Lumped Parameter Models for Heat Transfer and the Blot Number

Imagine a slab that has one dimension, of thickness 2d, that is much smaller than the other two dimensions; we also assume that the slab is homogeneous with temperature and position-independent physical parameters. At time t = 0, the slab is placed into contact with a liquid of constant temperature, T_c. Heat transfer from the slab to the fluid is governed by Newton's law of cooling:

$$Q = hA\left(T_s - T_c\right) \tag{1}$$

in which Q is the heat flux. A is the surface area for heat transfer, T_s is the surface temperature of the specimen, and $h\left(W - cm^{-2} - K^{-1}\right)$ is the heat transfer coefficient. Within the specimen itself, heat transfer is by conduction

$$Q = -kA\frac{\partial T}{\partial x} \tag{2}$$

$k\left(W - cm -^{1} - K -^{1}\right)$ is the thermal conductivity of the specimen. At the boundary of the sample, the heat flux given by the two expressions must be equal:

$$-h\left(T_s - T_c\right) = k\frac{\partial T_s}{\partial x} \tag{3}$$

$\frac{\partial T_s}{\partial x}$ is the temperature gradient at the specimen surface. Equation 3 is used as a one boundary condition in solving for the transient temperature distribution within the specimen, $T\left(x,t\right)$:

$$\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T_s}{\partial x^2}; \alpha \equiv \frac{k}{\rho Cp} \tag{4}$$

$\rho\left(g - cm^{-3}\right)$ is the density and $C_\rho\left(J - g^{-1} - K^{-1}\right)$ the heat capacity of the specimen. α is known as the thermal diffusivity and has units of cm^2-sec^{-1}. If the sample is cooled simultaneously from both sides, as is typically the case, the symmetry of the problem gives a second boundary condition:

$$\frac{\partial T}{\partial x}\left(0,t\right) = 0 \tag{5}$$

The specimen is initially assumed to be at a uniform temperature T_0:

$$T\left(x,0\right) = T_0 \tag{6}$$

To simplify the problems and to help in identifying important parameters, we define the following dimensionless variables:

$$\eta \equiv \frac{x}{d};\ \theta \equiv \frac{T - T_c}{T_0 - T_c};\ t \equiv \frac{t\alpha}{d^2}$$

which reduces the problem and the initial and boundary conditions to the following:

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial \eta^2} \tag{7}$$

$\theta(\eta, 0) = 1,$ Initial Condition (7a)

$\frac{\partial \theta}{\partial \eta}(0, \tau) = 0,$ Boundary Condition @ $x = 0$ (7b)

$\frac{\partial \theta}{\partial \eta}(1, \tau) = -\frac{hd}{k}\theta(1, \tau),$ Boundary Condition @ $x = 0$ (7c)

The dimensionless group hd/k is known as the Biot modulus, Bi. This modulus is a measure of the relative rates of convective to conductive heat transfer. With this choice of boundary conditions, the problem can be solved by a separation of variables technique. The exact solution is an infinite series expansion and is available in texts such as that of Carslaw and Jaeger (1959). A better insight into the physics of the problem can be obtained by examining certain limiting cases of this solution. First, let:

$$\theta(\eta, \tau) = N(\eta)\Phi(\tau) \tag{8}$$

N is a function of η and Φ is a function only of τ. Inserting Equation 8 into Equation 7 and rearranging:

$$\frac{\partial \Phi}{\partial \tau} / \Phi = \frac{\partial^2 N}{\partial \eta^2} / N \tag{9}$$

Because the left hand side of Equation 9 depends only on τ and the right hand side only on η, for the equality to hold both sides must be equal to a constant, $-\lambda^2$. Equation 7 separates into two ordinary differential equations:

$$\frac{\partial \Phi}{\partial \tau} + \lambda^2 \Phi = 0; \qquad \Phi(\tau) = C_1 \exp\left(-\lambda^2 \tau\right) \tag{10a}$$

$$\frac{\partial^2 N}{\partial \eta^2} + \lambda^2 N = 0; \qquad N(\eta) = C_2 \cos\ \lambda\eta + C_3\ \sin\ \lambda\eta \tag{10b}$$

$$\theta(\eta, \tau) = C_1\ \exp\ \left(-\lambda^2 \tau\right)\ \left[C_2\ \cos \lambda\eta + C_3\ \sin\ \lambda\eta\right] \tag{10c}$$

