REMARKS ON THE COMPLEX GEOMETRY OF THE 3-MONOPOLE

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Abstract. We develop the Ercolani-Sinha construction of $SU(2)$ monopoles and make this effective for (a five parameter family of centred) charge 3 monopoles. In particular we show how to solve the transcendental constraints arising on the spectral curve. For a class of symmetric curves the transcendental constraints become a number theoretic problem and a recently proven identity of Ramanujan provides a solution. The Ercolani-Sinha construction provides a gauge-transform of the Nahm data.

Contents

1. Introduction

Magnetic monopoles, or the topological soliton solutions of Yang-Mills-Higgs gauge theories in three space dimensions, have been objects of fascination for over a quarter of a century. BPS monopoles in particular have been the focus of much research (see [MS04] for a recent review). These monopoles arise as a limit in which the Higgs potential is removed and satisfy a first order Bogomolny equation

$$B_i = \frac{1}{2}\sum_{j,k=1}^{3} \epsilon_{ijk} F^{jk} = D_i \Phi$$

(together with certain boundary conditions, the remnant of the Higgs potential). Here F_{ij} is the field strength associated to a gauge field A, and Φ is the Higgs field. These equations may be viewed as a dimensional reduction of the four dimensional self-dual equations upon setting all functions independent of x_4 and identifying $\Phi = A_4$. Just as Ward's twistor transform relates instanton solutions in $\mathbb{R}^4$ to certain holomorphic vector bundles over the twistor space $\mathbb{CP}^3$, Hitchin showed [Hit82] that the dimensional reduction leading to BPS monopoles could be made at the twistor level as well. Mini-twistor space is a two dimensional complex manifold isomorphic to $\mathrm{T}\mathbb{P}^1$, and BPS monopoles may be identified with certain bundles over this space. In particular a curve $\mathcal{C} \subset \mathrm{T}\mathbb{P}^1$, the spectral curve, arises in this construction and, subject to certain nonsingularity conditions, Hitchin was able to prove all monopoles could be obtained by this approach [Hit83]. Nahm also gave a transform of the ADHM instanton construction to produce BPS monopoles [Nah82]. The resulting Nahm's equations have Lax form and the corresponding spectral curve is again $\mathcal{C}$. Many striking results are now known, yet, disappointingly, explicit solutions are rather few. This paper is directed towards constructing new solutions.

In a seminal paper, Ercolani and Sinha [ES89] sought to bring methods from integrable systems to bear upon the construction of solutions to Nahm's equations for the gauge group $SU(2)$. Integrable structures have long been associated with the self-dual equations and BPS monopoles: Ercolani and Sinha showed how one could solve (a gauge transform of) the Nahm equations in terms of a Baker-Akhiezer function for the curve $\mathcal{C}$. While conceptually simple, the Ercolani-Sinha construction is remarkably challenging to implement, and they noted that although their 'procedure, can in principle, be carried out for arbitrary monopole number, however, there are obvious technical difficulties in almost every step of the' construction. Here we follow the approach of Ercolani-Sinha for the particular case of charge 3 $SU(2)$ monopoles.

An outline of our paper is as follows. In section 2 we recall aspects of the Hitchin, Nahm and Ercolani-Sinha constructions and then proceed to extend the latter in section 3. Here we shall present new formulae and clarify the appearance of constant gauge transformations in the Ercolani-Sinha construction. Further we will highlight the ingredients needed to make effective the construction and show how these reduce to evaluating quantities intrinsic to the curve. As an illustration of our general theory we consider the charge 2 monopole in section 4. Key to expressing the Baker-Akhiezer function for a curve $\mathcal{C}$ is determining Riemann's theta function built from the period matrix of $\mathcal{C}$. The first hurdle in implementing the Ercolani-Sinha construction is to analytically determine the period matrix for $\mathcal{C}$ and then understand the theta divisor. In section 5 we will introduce a class of (genus 4) curves for which we can do this. They are of the form

$$\eta^3 + \hat{\chi}(\zeta-\lambda_1)(\zeta-\lambda_2)(\zeta-\lambda_3)(\zeta-\lambda_4)(\zeta-\lambda_5)(\zeta-\lambda_6) = 0, \tag{1.1}$$

where λ_i, $i = 1, \ldots, 6$ are distinct complex numbers. (For appropriate λ_i this yields a charge 3 monopole.) This class of curves was studied by Wellstein over one hundred years ago [Wel99] and more recently by Matsumoto [Mat01]. Here we will introduce our homology basis and define branch points in terms of θ-constants following [Mat01].

