quantity.

As  $\mu_u/\mu_d = e^{-\kappa\delta} = \beta_1$ , it is seen that  $\hat{\beta}_1$  is a consistent estimator of  $\beta_1$ . To get the asymptotic bias of  $\hat{\beta}_1$ , we need to establish  $\text{var}(t_1)$ ,  $\text{var}(t_2)$ ,  $\text{cov}(t_1, t_2)$ ,  $\text{cov}(t_1, t_3)$  and  $\text{cov}(t_2, t_3)$ . We see that for  $a, b \in \{1, 2, 3\}$   $\text{cov}(t_a, t_b) = n^{-2} \sum_{i=1}^n \sum_{j=1}^n \text{cov}(t_{ai}, t_{bj})$ . Thus, each pair of  $\text{cov}(t_{ai}, t_{bj})$  is needed. It is straightforward to show that

$$\begin{aligned} \text{cov}(t_{1i}, t_{1j}) &= \text{cov}(X_i, X_j e^{-\kappa\delta_{ij}}) \\ &= e^{-\kappa\delta_{ij}} \text{var}(X_i) = e^{-\kappa\delta_{ij}} \theta_\alpha / \theta_\beta^2. \end{aligned} \tag{A.2}$$

Similarly

$$\begin{aligned} \text{cov}(t_{2i}, t_{1j}) &= \text{cov}\{X_i^{-1}, E(X_j|X_i)\} \\ &= -e^{-\kappa\delta_{ij}} / (\theta_\alpha - 1). \end{aligned} \tag{A.3}$$

To establish  $\text{cov}(t_{2i}, t_{2j})$ , we note that

$$\begin{aligned} E(X_j^{-1}|X_i) &= c \sum_{k=0}^{\infty} \frac{(\lambda/2)^k}{k!} \frac{1/2e^{-\lambda/2}}{\nu/2 + k - 1} \\ &= \theta_\beta (1 - e^{-\kappa\delta_{ij}})^{-1} \sum_{k=0}^{\infty} \frac{(\lambda/2)^k}{k!} \frac{e^{-\lambda/2}}{\theta_\alpha + k - 1} \quad \text{and} \\ E\{X_i^{-1}(\lambda/2)^k e^{-\lambda/2}\} &= \frac{\theta_\beta \Gamma(\theta_\alpha + k - 1)}{\Gamma(\theta_\alpha)} (1 - e^{-\kappa\delta_{ij}})^{\theta_\alpha - 1} (e^{-\kappa\delta_{ij}})^k. \end{aligned}$$

These imply that

$$\begin{aligned} E(X_i^{-1}X_j^{-1}) &= \theta_\beta (1 - e^{-\kappa\delta_{ij}})^{-1} E \left\{ \sum_{k=0}^{\infty} \frac{(\lambda/2)^k}{k!} \frac{e^{-\lambda/2} X_i^{-1}}{\theta_\alpha + k - 1} \right\} \\ &= \frac{\theta_\beta^2}{(\theta_\alpha - 1)^2} F(1, 1, \theta_\alpha; e^{-\kappa\delta_{ij}}). \end{aligned} \tag{A.4}$$

Note the last equation in (A.4) applies the definition of a hypergeometric function; see Abramowitz and Stegun (1965) for details. Therefore,

$$\begin{aligned} \text{cov}(t_{2i}, t_{2j}) &= E(X_i^{-1}X_j^{-1}) - \mu_2^2 \\ &= \frac{\theta_\beta^2}{(\theta_\alpha - 1)^2} \{F(1, 1, \theta_\alpha; e^{-\kappa\delta_{ij}}) - 1\}. \end{aligned} \tag{A.5}$$

As  $F(a_1, a_2, b; Z)$  converges absolutely if  $b - a_1 - a_2 \geq 0$ ,  $E(X_i^{-1}X_j^{-1}) < \infty$  if  $\theta_\alpha \geq 2$ . To derive  $\text{cov}(t_{3i}, t_{1j})$ , we note for  $j < i$ ,

$$\begin{aligned} \text{cov}(t_{3i}, t_{1j}) &= \text{cov}\{X_{i-1}^{-1}X_i, X_{i-1}e^{-\kappa\delta_{ij}}e^{\kappa\delta}\} \\ &= e^{-\kappa\delta_{ij}}e^{\kappa\delta} \text{cov}(X_{i-1}^{-1}X_i, X_{i-1}) \\ &= -e^{-\kappa\delta_{ij}}e^{\kappa\delta} \alpha (1 - e^{-\kappa\delta}) / (\theta_\alpha - 1). \end{aligned} \tag{A.6}$$

