Bipartite graphs with every matching in a cycle

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Abstract

We give sufficient Ore-type conditions for a balanced bipartite graph to contain every matching in a hamiltonian cycle or a cycle not necessarily hamiltonian. Moreover, for the hamiltonian case we prove that the condition is almost best possible.

Keywords: Bipartite graph; Cycle; Hamiltonian cycle; Matching

1. Introduction

Let \( G = (B, W, E) \) be a bipartite graph. We will say that \( G \) is a balanced bipartite graph if \( |B| = |W| \).

In 1972, Las Vergnas obtained the following results [5]:

Theorem 1. Let \( G = (B, W, E) \) be a balanced bipartite graph of order \( 2n \). If for any \( x \in B, y \in W, xy \notin E \) we have \( d(x) + d(y) \geq n + 2 \), then every perfect matching in \( G \) is contained in a hamiltonian cycle.

For the existence of a perfect matching, he gave the sufficient condition:

Theorem 2. Let \( G = (B, W, E) \) be a balanced bipartite graph of order \( 2n \) and let \( q \geq 2 \). If for any \( x \in B, y \in W, xy \notin E \) we have \( d(x) + d(y) \geq n + q \), then every matching of cardinality \( q \) is contained in a perfect matching.

Using these two results he obtained the following corollary:

Corollary 3. Let \( G = (B, W, E) \) be a balanced bipartite graph of order \( 2n \) and let \( q \geq 2 \). If for any \( x \in B, y \in W, xy \notin E \) we have \( d(x) + d(y) \geq n + q \), then every matching of cardinality \( q \) is contained in a hamiltonian cycle.

About cycles through matchings in general graphs Berman proved in [1] the following result conjectured by Häggkvist in [3].
Theorem 4. Let $G$ be graph of order $n$. If for any $x, y \in V(G)$, $xy \notin E$ we have $d(x) + d(y) \geq n + 1$, then every matching lies in a cycle.

Theorem 4 has been improved by Jackson and Wormald in [4]. Häggkvist [3] gave also a sufficient condition for a general graph to contain any matching in a hamiltonian cycle. We give this theorem below in a slightly improved version obtained in [6].

For any integer $p \geq 1$, $K_p$ denotes a complete graph $K_p$ with empty set. Let $\mathcal{G}_n$ be the family of graphs $G = K_{(n + 2)/3} \ast H$, where $H$ is any graph of order $(2n - 3)/3$ containing a perfect matching, if $(n + 2)/3$ is an integer, and $\mathcal{G}_n = \emptyset$ otherwise (\ast denotes the join of graphs).

Theorem 5. Let $G$ be a graph of order $n \geq 3$, such that for every pair of nonadjacent vertices $x$ and $y$, $d(x) + d(y) \geq (4n - 2)/3$. Then, every matching of $G$ lies in a hamiltonian cycle, unless $G \in \mathcal{G}_n$.

We give sufficient conditions in a balanced bipartite graph for a matching to be contained in an hamiltonian cycle or a cycle not necessarily hamiltonian. Moreover, for the hamiltonian case we prove that the condition is almost best possible. Results are presented in Section 3 and will be proved in Sections 4 and 5.

2. Definitions

Let $G = (B, W, E)$ be a balanced bipartite graph and $M$ a matching in $G$.

A subgraph $H$ of $G$ is said to be a $\Theta$-graph compatible with $M$ if $H$ is a union of two cycles $C_1$ and $C_2$ satisfying the conditions:

1. The intersection of $C_1$ and $C_2$ is a path $R$ of length at least one.
2. Every edge of $M$ is an edge of $H$.
3. Every edge of $M$ incident with a vertex of $R$ lies in $R$.
4. $|V(R)|$ is even and the end vertices, say $x$ and $y$, of $R$ are in different partite sets.

We denote $P: x C_1 \setminus C_2 y$, $Q: x C_2 \setminus C_1 y$ and $H = (P, Q, R)$.

The notion of the $\Theta$-graph is based on the paper of Berman [1]. In Fig. 1 there is an example of a $\Theta$-graph.

![Fig. 1. A $\Theta$-graph compatible with $M$ and containing all the vertices of the graph $G$.](image-url)
A subgraph $H$ of $G$ is said to be a strict $\Theta$-graph compatible with $M$ if $H$ is a $\Theta$-graph $(P, Q, R)$ such that if we label the vertices of the paths as

\[
P : xp_1 \ldots p_\frac{1}{2} y, \quad Q : xq_1 \ldots q_\frac{1}{2} y, \quad R : xr_1 \ldots r_\gamma y,
\]

then $q_1 \in V(H) \setminus V(M)$, $p_\frac{1}{2} \in V(H) \setminus V(M)$, $xr_1 \in M$ and $r_\gamma y \in M$.

In Fig. 2 there is an example of a strict $\Theta$-graph.

If on a path $\pi : x_1 x_2 \ldots x_k$ of $G = (B, W, E)$ is given an orientation from $x_1$ to $x_k$, $\pi$ is said to be a $BB$-path if $x_1 \in B$, $x_k \in B$, a $WW$-path if $x_1 \in W$, $x_k \in W$, a $BW$-path if $x_1 \in B$, $x_k \in W$ and a $WB$-path if $x_1 \in W$, $x_k \in B$.

Let $C$ be a cycle or path with an arbitrary orientation and $x \in V(C)$, then $x^-$ is the predecessor of $x$ and $x^+$ is its successor according to the orientation of $C$.

Let $A$ be a subgraph of $G$, $v$ a vertex of $G$, then $d_A(v)$ is equal to the number of neighbors of $v$ in $A$, and for $S \subseteq V(G)$, we put $e(S, A) = \sum_{v \in S} d_A(v)$.