Corresponding to (some of) Hitchin's nonsingularity conditions Ercolani and Sinha obtain restrictions on the allowed period matrices for the spectral curve. Equivalent formulations of these conditions were given in [HMR00]. The Ercolani-Sinha conditions are transcendental constraints and to solve these is the next (perhaps *the*) major hurdle to overcome in the construction. In section 6 we do this for our curves. At this stage we have replaced the constraints by relations between various hypergeometric integrals. To simplify matters for the present paper we next demand more symmetry and consider in section 7 the genus 4 curves

$$\eta^3 + \chi(\zeta^6 + b\zeta^3 - 1) = 0 \tag{1.2}$$

where b is a certain real parameter. This restriction has the effect of reducing the number of hypergeometric integrals to be calculated to two. Interestingly the relations we demand of these integrals are assertions of Ramanujan only recently proven. We will denote curves of the form (1.2) as *symmetric monopole* curves (though in fact they may not satisfy all of Hitchin's nonsingularity conditions). The tetrahedrally symmetric charge 3 monopole is of this form.

The curve (1.2) covers a hyperelliptic curve of genus two and two elliptic curves. We discuss these coverings. Using Weierstrass-Poincaré reduction theory we are able to express the theta function behaviour of these symmetric monopoles in terms of elliptic functions and fairly comprehensive results may be obtained. Finally, in section 8, we shall consider the curve (1.2) associated with tetrahedrally symmetric 3-monopole when the above parameter $b = 2\sqrt{5}$. This genus 4 curve covers 4 elliptic curves and all entries to the period matrices are expressible in terms of elliptic moduli. The analytical means which we are using for our analysis involve Thomae-type formulae, Weierstrass-Poincaré reduction theory, multivariable hypergeometric function and higher hyperegeometric equalities of Goursat. Our conclusions in section 9 will highlight various of our results.

Part 1. **General Considerations**

2. Monopoles

In this section we shall recall various features of the spectral curve coming from Hitchin's and Nahm's construction and then describe the Ercolan-Sinha construction based on this curve.

2.1. **Hitchin Data.** Using twistor methods Hitchin [Hit83] has shown that each static $SU(2)$ Yang-Mills-Higgs monopole in the BPS limit with magnetic charge n is equivalent to a spectral curve of a restricted form. If ζ is the inhomogeneous coordinate on the Riemann sphere, and (ζ,η) are the standard local coordinates on $T\mathbb{P}^1$ (defined by $(\zeta,\eta)\to\eta\frac{d}{d\zeta}$), the spectral curve is an algebraic curve $\mathcal{C}\subset T\mathbb{P}^1$ which has the form

$$P(\eta,\zeta)=\eta^n+\eta^{n-1}a_1(\zeta)+\ldots+\eta^r a_{n-r}(\zeta)+\ldots+\eta\, a_{n-1}(\zeta)+a_n(\zeta)=0. \tag{2.1}$$

Here $a_r(\zeta)$ (for $1\le r\le n$) is a polynomial in ζ of maximum degree $2r$.

The Hitchin data constrains the curve $\mathcal{C}$ explicitly in terms of the polynomial $P(\eta,\zeta)$ and implicitly in terms of the behaviour of various line bundles on $\mathcal{C}$. If the homogeneous coordinates of $\mathbb{P}^1$ are $[\zeta_0,\zeta_1]$ we consider the standard covering of this by the open sets $U_0=\{[\zeta_0,\zeta_1]\,|\,\zeta_0\ne0\}$ and $U_1=\{[\zeta_0,\zeta_1]\,|\,\zeta_1\ne0\}$, with $\zeta=\zeta_1/\zeta_0$ the usual coordinate on U_0. We will denote by $\hat U_{0,1}$ the pre-images of these sets under the projection map $\pi:T\mathbb{P}^1\to\mathbb{P}^1$. Let L^λ denote the holomorphic line bundle on $T\mathbb{P}^1$ defined by the transition function $g_{01}=\exp(-\lambda\eta/\zeta)$ on $\hat U_0\cap\hat U_1$, and let $L^\lambda(m)\equiv L^\lambda\otimes\pi^*\mathcal{O}(m)$ be similarly defined in terms of the transition function $g_{01}=\zeta^m\exp(-\lambda\eta/\zeta)$. A holomorphic section of such line bundles is given in terms of holomorphic functions f_α on $\hat U_\alpha$ satisfying $f_\alpha=g_{\alpha\beta}f_\beta$. We denote line bundles on $\mathcal{C}$ in the same way, where now we have holomorphic functions f_α defined on $\mathcal{C}\cap\hat U_\alpha$.