If  $j \geq i$ ,

$$\begin{aligned} \text{cov}(t_{3i}, t_{1j}) &= \text{cov}\{X_{i-1}^{-1}X_i, X_i e^{-\kappa\delta_{ij}}\} \\ &= e^{-\kappa\delta_{ij}} \{E(X_{i-1}^{-1}X_i^2) - \mu_1\mu_3\}. \end{aligned} \tag{A.7}$$

As  $E(X_i^2|X_{i-1}) = X_{i-1}^2 e^{-2\kappa\delta} + X_{i-1} e^{-\kappa\delta} (1 - e^{-\kappa\delta}) \frac{2(\theta_\alpha + 1)}{\theta_\beta} + \frac{\theta_\alpha^2 + \theta_\alpha}{\theta_\beta^2} (1 - e^{-\kappa\delta})^2$ , (A.7) implies that for  $j \geq i$

$$\begin{aligned} \text{cov}(t_{3i}, t_{1j}) &= e^{-\kappa\delta_{ij}} (1 - e^{-\kappa\delta}) \left\{ \frac{2e^{-\kappa\delta}(\theta_\alpha + 1)}{\theta_\beta} - \alpha(1 + e^{-\kappa\delta}) \right. \\ &\quad \left. - \frac{\alpha}{\theta_\alpha - 1} + \frac{\theta_\alpha^2 + \theta_\alpha}{\theta_\beta(\theta_\alpha - 1)} (1 - e^{-\kappa\delta}) \right\}. \end{aligned}$$

The value of  $\text{cov}(t_{3i}, t_{2j})$  also depends on the order of  $i$  and  $j$ . When  $j < i$ ,

$$\begin{aligned} \text{cov}(t_{3i}, t_{2j}) &= \text{cov}\{X_{i-1}^{-1}E(X_i|X_{i-1}), X_j^{-1}\} \\ &= \alpha(1 - e^{-\kappa\delta}) \text{cov}(X_{i-1}^{-1}, X_j^{-1}) \\ &= \frac{\alpha(1 - e^{-\kappa\delta})\theta_\beta^2}{(\theta_\alpha - 1)^2} \{F(1, 1, \theta_\alpha; e^{-\kappa\delta(i-j-1)}) - 1\}. \end{aligned} \tag{A.8}$$

For  $j \geq i$ , by definition  $\text{cov}(t_{3i}, t_{2j}) = E(X_{i-1}^{-1}X_i X_j^{-1}) - \mu_2\mu_3$ . As

$$\begin{aligned} E(X_{i-1}^{-1}|X_i) &= c_1 / 2 \sum_{k=0}^{\infty} \frac{(\lambda_1/2)^k e^{-\lambda_1/2}}{k!(\theta_\alpha + k - 1)} \quad \text{and} \\ E(X_j^{-1}|X_i) &= c_2 / 2 \sum_{l=0}^{\infty} \frac{(\lambda_2/2)^l e^{-\lambda_2/2}}{l!(\theta_\alpha + l - 1)} \end{aligned} \tag{A.9}$$

where  $c_1 = 2\theta_\beta(1 - e^{\kappa\delta})^{-1}$ ,  $c_2 = 2\theta_\beta(1 - e^{-\kappa\delta_{ij}})^{-1}$ ,  $\lambda_1 = c_1 X_i e^{-\kappa\delta}$  and  $\lambda_2 = c_2 X_i e^{-\kappa\delta_{ij}}$ , we have

$$\begin{aligned} E(X_{i-1}^{-1}X_i X_j^{-1}) &= E\{X_i E(X_{i-1}^{-1}|X_i) E(X_j^{-1}|X_i)\} \\ &= \theta_\beta^2 (1 - e^{-\kappa\delta})^{-1} (1 - e^{-\kappa\delta_{ij}})^{-1} \\ &\quad \times \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{E\{X_i (\lambda_1/2)^k (\lambda_2/2)^l e^{-(\lambda_1/2 + \lambda_2/2)}\}}{k!l!(\theta_\alpha + k - 1)(\theta_\alpha + l - 1)} \\ &= C(\theta_\alpha) \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} S_{k,l} \end{aligned} \tag{A.10}$$