For notation and terminology not defined above a good reference should be [2].

3. Result

**Theorem 6.** Let $G = (B, W, E)$ be a balanced bipartite graph of order $2n$.

1. If for any $x \in B$, $y \in W$, $xy \notin E$ we have

\[
d(x) + d(y) > \frac{4n}{3},
\]

then every matching $M$ in $G$ is contained in a hamiltonian cycle.
2. If \( n > 4 \) and for any \( x \in B, y \in W, xy \notin E \) we have

\[
d(x) + d(y) \geq \frac{5n}{4},
\]

then every matching \( M \) in \( G \) is contained in a cycle of \( G \).

The first part of the theorem is almost best possible in the sense that if one decreases the sum of degrees of more than \( \frac{3}{2} \) then the theorem is no more true. By \( \overline{K}_l \) we denote the balanced bipartite graph of order \( 2l \) with empty edge set.

Let \( \overline{K}_{p+1} = (B_p, W_{p+1}, E_{p+1}) \) and \( K_{2p+1,2p+1} = (B_{2p+1}, W_{2p+1}, E_{2p+1}) \).

Consider the following bipartite graph \( G = (B, W, E) \) with \( B = B_{p+1} \cup B_{2p+1}, W = W_{p+1} \cup W_{2p+1} \) and \( E = E_{2p+1} \cup \{uv : u \in B_{p+1}, v \in W_{2p+1}\} \cup \{uv : u \in W_{p+1}, v \in B_{2p+1}\} \).

Note that \( G \) is a balanced bipartite graph of order \( 2n = 2(3p + 2) \). Let \( M \) be a perfect matching of \( K_{2p+1,2p+1} \). It is evident that there is no hamiltonian cycle containing \( M \) and that the minimum sum of degrees of two nonadjacent vertices is \( (4n - 2)/3 \).

Let now \( G' \) be the graph obtained from \( G \) by replacing \( \overline{K}_{p+1} \) by \( \overline{K}_{p,p} \). Then \( G' \) is a balanced bipartite graph which satisfies the hypothesis of part 1 of Theorem 6 and by consequence there is a hamiltonian cycle which contains \( M \). Notice however that \( M \) is not contained in any perfect matching of \( G' \), and the degree constraint in part 2 of Theorem 6 is clearly not sufficient to imply that any matching can be extended into a perfect matching.

3.1. Conjecture

During works on the proof of Theorem 6, D. Amar posed the following conjecture:

**Conjecture.** Let \( G = (B, W, E) \) be a balanced bipartite graph of order \( 2n \). If for any \( x \in B, y \in W, xy \notin E \) we have

\[
d(x) + d(y) \geq n + 2,
\]

then every matching \( M \) in \( G \) is contained in a cycle of \( G \).

It is not difficult to show the following:

**Remark.** If \( |M| = n - 1 \) and for any \( x \in B, y \in W, xy \notin E, d(x) + d(y) \geq n + 2 \), then \( M \) is contained in a hamiltonian cycle.

Suppose that \( G \) is not a complete graph (if \( G \) is complete then **Remark** is true). Let \( M \cup (pq) \), with \( p \in B, q \in W, pq \notin E \), be a perfect matching containing \( M \). From Theorem 1 it is contained in a hamiltonian cycle \( C \). Let \( D: qu_1u_2\ldots u_{2l}p \) be a hamiltonian path in \( G \) obtained from \( C \) by deleting the edge \( pq \). The edges \( u_1u_2, u_2u_3, \ldots, u_{2l-1}u_{2l} \) are edges of the matching \( M \). Since \( d(p) + d(q) \geq n + 2 \) then there exists a \( k \), such that \( qu_{k+1} \in E \) and \( pu_k \in E \). Note that \( p \in B, q \in W, u_k \in W \) and then \( k \) is even. The edge \( u_ku_{k+1} \) is not in \( M \). The cycle \( C: qu_1\ldots u_kpu_{k-1}\ldots u_{k+1}q \) is a hamiltonian cycle of \( G \) which contains \( M \).

4. Proof of part 1 of Theorem 6

Let \( G = (B, W, E) \) be a bipartite graph satisfying the conditions of part 1 of Theorem 6 and let us suppose that there is a matching \( M \) in \( G \) such that there is no hamiltonian cycle through \( M \). Without loss of generality we may suppose that:

(i) \( M \) is maximal, i.e. \( M \) is the only matching which contains \( M \).

(ii) \( G \) is maximal without a hamiltonian cycle through \( M \) (any addition of an edge \( uv, u \in B, v \in W, uv \notin E \) creates a hamiltonian cycle containing \( M \)).

So we have a hamiltonian path \( P_H: u_p \ldots u_{2n-2}v \) containing \( M \). Since \( uv \notin E \), we have \( d(u) + d(v) > 4n/3 \) and this implies that we have at least two vertices \( p_i, p_{i+1} \) satisfying \( u_{p_{i+1}}, vp_i \in E \).
Then the Hamiltonian cycle:

\[ C': u_{i_1+1} p_{i_1+1} \ldots u_{i_l+1} p_{i_l+1} \ldots \]

contains all edges of the path \( P_{qq} \) except \( p_i p_{i+1} \). Since there is no Hamiltonian cycle containing \( M \) in \( G \) we have \( p_i p_{i+1} \in M \). Now take the cycles: \( C_1: u_{i_1+1} p_{i_1+1} \ldots u_{i_l+1} p_{i_l+1} \ldots v \) and \( C_2: v_{i_1} p_{i_1+1} p_{i_1+2} \ldots v \). The subgraph \( H = C_1 \cup C_2 \) is a \( \Theta \)-graph compatible with \( M \) and containing all the vertices of the graph \( G \). We can see an example of such \( \Theta \)-graph which is not a strict \( \Theta \)-graph in Fig. 1.