The Hitchin data constrains the curve to satisfy:

A1. Reality conditions

$$a_r(\zeta)=(-1)^r\zeta^{2r}\overline{a_r(-\frac{1}{\bar\zeta})}. \tag{2.2}$$

This is the requirement that $\mathcal{C}$ is real with respect to the standard real structure on $T\mathbb{P}^1$

$$\tau:(\zeta,\eta)\mapsto(-\frac{1}{\bar\zeta},-\frac{\bar\eta}{\bar\zeta^2}). \tag{2.3}$$

This is the anti-holomorphic involution defined by reversing the orientation of the lines in $\mathbb{R}^3$. A consequence of the reality condition is that we may parameterize $a_r(\zeta)$ as follows,

$$a_r(\zeta)=\sum_{k=0}^{2r}a_{rk}\,\zeta^k=\chi_r\left[\prod_{l=1}^{r}\left(\frac{\overline{\alpha}_l}{\alpha_l}\right)^{1/2}\right]\prod_{k=1}^{r}(\zeta-\alpha_k)(\zeta+\frac{1}{\overline{\alpha}_k}),\qquad \alpha_r\in\mathbb{C},\ \chi_r\in\mathbb{R}. \tag{2.4}$$

Thus each $a_r(\zeta)$ contributes $2r+1$ (real) parameters.

A2. L^2 is trivial on $\mathcal{C}$ and $L(n-1)$ is real. The triviality of L^2 on $\mathcal{C}$ means that there exists a nowhere-vanishing holomorphic section. In terms of our open sets $\hat U_{0,1}$ we then have two, nowhere-vanishing holomorphic functions, f_0 on $\hat U_0\cap\mathcal{C}$ and f_1 on $\hat U_1\cap\mathcal{C}$, such that on $\hat U_0\cap\hat U_1\cap\mathcal{C}$

$$f_0(\eta,\zeta)=\exp\left\{-2\frac{\eta}{\zeta}\right\}f_1(\eta,\zeta). \tag{2.5}$$

A3. $H^0(\mathcal{C},L^\lambda(n-2))=0$ for $\lambda\in(0,2)$.

For a generic n-monopole the spectral curve is irreducible and has genus $g_{\mathcal{C}}=(n-1)^2$. This may be calculated as follows. For fixed ζ the n roots of $P(\eta,\zeta)=0$ yield an n-fold covering of the Riemann sphere. The branch points of this covering are given by

$$0 = \ \text{Resultant}_\eta(P(\eta,\zeta),\partial_\eta P(\eta,\zeta)) = \prod_{i=1}^{n} \partial_\eta P(\eta_i,\zeta), \qquad \text{where } P(\eta_i,\zeta)=0.$$

This expression is of degree $n\times \deg a_{n-1} = n(2n-2)$ in ζ and so by the Riemann-Hurwitz theorem we have that

$$2g_{\mathcal{C}} - 2 = 2n(g_{\mathbb{P}^1}-1) + n(2n-2) = 2(n-1)^2 - 2,$$

whence the genus as stated.

The $n=1$ monopole spectral curve is given by

$$\eta = (x_1+ix_2) - 2x_3\zeta - (x_1-ix_2)\zeta^2,$$

where $x=(x_1,x_2,x_3)$ is any point in $\mathbb{R}^3$. In general the three independent real coefficients of $a_1(\zeta)$ may be interpreted as the centre of the monopole in $\mathbb{R}^3$. Strongly centred monopoles have the origin as center and hence $a_1(\zeta)=0$. The group $SO(3)$ of rotations of $\mathbb{R}^3$ induces an action on $T\mathbb{P}^1$ via the corresponding $PSU(2)$ transformations. If

$$\begin{pmatrix} p & q \\ -\bar q & \bar p\end{pmatrix} \in PSU(2), \qquad |p|^2+|q|^2=1,$$

the transformation on $T\mathbb{P}^1$ given by

$$\zeta \to \frac{\bar p\,\zeta - \bar q}{q\,\zeta + p}, \qquad \eta \to \frac{\eta}{(q\,\zeta+p)^2}$$