where  $C(\theta_\alpha) = \frac{\theta_\beta}{\Gamma(\theta_\alpha)}(1-e^{-\kappa\delta})^{\theta_\alpha}(1-e^{-\kappa\delta_{ij}})^{\theta_\alpha}(1-e^{-\kappa\delta}e^{-\kappa\delta_{ij}})^{-(\theta_\alpha+1)}$ ,  
 $S_{k,l} = \frac{\Gamma(\theta_\alpha+l+k+1)Z_1^k Z_2^l}{(\theta_\alpha+k-1)k!(\theta_\alpha+l-1)l!}$ ,  $Z_1 = \frac{(1-e^{-\kappa\delta_{ij}})e^{-\kappa\delta}}{1-e^{-\kappa\delta}e^{-\kappa\delta_{ij}}}$  and  $Z_2 = \frac{(1-e^{-\kappa\delta})e^{-\kappa\delta_{ij}}}{1-e^{-\kappa\delta_{ij}}e^{-\kappa\delta}}$ .

Simplifying the double sum over  $k$  and  $l$  using the properties of the hypergeometric function, we show that when  $j \geq i$

$$\text{cov}(t_{3i}, t_{2j}) = A_{1,ij} + A_{2,ij}, \tag{A.11}$$

where

$$A_{1,ij} = -e^{\kappa\delta}(1-e^{-\kappa\delta}) \frac{\theta_\beta(\theta_\alpha-2)}{(\theta_\alpha-1)} \times \sum_{l=1}^{\infty} \frac{(e^{-\kappa\delta_{ij}}e^{-\kappa\delta})^l \Gamma(l+1)\Gamma(\theta_\alpha-2)}{\Gamma(\theta_\alpha-1+l)}$$

$$A_{2,ij} = (1-e^{-\kappa\delta}) \frac{\theta_\beta(\theta_\alpha-2)}{\theta_\alpha-1} \times \sum_{l=1}^{\infty} \frac{(e^{-\kappa\delta_{ij}}e^{-\kappa\delta})^l \Gamma(l+2)\Gamma(\theta_\alpha-2)}{\Gamma(\theta_\alpha+l)}.$$

Summarizing results in (A.2), (A.3), (A.5)–(A.8) and (A.11), the bias of  $\hat{\beta}_1$  for a fixed  $\delta$  is given by

$$E(\hat{\beta}_1) - \beta_1 = n^{-2}(1-e^{-\kappa\delta}) \left( \sum_{i=1}^n \sum_{j=1}^n [-e^{-\kappa\delta_{ij}} - \theta_\alpha^2 \{F(1, 1, \theta_\alpha; e^{-\kappa\delta_{ij}}) - 1\} + \theta_\alpha e^{-\kappa\delta_{ij}}] + \sum_{j=1}^n \sum_{i=j+1}^n [-\theta_\alpha e^{\kappa\delta} e^{-\kappa\delta_{ij}} + \theta_\alpha^2 \{F(1, 1, \theta_\alpha; e^{-\kappa\delta(i-1)j}) - 1\}] + \sum_{i=1}^n \sum_{j=i}^n \{C_{1,\theta} e^{-\kappa\delta_{ij}} - \theta_\alpha(\theta_\alpha-1)(\theta_\alpha-2) \times (e^{\kappa\delta} S_{1,ij} - S_{2,ij})\} \right) + O(n^{-2}), \tag{A.12}$$

where  $C_{1,\theta} = (\theta_\alpha-1)\{2e^{-\kappa\delta}(\theta_\alpha+1) - \theta_\alpha(1+e^{-\kappa\delta}) - \frac{\theta_\alpha}{\theta_\alpha-1} + \frac{\theta_\alpha^2+\theta_\alpha}{(\theta_\alpha-1)}(1-e^{-\kappa\delta})\}$ ,  $S_{1,ij} = \sum_{l=1}^{\infty} \frac{(e^{-\kappa\delta_{ij}}e^{-\kappa\delta})^l \Gamma(l+1)\Gamma(\theta_\alpha-2)}{\Gamma(\theta_\alpha-1+l)}$  and  $S_{2,ij} = \sum_{l=1}^{\infty} \frac{(e^{-\kappa\delta_{ij}}e^{-\kappa\delta})^l \Gamma(l+2)\Gamma(\theta_\alpha-2)}{\Gamma(\theta_\alpha+l)}$ .