Following the notations from Section 2, label the vertices of the paths \( P, Q \) and \( R \) as follows:

\[ P : x p_1 \ldots p_x y, \]
\[ Q : x q_1 \ldots q_x y, \]
\[ R : x r_1 \ldots r_x y, \]

and denote by \( P_i, i = 1, \ldots, n_P, Q_j, j = 1, \ldots, n_Q, \) and \( R_k, k = 1, \ldots, n_R, \) the paths obtained, respectively, from \( P, Q, R \) by removal of the edges of \( M \). Without loss of generality we may assume \( x \in B, y \in W \).

We may assume that \( H = (P, Q, R) \) is a \( \Theta \)-graph compatible with \( M \) such that \( |V(R)| \) is maximum.

Remark. Since \( M \) is maximal, for any \( i, j \) and \( k \) we have\( 2 \leq |V(P_i)| \leq 3, 2 \leq |V(Q_j)| \leq 3, \) and \( 1 \leq |V(R_k)| \leq 3. \)

From the assumption that every edge of \( M \) incident with a vertex of \( R \) lies in \( R \), if one of the edges \( p_x q_1, p_1 q_y \) exists then there is a Hamiltonian cycle in \( G \) containing every edge of \( M \), so we may assume that \( p_x q_1 \notin E \) and \( p_1 q_y \notin E \) and then we have

\[
\frac{d(p_1) + d(q_1) + d(p_x) + d(q_y)}{3} > \frac{8n}{3}. \tag{1}
\]

4.1. Neighbors of \( p_1, p_x, q_1, q_y \) on \( Q \) and \( P \)

Claim 1. If \( p_1 q_1 \in E \) and \( l > 1 \) (\( p_1 \) and \( q_1 \) are in the same partite set), then \( q_1 q_{l+1} \in M \). Moreover, for \( i = 2, \ldots, n_Q, e(p_1, Q_i) \leq 1 \) and if \( e(p_1, Q_i) = 1 \), then \( e(q_y, Q_i) = 0. \)

Proof. In fact if \( p_1 q_1 \in E \), then \( H' = (P', Q', R') \) with \( P': q_1 p_1 p_2 \ldots p_x y, Q': q_1 q_{l+1} \ldots q_y y \) and \( R': q_{l+1} \ldots q_1 x r_1 r_2 \ldots r_y y \) is a \( \Theta \)-graph compatible with \( M \) with \( |V(R')| > |V(R)| \) unless \( q_1 q_{l+1} \in M. \)

So let us suppose that \( p_1 q_1 \in E \) and \( q_1 q_{l+1} \in M \), with \( q_1 \in Q_{i_0} \). Then \( q_1 \in B \) for \( p_1 \in W \). The vertex \( q_{l+1} \) is the only vertex of \( V(Q_{i_0}) \) in \( W \).

If \( q_y q_{l+1} \in E \) then the cycle

\[
C': q_1 q_2 \ldots q_{l+1} x r_1 \ldots r_y y p_x p_{x-1} \ldots p_{l+1} q_{l+1} \quad \quad \tag{2}
\]

is a Hamiltonian cycle of \( G \) containing \( M \) and Claim 1 is proved. \qed

Claim 2. \( 1 \leq e(p_1, Q_1) \leq 2 \) and if \( e(p_1, Q_1) = 1 \), then \( e(q_y, Q_1) \leq 1 \). If \( e(p_1, Q_1) = 2 \) then \( e(q_y, Q_1) = 0. \)

Proof. Since \( x \in N(p_1) \cap Q_1 \) we have \( e(p_1, Q_1) \leq 1. \) Note that \( |V(Q_1)| = 2 \) or \( |V(Q_1)| = 3. \) When \( |V(Q_1)| = 2 \) then \( q_y \) may be adjacent to \( q_2 \) and \( e(q_y, Q_1) \leq 1. \) If \( |V(Q_1)| = 3 \) and \( p_1 q_2 \in E \) then \( e(q_y, Q_1) = 0. \) because otherwise the cycle \( C' \) given by (2) for \( l = 2 \) is a Hamiltonian cycle of \( G \) containing \( M \) and Claim 2 is proved. \qed

Claim 3. 1. If \( Q_{i_0} \) is a BB-path and \( Q_{j_0} \) is a WW-path, \( 2 \leq i_0, j_0 \leq n_Q, \) then

\[
e((p_1, q_y), Q_{i_0} \cup Q_{j_0}) \leq 3 = \frac{|V(Q_{i_0})| + |V(Q_{j_0})|}{2}. \tag{3}
\]

2. If \( Q_k, 2 \leq k \leq n_Q, \) is a BB-path or a WB-path then

\[
e((p_1, q_y), Q_k) \leq 1 = \frac{|Q_k|}{2}. \tag{4}
\]
3. In any case

\[ e(\{p_1, q_\beta\}, Q_1) \leq 2. \] (5)

**Proof.** For any \( i \), since the matching \( M \) is maximal we have \(|Q_i| = 3\), if and only if \( Q_i \) is a BB-path or a WW-path, and \(|Q_j| = 2\), if and only if \( Q_j \) is a BW-path or a WB-path. Consider a BB-path \( Q_{i_0} \) and a WW-path \( Q_{j_0} \) \((2 \leq i_0, j_0 \leq n_Q)\). From Claim 1 for \( 2 \leq i_0, j_0 \leq n_Q \), we have \( e(\{p_1, q_\beta\}, Q_{i_0}) \leq 1 \) and \( e(\{p_1, q_\beta\}, Q_{j_0}) \leq 2 \). These prove inequality (3). If \(|Q_i| = 2\) from Claim 1 we have (4). Inequality (5) is an immediate consequence of Claim 2. \( \square \)

Let us denote \( v_3(Q) \) the number of paths \( Q_i \) with odd number of vertices and \( v_2(Q) \) the number of paths \( Q_k \) with an even number of vertices, \( 1 \leq i, k \leq n_Q \).