corresponds to a rotation by θ around $\mathbf{n}\in S^2$, where $n_1\sin(\theta/2) = \operatorname{Im} q$, $n_2 \sin(\theta/2) = -\operatorname{Re} q$, $n_3\sin(\theta/2) = -\operatorname{Im} q$, and $\cos(\theta/2) = \operatorname{Re} p$. (Here the η transformation is given by the derivative of the ζ transformation.) The $SO(3)$ action commutes with the real structure τ. Although a general Möbius transformation does not change the period matrix of a curve $\mathcal{C}$ only the subgroup $PSU(2) < PSL(2,\mathbb{C})$ preserves the desired reality properties . We have that

$$\alpha_k \to \tilde\alpha_k \equiv \frac{p\alpha_k+\bar q}{\bar p - \alpha_k q}, \qquad \chi_r \to \tilde\chi_r \equiv \chi_r \prod_{k=1}^{r}\left[\frac{(\bar p - \alpha_k q)(p-\bar\alpha_k\bar q)(\bar\alpha_k\bar p + q)(\alpha_k p + \bar q)}{\alpha_k\bar\alpha_k}\right]^{1/2},$$

and

$$a_r \to \frac{\tilde a_r}{(q\,\zeta+p)^{2r}} \equiv \frac{\tilde\chi_r}{(q\,\zeta+p)^{2r}}\left[\prod_{l=1}^{r}\left(\frac{\bar{\tilde\alpha}_l}{\tilde\alpha_l}\right)^{1/2}\right]\prod_{k=1}^{r}(\zeta-\tilde\alpha_k)(\zeta+\frac{1}{\bar{\tilde\alpha}_k}).$$

In particular the form of the curve does not change: that is, if $a_r=0$ then so also $\tilde a_r = 0$. It is perhaps worth emphasizing that the reality conditions are an extrinsic feature of the curve (encoding the space-time aspect of the problem) whereas the intrinsic properties of the curve are invariant under birational transformations or the full Möbius group. Such extrinsic aspects are not a part of the usual integrable system story.

2.2. **Nahm Data.** The Nahm construction of charge n $SU(2)$ monopoles is in terms of $n\times n$ matrices (T_1,T_2,T_3) depending on a real parameter $s\in[0,2]$ and satisfying the following:

B1. Nahm's equation

$$\frac{dT_i}{ds} = \frac{1}{2}\sum_{j,k=1}^{3}\epsilon_{ijk}[T_j,T_k]. \qquad (2.6)$$

B2. $T_i(s)$ is regular for $s \in (0,2)$ and has simple poles at $s=0$ and $s=2$, the residues of which form the irreducible n-dimensional representation of $su(2)$.

B3. $T_i(s) = -T_i^\dagger(s), \qquad T_i(s) = T_i^t(2-s).$

A caution is perhaps worth giving regarding the second of the constraints B3. All that really is required is that the matrices $T_i(s)$ are conjugate to the matrices $T_i^t(2-s)$. Many explicit examples often take this for granted. Thinking of the Nahm equations as a one-dimensional gauge theory then we still have some gauge freedom left, associated with constant gauge transformations. The spectral curve is gauge invariant, so if it has the correct reality properties this guarantees that there is a gauge in which the $s \to 2-s$ relation is explicit even if we do not happen to be in that gauge at the moment.[1]

The Nahm equations admit a Lax formulation. Upon setting

$$A_{-1} = T_1 + iT_2,\ A_0 = -2iT_3,\ A_1 = T_1 - iT_2,$$

$$A = A_{-1}\zeta^{-1} + A_0 + A_1\zeta, \qquad M = \frac{1}{2}A_0 + A_1\zeta,$$

then

$$\frac{dA}{ds} = [A, M],\ \text{or equivalently}\ [\frac{d}{ds} + M, A] = 0. \tag{2.7}$$

Nahm's equation (2.6) describes linear flow on a complex torus, which is the Jacobian of an algebraic curve. This algebraic curve is in fact the monopole spectral curve $\mathcal{C}$ and may be explicitly read off from the Lax equation

$$P(\eta,\zeta) = \det(\eta + (T_1 + iT_2) - 2iT_3\zeta + (T_1 - iT_2)\zeta^2) = 0. \tag{2.8}$$

The regularity condition B2 for $s \in (0,2)$ is the manifestation in the ADHMN approach of the condition A3 for spectral curves.