We derive  $\text{var}(\hat{\beta}_1)$  next. For taking variance operation on (A.1), we note that  $\text{var}(t_3)$  is needed. For  $j > i$

$$\text{cov}(t_{3i}, t_{3j}) = \text{cov}\{X_{i-1}^{-1}X_i, X_{j-1}^{-1}E(X_j|X_{j-1})\} = \alpha(1-e^{-\kappa\delta})\text{cov}(t_{3i}, t_{2j}) = (1-e^{-\kappa\delta})\alpha(A_{1,ij} + A_{2,ij}). \tag{A.13}$$

By the symmetric of  $i$  and  $j$ ,  $\text{cov}(t_{3i}, t_{3j})$  has the sample expression (A.13) for  $j < i$ . When  $i = j$ ,

$$\text{var}(t_{3i}) = E \left[ X_{i-1}^{-2} \left\{ X_{i-1}^2 e^{-2\kappa\delta} + X_{i-1} \frac{2(\theta_\alpha+1)}{\theta_\beta} (1-e^{-\kappa\delta}) + \frac{\theta_\alpha^2+\theta_\alpha}{\theta_\beta^2} (1-e^{-2\kappa\delta})^2 \right\} \right] - \mu_3^2 = (1-e^{\kappa\delta})\tilde{C}_{2,\theta},$$

where  $\tilde{C}_{2,\theta} = \frac{2e^{-\kappa\delta}(\theta_\alpha-1)-\theta_\alpha^2(1-e^{-\kappa\delta})}{(\theta_\alpha-1)^2} + \frac{\theta_\alpha^2+\theta_\alpha}{(\theta_\alpha-1)(\theta_\alpha-2)}(1-e^{-\kappa\delta})$ . Thus, (A.13) and (A.14) together imply that

$$\text{var}(t_3) = n^{-2}(1-e^{-\kappa\delta}) \left\{ n\tilde{C}_{2,\theta} + 2 \sum_{i=1}^n \sum_{j=i+1}^n \alpha(A_{1,ij} + A_{2,ij}) \right\}. \tag{A.14}$$

Substitute results in (A.14), (A.2), (A.3), (A.5)–(A.8) and (A.11) into variance of (A.1),

$$\text{var}(\hat{\beta}_1) = n^{-2} \left( (1-e^{-\kappa\delta}) \left\{ nC_{2,\theta} - 2\theta_\alpha(\theta_\alpha-1)(\theta_\alpha-2)(1-e^{-\kappa\delta}) \times \sum_{i=1}^n \sum_{j=i+1}^n (e^{\kappa\delta} S_{1,ij} - S_{2,ij}) \right\} + (1-e^{-\kappa\delta})^2 \times \left[ \sum_{i=1}^n \sum_{j=1}^n \{ \theta_\alpha^2 (F(1, 1, \theta_\alpha; e^{-\kappa\delta_{ij}}) - 1) - \theta_\alpha e^{-\kappa\delta_{ij}} \} - 2 \sum_{j=1}^n \sum_{i=j+1}^n \{ -\theta_\alpha e^{\kappa\delta} e^{-\kappa\delta_{ij}} + \theta_\alpha^2 \{F(1, 1, \theta_\alpha; e^{-\kappa\delta(i-1)j}) - 1\} \} - 2 \sum_{i=1}^n \sum_{j=i}^n \{ C_{1,\theta} e^{-\kappa\delta_{ij}} - \theta_\alpha(\theta_\alpha-1)(\theta_\alpha-2) \times (e^{\kappa\delta} S_{1,ij} - S_{2,ij}) \} \right] \right) + O(n^{-2}), \tag{A.15}$$

where  $C_{2,\theta} = 2e^{-\kappa\delta}(\theta_\alpha-1) - \theta_\alpha^2(1-e^{-\kappa\delta}) + \frac{(\theta_\alpha-1)(\theta_\alpha^2+\theta_\alpha)}{(\theta_\alpha-2)}(1-e^{-\kappa\delta})$ .