As \(|V(Q)|\) is even, the number of BB-paths is equal to the number of WW-paths and so \( v_3(Q) \) is even i.e. \( v_3(Q) = 2\mu \). Clearly, \(|V(Q)| = \beta + 2 = 3 v_3(Q) + 2 v_2(Q) = 6\mu + 2 v_2(Q) \).

Now we shall estimate \( e(\{p_1, q_\beta\}, Q) \). From Claims 1–3 we have

\[ e(\{p_1, q_\beta\}, Q) = \sum_{|V(Q_i)|=3} e(\{p_1, q_\beta\}, Q_i) + \sum_{|V(Q_j)|=2} e(\{p_1, q_\beta\}, Q_j) \]

\[ \leq 3 \mu + v_2(Q) + 1 = \frac{\beta}{2} + 2. \] (6)

Similarly, we obtain the following three inequalities:

\[ e(\{q_1, p_2\}, Q) \leq \frac{\beta}{2} + 2, \] (7)

\[ e(\{p_1, q_\beta\}, P) \leq \frac{\alpha}{2} + 2, \] (8)

\[ e(\{q_1, p_2\}, P) \leq \frac{\alpha}{2} + 2. \] (9)

4.2. Neighbors of \( p_1, p_2, q_1, q_\beta \) on \( R \)

Note that, for any \( k = 1, \ldots, n_R \), we have \( 1 \leq |V(R_k)| \leq 3 \). If \( xr_1 \in M \) then \( R_1 = \{x\} \) and \(|V(R_1)| = 1 \). If \( r_1 y \in M \) then \( R_1 = \{x\} \) and \(|V(R_1)| = 1 \). For \( k = 2, \ldots, n_R - 1 \), we have \( 2 \leq |V(R_k)| \leq 3 \).

It is easy to check that if \(|V(R_j)| = 2 \) then \( e(\{p_1, p_2\}, R_j) \leq 1 \) and if \(|V(R_j)| = 3 \) then \( e(\{p_1, p_2\}, R_j) \leq 2 \).

If \(|V(R_j)| = 1 \) then \( e(\{p_1, p_2\}, R_j) = 1 \).

Denote by \( v_3(R) \) the number of paths \( R_j \) with three vertices, by \( v_2(R) \) the number of paths \( R_j \) with two vertices and by \( v_1(R) \) the number of paths \( R_k \) with one vertex.

Note that \( v_1(R) + v_3(R) \) is even and \( \gamma + 2 = 3 v_3(R) + 2 v_2(R) + v_1(R) \).

We have

\[ e(\{p_1, p_2\}, R_j) = \sum_{|V(R_j)|=3} e(\{p_1, p_2\}, R_j) + \sum_{|V(R_j)|=2} e(\{p_1, p_2\}, R_j) \]

\[ + \sum_{|V(R_k)|=1} e(\{p_1, p_2\}, R_k) \]

\[ \leq 2 v_3(R) + v_2(R) + v_1(R) \]

\[ = \frac{2\gamma + 4 + v_1(R) - v_2(R)}{3} \]

\[ \leq \frac{2\gamma + 6}{3}. \] (10)
Similarly, we have
\[ e((q_1, q_\beta), R) \leq \frac{2\gamma + 6}{3}. \] (11)

4.3. Conclusion

Now we shall estimate the sum \( d(p_1) + d(p_2) + d(q_1) + d(q_\beta) \).

From (6)–(11) we have
\[
d(p_1) + d(p_2) + d(q_1) + d(q_\beta) \\
= e((p_1, q_\beta), Q) + e((q_1, p_2), Q) + e((q_1, p_2), P) + e((p_1, q_\beta), P) \\
+ e((p_1, p_2), R) + e((q_1, q_\beta), R) - 2e((p_1, q_1, p_2, q_\beta), \{x, y\}) \\
\leq \alpha + \beta + 8 + \frac{4\gamma + 12}{3} - 8 = \frac{3\alpha + 3\beta + 4\gamma}{3} + 4.
\]

As \( \alpha \geq 2 \) and \( \beta \geq 2 \), we obtain the following inequality:
\[
d(p_1) + d(p_2) + d(q_1) + d(q_\beta) \leq \frac{4(\alpha + \beta + \gamma) + 8}{3} = \frac{8n}{3},
\]
which contradicts (1) and part 1 of Theorem 6 is proven.

5. Proof of part 2 of Theorem 6

Let \( G = (B, W, E) \) be a balanced bipartite graph with \( |B| = |W| = n, n > 4 \) satisfying the conditions of Theorem 6. For \( n \geq 8 \) we have \( 5n/4 \geq n + 2 \) and so from assumptions of Theorem 6 we have
\[
d(x) + d(y) \geq \frac{5n}{4} \geq n + 2,
\] (12)
for any \( x \in B, y \in W, xy \notin E \).

Note that, for \( n = 5, 6 \) and \( 7, 5n/4 \) is not an integer, in fact \( d(x) + d(y) \geq \lceil 5n/4 \rceil \). It is easy to verify that for \( n = 5, 6 \) and \( 7 \) we have \( \lceil 5n/4 \rceil = n + 2 \).

From the above and (12) we have
\[
d(x) + d(y) \geq n + 2,
\] (13)
for any \( x \in B, y \in W, xy \notin E \).