Equation 10(c) is the general solution of Equation 7. For this particular problem, the initial and boundary conditions must be met. Starting with Equation 7(b):

$$\frac{\partial \theta}{\partial \eta}(0, \tau) = 0 = C_1\ \exp\ \left(-\lambda^2 \tau\right)\left[\lambda C_3\right];\ \text{or } C_3 = 0 \tag{11a}$$

$$\theta(\eta,\tau) = C\ \exp\ \left(-\lambda^2\tau\right)\ \cos\ \lambda\eta$$

in which $C_1C_2 \equiv C$. Imposing the second boundary condition by inserting Equation 11(b) into Equation 7(c) gives:

$$C\exp\left(-\lambda^2\tau\right)\lambda\sin\lambda = C\exp\left(-\lambda^2\tau\right)Bi\cos\lambda \qquad (12a)$$

$$\lambda\sin\lambda = Bi\cos\lambda \qquad (12b)$$

$$\tan\lambda = \frac{Bi}{\lambda} \qquad (12c)$$

There are an infinite number of solutions to Equation 12(b) which are known as eigenvalues, λ_n. The solution can be written as an infinite series in which the C_n are chosen to match the initial condition, Equation 7a, and the λ_n are the solutions of Equation 12(b).

$$\theta(\eta,\tau) = \Sigma_{n=1}^{\infty}\ C_n\exp\left(-\lambda_n^2\tau\right)\cos\lambda_n\eta \qquad (13)$$

For large λ_n (finite Bi), tan $\lambda_n \approx 0$ and $\lambda_n = (n-1)\pi$. However, these terms do not contribute much to the solution because they are damped by the exp $\left(-\lambda_n^2\tau\right)$ factor in Equation 13. Often, the first term of Equation 13 is the only significant term in the solution (see Equation 15). For our purposes, it is best to examine two limiting cases:

Case I: Bi<1
Case II: Bi $\to\infty$

When Bi < 1, convection from the sample to the cryogen is slow compared to conduction. For Bi $\to\infty$ convection is fast compared to conduction and the surface temperature of the sample is always the same as the cryogen.

Case I: Bi < I, Convection Limited

For $Bi<1$, λ_1 will also be much less than one. Expanding $\tan\ \lambda$ in a Taylor series for small λ_1 gives

$$\tan\lambda_1 \approx \lambda_1 + 0\left(\lambda_1^3\right) = \frac{Bi}{\lambda_1} \qquad (14a)$$

The term $0\left(\lambda_1^3\right)$ means that the error in this approximation is of order λ_1^3. For λ_1 small enough (for λ_1 = 0.5, the error is less than 10%) only the linear term is important and

$$\lambda_1 \approx \frac{Bi}{\lambda_1}\ \text{or}\ \lambda_1 \approx (Bi)^{1/2} \qquad (14b)$$

The second eigenvalue, λ_2, can be seen from Figure 2 to be approximately equal to π. It is useful to compare this second term in the solution to the first $(\tau > 0)$:

$$\frac{e-^{(\lambda_2)^2}}{e-^{(\lambda_1)^2}} \approx \frac{e-^{\pi^2}}{e-^{Bi}} \approx 5x10^{-5} \tag{15}$$

Clearly, only the first term in the solution is significant for reasonable τ. Therefore, the solution to Equation 7 in this approximation is

$$\theta(\eta,\tau) \approx C_1 \exp(-Bi\ \tau)\cos(Bi)^{1/2\eta} \tag{16}$$

The cosine term above can also be expanded in a Taylor series:

$$\cos(Bi)^{1/2}\eta \approx 1-\frac{(Bi)\eta^2}{2} \approx 1 \tag{17}$$

For Bi sufficiently small there are no spatial gradients within the sample. Matching the initial condition dictates that $C_1 = 1$ and the final approximate solution for small *Bi* is:

$$\theta \approx \exp(-Bi\tau) \tag{18a}$$

or, in terms of the physical parameters of the problem

$$T(x,t) = T_C + (T_0 - T_C)\exp\left(\frac{-ht}{\rho C_p d}\right) \tag{18b}$$

Because the internal temperature of the specimen is position independent and the only mechanism of heat transfer is convection, it is simple to generalize this approach to cover arbitrary shapes. The average cooling rate is then proportional to:

$$\frac{dT}{dt} \approx \frac{-A}{V} h(T_0 - T_C)\frac{1}{\rho C_p} \tag{19}$$

It is necessary to recognize two consequences of Equation 19 $(Bi < 1)$. First, the cooling rate (Equation 19) is proportional to $\frac{A}{V}$, or the inverse first power of the characteristic dimension of the object (d the half thickness of a slab, $\frac{R}{2}$ for a cylinder, and $\frac{R}{3}$ for a sphere). Second, the cooling rate is independent of the thermal conductivity of the specimen; the only physical property of the sample that is important is the "thermal density," ρC_p. The cooling rate is also linearly dependent on the temperature difference between the specimen and the liquid cryogen, and the heat transfer coefficient, h.

$Bi \to \infty$ Conduction Limited

For $Bi \to \infty$, Equation 12b becomes

$$\tan\lambda \approx \infty, \lambda_n = \frac{1}{2}(2n+1)\pi, n = 0,1,2... \qquad (20)$$

Physically, as $Bi \to \infty$, the specimen surface temperature must approach the cryogen temperature, T_c Tc. This gives us a new boundary condition, $\theta(1,\tau) = 0$, to replace Equation 7c and Equation 20 now reads:

$$\cos\lambda_n = 0 \qquad (21)$$

which gives the same eigenvalues as Equation 20. The first term in the solution for $\theta(\eta,\tau)$ is then;

$$\theta(\eta,\tau) \approx C\exp\left[-\left(\frac{\pi}{2}\right)^2\tau\right]\cos\left(\frac{\pi\eta}{2}\right) \qquad (22a)$$

Because the conduction-limited solution (Equation 22a) varies with position, the initial condition cannot be matched by a single term as in Equation 18: the full infinite series solution (Equation 22b) is necessary to match this condition exactly. The full solution in terms of the physical parameters of the problem is:

$$T(x,t) = T_c + 2(T_0 - T_c)\Sigma_{n=0}^{\infty}\frac{(-1)^n}{\left(n+\frac{1}{2}\right)\pi}\exp\left[\frac{\left(-n+\frac{1}{2}\right)^2\pi^2 kt}{\rho C_p d^2}\right]\cos\left[\frac{\left(n+\frac{1}{2}\right)\pi x}{d}\right] \qquad (22b)$$

However, by the same reasoning as used in Equation 15, the first term of the infinite series solution dominates all others for $\tau > 0$:

$$T(x,t) = T_c + \frac{4}{\pi}(T_0 - T_c)\exp\left[\frac{-\pi^2 kt}{4\rho\, C_p d^2}\right]\cos\left[\frac{\pi x}{2d}\right] \qquad (22c)$$

To compare Equation 22c with Equation 18. it is necessary to take its spatial average. This involves integrating Equation 22(c) over x from zero to d, and dividing by d, which gives the following:

$$T(x,t) = T_c + \frac{8}{\pi^2}(T_0 - T_c)\exp\left[\frac{-\pi^2 kt}{4\rho\, Cpd^2}\right]$$

Hence, the average cooling rate for the slab is proportional to:

$$\frac{dT}{dt} \approx \frac{1}{d^2}k(T_0 - T_C)\frac{1}{\rho C_p}$$

and the average cooling rate for arbitrary shapes is proportional to:

(23a)

$$\frac{dT}{dt} \approx \left(\frac{A}{V}\right)^2 k\left(T_0 - T_C\right)\frac{1}{\rho C_p} \qquad (23b)$$

In comparison to Case I, the cooling rate is proportional to the thermal conductivity of the slab, k, and the inverse square of the characteristic dimension, $\left(\frac{A}{V}\right)^2$, of the slab. The heat transfer coefficient, h, does not appear in the cooling rate.

	Conduction Limited	**Convection Limited**
Characteristic Time	$\frac{kt}{\rho C_p (V/A)^2}$	$\frac{ht}{\rho C_p V/A}$
Characteristic Scaling	$(V/A)^2$	(V/A)
	Spatial dependent T	Spatial independent T
	Material dependent T	Material independent T
	Distribution	distribution