2.3. **The Ercolani-Sinha construction.** We shall now present a short overview of the Ercolani-Sinha construction which expresses a gauge transform of the Nahm data in terms of Baker-Akhiezer functions on the spectral curve $\mathcal{C}$. An explicit representation of these functions will be given after that. Extensions to the theory of Ercolani and Sinha will be presented in the next subsection.

2.3.1. *Overview.* Let $z = s-1$. Then $z \in [-1,1]$ for $s \in [0,2]$ and we have that[2]

$$A_0(z) = A_0^\dagger(z), \qquad A_1(z) = -A_{-1}^\dagger(z), \qquad A_\alpha(z) = A_\alpha^t(-z),\ \alpha = 1,2,3. \tag{2.9}$$

Ercolani and Sinha begin by focussing attention on the differential operator

$$\frac{d}{dz} + M(z) = \frac{d}{dz} + \frac{1}{2}A_0(z) + A_1(z)\zeta$$

related to the Lax equation (2.7). The spectral theory of this equation enables the integration of the Lax equation. The z-dependence of the term $A_1(z)$ means that

$$\left(\frac{d}{dz} + \frac{1}{2}A_0(z)\right)\varphi = -\zeta A_1(z)\varphi$$

[1]We thank Paul Sutcliffe for discussions on this point.

[2]The matrices of Ercolani-Sinha and Nahm are related by $T_i^{\mathrm{ES}}(z) = T_i^{\mathrm{Nahm}}(z+1)$, whence

$$T_i^{\mathrm{ES}}(z) = T_i^{\mathrm{Nahm}}(z+1) = T_i^{\mathrm{Nahm}\,t}(2-[z+1]) = T_i^{\mathrm{Nahm}\,t}(1-z) = T_i^{\mathrm{ES}\,t}(-z).$$

is not of standard eigenvalue form. By considering the gauge transformation

$$Q_\alpha(z) = C^{-1}(z)A_\alpha(z)C(z), \qquad \varphi = C(z)\Phi,$$

they obtain the standard eigenvalue equation

$$\left(\frac{d}{dz} + Q_0(z)\right)\Phi = -\zeta Q_1(0)\Phi, \tag{2.10}$$

if and only if $C(z)$ satisfies

$$(C(z)^{-1}C'(z) = \frac{1}{2}Q_0(z), \text{ equivalently } \left(\frac{d}{dz} + \frac{1}{2}Q_0(z)\right)C^{-1} = 0. \tag{2.11}$$

The gauge transform was chosen so that $Q_1(z) = A_1(0) = Q_1(0)$. From (2.9) we see that this is a symmetric matrix, and (by an overall constant gauge transformation) we may assume this is diagonal,

$$A_1(0) = Q_1(0) = \text{diag}(\rho_1, \ldots, \rho_n). \tag{2.12}$$

We see from (2.10) that the ρ_j here (which may be assumed distinct) correspond to the roots of $P(\eta,\zeta)/\zeta^{2n}$ near $\zeta = \infty$,

$$\frac{P(\eta,\zeta)}{\zeta^{2n}} \sim \prod_{j=1}^{n}\left(\frac{\eta}{\zeta^2} - \rho_j\right). \tag{2.13}$$

As a consequence we see that at ∞_j we have

$$\frac{\eta}{\zeta} \sim \rho_j\,\zeta, \qquad d\left(\frac{\eta}{\zeta}\right) \sim \rho_j\,d\zeta = \left(-\frac{\rho_j}{t^2} + O(1)\right)dt, \tag{2.14}$$

where $t = 1/\zeta$ is a local coordinate. From (2.4) we see that at $\zeta = 0$ we also have that

$$P(\eta, 0) = \prod_{j=1}^{n}\left(\eta + \overline{\rho_j}\right). \tag{2.15}$$

The spectral curves $0 = \det(\eta - A) = \det(\eta - Q)$ agree being related by a gauge transformation $Q = C^{-1}AC$. Ercolani and Sinha now construct $Q_0(z)$ in terms of a Baker-Akhiezer function on $\mathcal{C}$. Baker-Akhiezer functions are a slight extension to the class of meromorphic functions that allow essential singularities at a finite number of points; they have many properties similar to those of meromorphic functions. While for a meromorphic function one needs to prescribe $g_{\mathcal{C}} + 1$ poles in the generic situation, a non-trivial Baker-Akhiezer function exists with $g_{\mathcal{C}}$ arbitrarily prescribed poles on a surface of genus $g_{\mathcal{C}}$. The key result is the following theorem.