Let \( M \) be a matching in \( G \). We may assume that \( M \) is a maximal matching. If \( M \) is a perfect matching, then Theorem 1 implies that \( M \) is contained in a hamiltonian cycle. We can assume that \( M \) is not a perfect matching and consider a maximal counterexample, i.e. a balanced bipartite graph \( G \) and a maximal matching \( M \) such that

1. There is no cycle in \( G \) containing \( M \).
2. For every pair of vertices \( (p, q), p \in B, q \in W, pq \notin E, p, q \notin V(M) \), then \( M \) is contained in a cycle in \( G \cup (pq) \).

Note that, since \( M \) is not a perfect matching, thus we have at least two vertices \( p, q \) such that \( p, q \notin V(M) \).

Thus there is a path
\[ D: qu_1u_2 \ldots u_ip \] (14)
containing \( M \) and oriented from \( q \) to \( p \).

Since \( qp \notin E \) then from (13) there exists an \( i \) such that \( 1 \leq i \leq l-1, qu_{i+1} \in E \) and \( pu_i \in E \).

The cycle
\[ C': qu_iu_{i+1}u_{i+2} \ldots u_ipu_iu_{i-1} \ldots u_1q \]
cannot contain the matching \( M \), so \( u_1u_{i+1} \in M \).
Consider the paths
\[ P : u_1 p u_I \ldots u_i u_{i+1}, \]
\[ Q : u_i u_{i-1} \ldots u_1 q u_{i+1}, \]
\[ R : u_i u_{i+1}, \]
and note that \( H = (P, Q, R) \) is a strict \( \Theta \)-graph compatible with the matching \( M \). (For an example cf. Fig. 2.)

Let \( u_s, u_r \in V(D) \), \( s < r \) be such that \( pu_r \in E, qu_r \in E, u_i u_{s+1} \in M, u_r u_{r-1} \in M \) (note that \( s = i, r = i + 1 \) satisfy these conditions) and \( r - s \) is maximal.

The graph \( H = (P, Q, R) \):
\[ P : u_s p u_{i-1} \ldots u_r, \]
\[ Q : u_s u_{s-1} \ldots u_1 q u_r, \]
\[ R : u_s \ldots u_r, \]
is a strict \( \Theta \)-graph compatible with the matching \( M \) such that \( |V(R)| \) is maximum.

Since there is no cycle containing \( M \) we have \( E(P) \cap M \neq \emptyset, E(Q) \cap M \neq \emptyset \) and since \( H \) is a strict \( \Theta \)-graph \( |V(P)|, |V(Q)| \geq 6 \).

We label the vertices of \( H \) as follows:
\[ P : xp_{1} \ldots p_{2} y, \]
\[ Q : xq_{1} \ldots q_{2} y, \]
\[ R : xr_{1} \ldots r_{y} y. \]

We assume that \( x \in B, y \in W, q = q_{1} \in W, a = q_{2} \in B, p = p_{2} \in B \) and \( b = p_{2} \in W \).

Let \( G_M \) be the subgraph of \( G \) induced by \( V(G) \backslash V(M) \) and let \( Z \) be the subgraph of \( G \) induced by \( V(G) \backslash V(D) \).

Subgraphs \( G_M \) and \( Z \) are independent i.e. \( e(V(G_M), Z) = 0 \).

Since \( V(G) = V(P \backslash \{y\}) \cup V(Q \backslash \{x\}) \cup V(R \backslash \{x, y\}) \cup V(Z) \) and the sets \( V(P \backslash \{y\}), V(Q \backslash \{x\}), V(R \backslash \{x, y\}) \) and \( V(Z) \) are vertex-disjoint for every vertex \( v \in V(G) \), we have
\[ d(v) = d_{P \backslash \{y\}}(v) + d_{Q \backslash \{x\}}(v) + d_{R \backslash \{x, y\}}(v) + d_{Z}(v). \]

Let \( |M| = m, |V(M)| = 2m, |V(D \backslash M)| = 2\delta \) and \( |V(Z)| = 2t, \) then \( n = m + \delta + t \).

**Remark.** As \( p_{2} \notin V(M), q_{1} \notin V(M), \) \( |V(P)| \) and \( |V(Q)| \) are even, then \( \delta \geq 2 \). (There are at least two vertices of \( V(G) \backslash V(M) \) on \( P \) and on \( Q \).)

Denote by \( P_i, i = 1, \ldots, n_P, Q_j, j = 1, \ldots, n_Q, \) and \( R_k, k = 1, \ldots, n_R, \) the paths obtained, respectively, from \( P, Q \) and \( R \) by removal of the edges of \( M \).

Take an \( i \in \{1, \ldots, n_P\} \). Note that since \( M \) is maximal then if \( P_i \) is a path with an odd number of vertices, then \( |V(P_i)| = 3 \) and if \( P_i \) is a path with an even number of vertices, then \( |V(P_i)| = 2 \). Moreover, if \( |V(P_i)| = 3 \) then \( P_i \) is a \( BB \)-path or a \( WW \)-path. If \( |V(P_i)| = 2 \) then \( P_i \) is a \( BW \)-path or \( WB \)-path. As \( |V(P)| \) is even, the number of \( BB \)-paths is equal to the number of \( WW \)-paths. Let \( n_3(P) \) be the number of paths \( P_i \) with an odd number of vertices, \( v_2^B(P) \) the number of \( BW \)-paths \( P_i, v_2^W(P) \) the number of \( WB \)-paths \( P_i \) and \( v_2(P) = v_2^B(P) + v_2^W(P) \) the number of paths \( P_i \) with an even number of vertices.