Transforming  $\hat{\beta}_1$  back to  $\hat{\kappa}$  we have for  $\delta$  fixed,

$$E(\hat{\kappa}) - \kappa = (n\delta)^{-1}B_3(\theta, \delta) + O(n^{-2}) \quad \text{and} \quad \text{var}(\hat{\kappa}) = (n\delta)^{-1}V_4(\theta, \delta) + O(n^{-2}) \tag{A.16}$$

where  $B_3(\theta, \delta)$  and  $V_4(\theta, \delta)$  are defined before Theorem 3.1.3. This completes proving the part of the theorem regarding  $\hat{\kappa}$ . The proofs for  $\hat{\alpha}$  and  $\hat{\sigma}^2$  are almost the same, by first carrying out Taylor expansions for the estimators and then applying those intermediate results given from (A.2) to (A.11).  $\square$

**Proof of Theorem 3.1.4.** Let  $\beta_1 = e^{-\kappa\delta}$ ,  $\beta_2 = \alpha$ ,  $\beta_3 = \sigma^2(2\kappa)^{-1}(1-e^{-2\kappa\delta})$  and  $\beta = (\beta_1, \beta_2, \beta_3)$  be the 1–1 mapping from  $\theta = (\kappa, \alpha, \sigma^2)$ . The  $\ell(\theta)$  defined by (2.12) can be regarded as  $\ell(\beta)$  after the re-parametrization. Then the pseudo-MLE  $\hat{\beta}$  is the root of  $\frac{\partial^T \ell(\hat{\beta})}{\partial \beta} = 0$ . It can be shown that  $E\hat{\beta}_3 = \beta_3 + b(\theta, \delta) + O(n^{-1})$  where  $b(\theta, \delta) = (\theta_\alpha-1)^{-1}(1-e^{-2\kappa\delta})^2\sigma^2$  is a bias term which does not converge to 0 unless  $\delta \rightarrow 0$ . Let  $\tilde{\beta} = (\beta_1, \beta_2, \beta_3 + b(\theta, \delta))^T$  and applying Taylor's expansion to the pseudo-likelihood score equations, we have

$$0 = \frac{\partial^T \ell(\hat{\beta})}{\partial \beta} = \frac{\partial^T \ell(\tilde{\beta})}{\partial \beta} + \frac{\partial^2 \ell(\tilde{\beta})}{\partial \beta \partial \beta^T} (\hat{\beta} - \tilde{\beta}) + \xi_n,$$

where  $\xi_n$  is the remainder term. This implies that

$$\sqrt{n}(\hat{\beta} - \tilde{\beta}) = \left\{ -\frac{\partial^2 \ell(\tilde{\beta})}{\partial \beta \partial \beta^T} \right\}^{-1} \left\{ \sqrt{n} \left( \frac{\partial^T \ell(\tilde{\beta})}{\partial \beta} + \xi_n \right) \right\}.$$

Utilizing the central limiting theorem for mixing sequences (Bosq, 1998), it can be shown that

$$\frac{1}{\sqrt{n}} \frac{\partial^T \ell(\tilde{\beta})}{\partial \beta} \xrightarrow{d} N(0, \Sigma^{-1}),$$

$$\text{where } \Sigma = \begin{pmatrix} \delta V_4(\theta) & -(1+e^{-\kappa\delta}) & 0 \\ -(1+e^{\kappa\delta}) & 2\alpha\theta_\beta^{-1}(1-e^{-\kappa\delta})^{-1} & 0 \\ 0 & 0 & Z_2(\theta, \delta) \end{pmatrix},$$

$$Z_2(\theta, \delta) = \frac{1}{4\beta_3^2} \left[ 1 + \frac{1}{1+e^{-\kappa\delta}} \left\{ 12e^{-2\kappa\delta} + (12\nu + 48)c(\theta, \delta)^{-1} \frac{e^{-\kappa\delta}\theta_\beta}{\theta_\alpha - 1} \right\} \right]$$