The paths \( Q_j, i = 1, \ldots, n_Q, \) and \( R_i, i = 1, \ldots, n_R, \) have the same properties as the paths \( P_i \) and in the same way as above, we define \( v_2^B(P), v_2^W, v_2 = v_2^B + v_2^W \) and \( v_3 \) for paths \( Q \) and \( R \) (in which the number of \( BB \)-paths is also equal to the number of \( WW \)-paths).

From the maximality of \( G \) and \( M \) the graph induced by \( V(D) \backslash V(M) \) is independent. Thus, since \( bp_{x,y} \) is a \( WW \)-path we have
\[ n_P = v_3(P) + v_2(P) = |M \cap E(P)| + 1. \]
Similarly, since \( xq_1a \) is a BB-path we have
\[
n_Q = v_3(Q) + v_2(Q) = |M \cap E(Q)| + 1.
\] (17)

Note that on the path \( R \{x, y\} \) we have
\[
n_R = v_3(R) + v_2(R) = |M \cap E(R)| - 1.
\] (18)

From (16)–(18) we have
\[
3 \sum_{i=2} (v_i(P) + v_i(Q) + v_i(R)) = m + 1.
\] (19)

In every path \( P_i \) with an odd number of vertices, there is one vertex of \( V(\overline{D}) \setminus V(M) \) and since \(|V(R)|\) is even we have
\[
v_3(P) = |V(P \setminus M)|.
\] (20)

Similarly, we have
\[
v_3(Q) = |V(Q \setminus M)|.
\] (21)

\[
v_3(R) = |V(R \setminus M)|.
\] (22)

From (20)–(22) we have
\[
v_3(P) + v_3(Q) + v_3(R) = 2\delta.
\] (23)

5.1. Lower bound of the sums of degrees

If one of the edges \( ab, p_2q_1, p_1q_\beta \) exists, we have a cycle in \( G \) containing every edge of \( M \). For example, if \( p_1q_\beta \in E \) then the cycle
\[
C: p_1q_\beta q_{\beta - 1} \ldots q_1x1 \ldots r_\gamma y p_2 \ldots p_1
\]
is containing \( M \).

We may assume that \( ab \notin E, p_2q_1 \notin E, p_1q_\beta \notin E \) and then
\[
d(a) + d(b) \geq \frac{5n}{4},
\] (24)
\[
d(q_\beta) + d(p_1) \geq \frac{5n}{4},
\] (25)
\[
d(p_2) + d(q_1) \geq \frac{5n}{4}.
\] (26)

5.2. Upper bound of the sum of degrees

5.2.1. Neighbors of \( a, b, p_2, q_1, q_\beta, p_1 \) on \( R \{x, y\} \)

1. Consider a WB-path \( R_i: vu \) on \( R, u \in B, v \in W, v = u^-, uv \notin M \). Since there is no cycle containing every edge of \( M \), the following inequalities are satisfied: \( e([p_2, p_1], R_i) \leq 1, e([q_1, q_\beta], R_i) \leq 1 \) and \( e([a, b], R_i) \leq 1 \).

Suppose that \( e([a, b], R_i) = 2 \), then \( av, bu \in E \) and the following cycle \( C \):
\[
C: avu^- \ldots r_1xp_1 \ldots p_2buu^+ \ldots r_\gamma y q_\beta \ldots a
\]
contains \( M \), a contradiction.
Now suppose that \( e([p_1, p_2], R_i) = 2 \). In this case \( p_1u, p_2v \in E(G) \) and the following cycle \( C \):
\[
C: p_2v^+ \ldots r_1xq_1 \ldots q_\beta yr_\gamma \ldots up_1p_2 \ldots p_2
\]
contains \( M \), a contradiction.

The case \( e([q_1, q_\beta], R_i) = 2 \) is the same as \( e([p_1, p_2], R_i) = 2 \) and so we have
\[
e([a, b, p_1, p_2, q_1, q_\beta], R_i) \leq 3.
\]

2. Consider a BW-path \( R_i: uv \) on \( R, u \in B, v \in W, v = u^+, uv \notin M \). The following inequalities hold: \( e([p_1, p_2], R_i) \leq 1, e([q_1, q_\beta], R_i) \leq 1, e([a, b], R_i) \leq 2 \).

Since \( a, u \in B \) and \( b, v \in W \) it is clear that \( e([a, b], R_i) \leq 2 \).

Suppose that \( e([p_1, p_2], R_i) = 2 \), then \( p_1u, vp_2 \in E \) and the following cycle \( C \):
\[
C: p_2v^+ \ldots r_\gamma q_\beta \ldots q_1xr_1 \ldots up_1 \ldots p_2
\]
contains \( M \), a contradiction.

The case \( e([q_1, q_\beta], R_i) = 2 \) is the same as \( e([p_1, p_2], R_i) = 2 \). Thus
\[
e([a, b, p_1, p_2, q_1, q_\beta], R_i) \leq 4.
\]

3. Consider a WW-path \( R_i: v_1uv_2, u \in B, v_1, v_2 \in W, u \in V(D \setminus M), u = v_1^+ = v_2^- \). As \( q_1 \notin V(M), u \notin V(M) \) and \( M \) is maximal, we have \( q_1u \notin E \). Since there is no cycle containing \( M \), the following inequalities hold: \( e([p_1, p_2], R_i) \leq 2, e([a, q_\beta], R_i) \leq 2 \).

We will start to compute \( e([p_1, p_2], R_i) \).

If \( p_1u \notin E \) then \( e([p_1, p_2], R_i) \leq 2 \).

Suppose now that \( p_1u \in E \) and \( e(p_2, R_i) \neq 0 \). \( e(p_2, R_i) \neq 0 \) implies that \( p_2v_1 \in E \) or \( p_2v_2 \in E \).

If \( p_2v_1 \in E \), then the following cycle \( C \):
\[
C: p_2v_1v_1^- \ldots r_1xq_1 \ldots q_\beta yr_\gamma \ldots up_1 \ldots p_2
\]
contains \( M \), a contradiction.

If \( p_2v_2 \in E \), then the following cycle \( C \):
\[
C: p_2v_2v_2^+ \ldots r_\gamma q_\beta \ldots q_1xr_1 \ldots up_1 \ldots p_2
\]
contains \( M \), a contradiction. So if \( p_1u \in E \) we have \( e([p_1, p_2], R_i) = 1 \).

Thus in any case we have \( e([p_1, p_2], R_i) \leq 2 \).

Now, we shall compute \( e([a, q_\beta], R_i) \). Note that \( a \) and \( q_\beta \) cannot be adjacent to two different vertices on \( R_i \). Since \( a, u, q_\beta \in B \) and \( v_1, v_2 \in W \), we shall consider the existence of four edges: \( av_1, q_\beta v_1, av_2 \) and \( q_\beta v_2 \).

Suppose that \( av_1, q_\beta v_1 \in E \), then the following cycle \( C \):
\[
C: av_1v_1^- \ldots r_1xp_1 \ldots p_2yr_\gamma \ldots v_2q_\beta \ldots a
\]
contains \( M \), a contradiction.

If \( av_2, q_\beta v_1 \in E \), then the following cycle \( C \):
\[
C: av_2v_2^+ \ldots r_\gamma xp_2 \ldots p_1xr_1 \ldots v_1a
\]
contains \( M \), a contradiction.

So we have \( e([a, q_\beta], R_i) \leq 2 \) and since it may happen that \( bu \in E \), we have
\[
e([a, b, p_1, p_2, q_1, q_\beta], R_i) \leq 5.
\]

4. Consider a BB-path \( R_i: u_1u_2, u_1, u_2 \in B, v \in W, v \in V(D \setminus M), v = u_1^+ = u_2^- \). Since \( p_2 \notin V(M), v \notin V(M) \) and \( M \) is maximal, we have \( p_2v \notin E \). Using the same arguments as in case 3, since there is no cycle containing \( M \), the following inequalities hold: \( e([q_1, q_\beta], R_i) \leq 2, e([b, p_1], R_i) \leq 2 \), and since it may happen that \( av \in E \), we have
\[
e([a, b, p_1, p_2, q_1, q_\beta], R_i) \leq 5.
\]
By summing over all the paths $R_i$ from (27)--(30) we have
\[ e([a, b, p_x, q_1, q_\beta, p_1], R - \{x, y\}) \leq 3v_2(R) + v_2^W(R) + 5v_3(R). \] (31)

5.2.2. Neighbors of $a$, $b$, $p_x$, $q_1$, $q_\beta$, $p_1$ on $Q\setminus\{x\}$

1. Consider the vertices $\{q_1, a\}$. Since there is no cycle containing $M$ we have $e([p_1, q_\beta], \{q_1, a\}) \leq 1$, $aq_1 \in E$, $p_xq_1, ab \notin E$ and thus
\[ e([a, b, p_1, p_x, q_1, q_\beta], \{q_1, a\}) \leq 3. \] (32)

2. Consider a $BW$-path $Q_1: uv, u \in B, v \in W, v = u^+, uv \notin M$. Since there is no cycle containing $M$ we have $e([p_1, p_2], Q_1) \leq 1$ and $e([a, b], Q_1) \leq 1$.
Suppose that $e([p_1, p_2], Q_1) = 2$, then $p_1u, p_xv \in E$ and the following cycle $C$:
\[ C: p_1uu^- \ldots q_1xr_1 \ldots r_2yq_\beta \ldots v_xp_x \ldots p_1 \]
contains $M$, a contradiction.

If $e([a, b], Q_1) = 2$, then $bu, av \in E$ and the following cycle $C$:
\[ C: buu^- \ldots avv^+ \ldots q_\beta yr_\gamma \ldots r_1xp_1 \ldots b \]
contains $M$, a contradiction.

Note that since $M$ is maximal, we have $q_1u \notin E$ and from this: $e([q_1, q_\beta], Q_1) \leq 2$.

3. Consider a $WB$-path $Q_2: v_1uv_2, v_2 \neq y, u \in B, v_1, v_2 \in W, u = v_1^+ = v_2^-$.
Since $v_2 \neq y$ and as $R$ is maximal $p_xv_2 \notin E$. Suppose that $p_xv_2 \in E$, then the graph $H' = (P', Q, R')$ with
\[ P': xp_1 \ldots p_xv_2, \]
\[ Q': xq_1 \ldots v_2, \]
\[ R': xr_1 \ldots r_2yq_\beta \ldots v_2 \]
is a strict $\Theta$-graph compatible with $M$ with $|V(R')| > |V(R)|$.
Since there is no cycle containing $M$, using similar arguments as in the case 2, we have $e([p_1, p_2], \{u_1, v\}) \leq 1$, $e([a, b], \{u, v\}) \leq 1$. Hence, $e([p_1, p_2], Q_1) \leq 1$ and since it is possible that $av_1 \in E$ we have $e([a, b], Q_1) \leq 2$.

From these inequalities we have
\[ e([a, b, p_1, p_x, q_1, q_\beta], Q_1) \leq 5. \] (35)

5. In Case 4 we have assumed that $v_2 \neq y$. If $v_2 = y$, then $i = N_Q$ and the path $Q_{N_Q}$ is a $WW$-path $Q_{N_Q}: q_\beta \ldots q_\beta y$. In fact, it is the same case as case 4, but since $p_xy \in E$, we have
\[ e([a, b, p_1, p_x, q_1, q_\beta], Q_1) \leq 6. \] (36)

6. Consider a $BB$-path $Q_1: u_1uv_2, u_1, u_2 \in B, v \in W, v = u_1^+ = u_2^-$. Note that since $p_x, v \notin V(M)$ and since $M$ is maximal we have $p_xv \notin E$.
Since there is no cycle containing $M$ we have $e([a, b], \{u_1, v\}) \leq 1$, $e([p_1, q_\beta], \{v, u_2\}) \leq 1$.
Suppose that $e([a, b], \{u_1, v\}) = 2$, then $av, bu_1 \in E$ and the following cycle $C$:
\[ C: bu_1u_1^- \ldots avv^+ \ldots q_\beta yr_\gamma \ldots r_1xp_1 \ldots b \]
contains $M$, a contradiction.
Suppose that $e([p_1, q_\beta], \{v, u_2\}) = 2$, then $p_1u_2, q_\beta v \in E$ and the following cycle $C$:

$$C: p_1u_2u_2^+ \ldots q_\beta vu_1 \ldots q_1xr_1 \ldots r_\gamma p_2 \ldots p_1$$

contains $M$, a contradiction.

Note that $p_1u_1 \notin E$, because if $p_1u_1 \in E$, then the graph $H' = (P', Q, R')$ with

$$P': u_1p_1 \ldots p_2y,$$

$$Q': u_1vu_2 \ldots q_\beta y,$$

$$R': u_1u_1^- \ldots q_1xr_1 \ldots r_\gamma y$$

is a strict $\Theta$-graph compatible with $M$ with $|V(R')| > |V(R)|$, a contradiction.

From the above we have $e([p_1, q_\beta], Q_i) \leq 1$, $e([a, b], Q_i) \leq 2$, and since $e([q_1], Q_i) \leq 2$ we have

$$e([a, b, p_1, p_2, q_1, q_\beta], Q_i) \leq 5. \quad (37)$$

By summing over all the paths $Q_i$ from (32)–(37) we have

$$e([a, b, p_1, p_2, q_1, q_\beta], Q\setminus\{x\}) \leq 4v_2(Q) + 5v_3(Q) - 1. \quad (38)$$

5.2.3. Neighbors of $a, b, p_2, q_1, q_\beta, p_1$ on $P\setminus\{y\}$

Using the similar arguments as in Section 5.2.2 we have

$$e([a, b, p_1, p_2, q_1, q_\beta], P\setminus\{y\}) \leq 4v_2(P) + 5v_3(P) - 1. \quad (39)$$

5.2.4. Neighbors of $a, b, p_2, q_1, q_\beta, p_1$ in $Z$

Since $G_M$ and $Z$ are independent we have

$$d_Z(p_2) = d_Z(q_1) = 0$$

and thus

$$e([a, b, p_1, p_2, q_1, q_\beta], Z) \leq 4t. \quad (40)$$

5.2.5. Neighbors of $p_2$ and $q_1$ on $R \cup Q \cup P$

Using similar methods as those in Sections 5.2.1–5.2.3 we get the following inequalities:

$$e([p_2, q_1], R\setminus\{x, y\}) \leq v_2^{BW}(R) + 2v_2^{WB}(R) + 2v_3(R), \quad (41)$$

$$e([p_2, q_1], Q\setminus\{x\}) \leq v_2(Q) + 2(v_3(Q) - 1) + 1 = v_2(Q) + 2v_3(Q) - 1, \quad (42)$$

$$e([p_2, q_1], P\setminus\{y\}) \leq v_2(P) + 2(v_3(P) - 1) + 1 = v_2(P) + 2v_3(P) - 1. \quad (43)$$

Now we shall estimate the sum of degrees. From (41)–(43) we have

$$d(p_2) + d(q_1) \leq v_2^{BW}(R) + m + 2\delta - 1 = v_2^{WB}(R) + n - t + \delta - 1. \quad (44)$$

5.3. Conclusion

From (31), (38)–(40) we have

$$d(a) + d(b) + d(p_1) + d(p_2) + d(q_1) + d(q_\beta)$$

$$\leq 4\left(\sum_{i=2}^{3} (v_i(P) + v_i(Q) + v_i(R)) + v_3(P) + v_3(Q) + v_3(R) - 2 + 4t - 2v_2^{WB}(R). \quad (45)$$
From (19), (23) and (45) we deduce that
\[ d(a) + d(b) + d(p_1) + d(p_2) + d(q_1) + d(q_\beta) \leq -2 + 4(m + 1) + 2\delta + 4t - v^w_B(R). \] (46)

Since \( n = m + \delta + t \), from (46) we have
\[ d(a) + d(b) + d(p_1) + d(p_2) + d(q_1) + d(q_\beta) \leq 4n + 2 - v^w_B(R) - 2\delta. \] (47)

From (44) and (47) we can deduce that
\[ d(a) + d(b) + d(q_\beta) + d(p_1) + 2d(p_2) + 2d(q_1) \leq 5n - t - \delta + 1. \] (48)

Note that \( \delta \geq 2 \) and from (48) we have
\[ d(a) + d(b) + d(q_\beta) + d(p_1) + 2d(p_2) + 2d(q_1) \leq 5n - 1. \] (49)

Now we shall give the lower bound of the sum of degrees. From (24)–(26) we have
\[ 4 \frac{5n}{4} \leq d(q_\beta) + d(p_1) + d(a) + d(b) + 2d(p_2) + 2d(q_1). \] (50)

Assuming that there does not exist a cycle in \( G \) which contains every edge of the matching \( M \), we have obtained (49) and (50). Hence,
\[ 5n \leq 5n - 1, \]
a contradiction. Part 2 of Theorem 6 is proven.

References