Theorem 2.1 (Krichever, 1977)**.** *Let $\mathcal{C}$ be a smooth algebraic curve of genus $g_{\mathcal{C}}$ with $n \geqslant 1$ punctures P_j, $j = 1, \ldots, n$. Then for each set of $g_S + n - 1$ points $\delta_1, \ldots, \delta_{g_S+n-1}$ in general position, there exists a unique function $\Psi_j\,(t, P)$ and local coordinates $w_j(P)$ for which $w_j(P_j) = 0$, such that*

1. *The function Ψ_j of $P \in \mathcal{C}$ is meromorphic outside the punctures and has at most simple poles at δ_s (if all of them are distinct);*
2. *In the neighbourhood of the puncture P_l the function Ψ_j has the form*

$$\Psi_j\,(z, P) = e^{z\,{w_l}^{-m}}\left(\delta_{jl} + \sum_{k=1}^{\infty}\alpha_{jl}^k\,(z)\,w_l^k\right), \qquad w_l = w_l\,(P), \qquad m \in \mathbb{N}^+. \tag{2.16}$$

The integer $m \geq 1$ in the theorem is arbitrary and in applications is determined by a given flow. Let $\tilde{w}_j(P)$ be any local coordinates on $\mathcal{C}$ such that $\tilde{w}_j(P_j) = 0$. To a particular flow we associate the unique meromorphic differential $d\Omega^{[m]}$ on $\mathcal{C}$, holomorphic outside the punctures P_j, with form

$$d\Omega^{[m]} = d\left(\tilde{w}_j^{-m} + 0\left(\tilde{w}_j\right)\right) \tag{2.17}$$

near the puncture P_j, and normalized with vanishing $\mathfrak{a}$-periods

$$\oint_{\mathfrak{a}_k} d\Omega^{[m]} = 0. \tag{2.18}$$

We may utilise the Baker-Akhiezer function in the monopole setting as follows. Let $\Phi_1, \ldots, \Phi_n$ be the columns of the fundamental matrix solution Ω to (2.10), normalized so that

$$\exp(\zeta A_1(0)z)\Omega\big|_{\infty} = \mathrm{Id}_n. \tag{2.19}$$

In view of (2.14) we consider a differential γ_∞ of the second kind such that

$$\gamma_\infty(P) = \left(\frac{\rho_l}{t^2} + O(1)\right)dt, \quad \text{as} \quad P \to \infty_l, \tag{2.20}$$

$$\oint_{\mathfrak{a}_k} \gamma_\infty(P) = 0, \quad \forall k = 1, \ldots, g, \tag{2.21}$$

and take as punctures $\{P_l = \infty_l\}$ $(l = 1, \ldots, n)$, the n points on $\mathcal{C}$ which lie above the point $\zeta = \infty$. Then

Theorem 2.2 (Ercolani and Sinha, 1989)**.** *The j-th component of Φ_l normalised by (2.19) is given by the expansion (2.16) of Ψ_j at ∞_l. Further the matrix Q_0 has vanishing diagonal entries and may be reconstructed from*

$$\left(Q_0\right)_{jl} = -(\rho_j - \rho_l)\,\alpha^1_{jl} = -(\rho_j - \rho_l)\lim_{P\to\infty_l} \zeta \exp(\zeta\rho_l z)\Psi_j(z, P). \tag{2.22}$$

The steps involved to obtain Nahm data in the Ercolani-Sinha construction, are as follows:

1. From the asymptotic properties of the curve solve for ρ_j $(j = 1, \ldots, g_{\mathcal{C}})$. Then $A_1(0) = Q_1(0) = \mathrm{diag}(\rho_j)$.
2. Determine $Q_0(z)$ from (2.10) in terms of the Baker-Akhiezer function (2.22).
3. Determine $C(z)$ from (2.11). Then
   - (a) $A_0(z) = C(z)Q_0C^{-1}(z)$.
   - (b) $A_1(z) = C(z)A_1(0)C^{-1}(z)$.
   - (c) $A_{-1}(z) = -A_1^\dagger(z)$.
4. From $A(z)$ reconstruct $T_i^{\mathrm{ES}}(z) = T_i^{\mathrm{Nahm}}(z+1)$.

The constraints (B2, B3) or (A2, A3) arise in the Ercolani-Sinha construction as constraints on the Baker-Akhiezer functions in step (2). Thus to implement the approach we need to be able to concretely express Baker-Akhieser functions. We shall consider these in more detail in the next subsection, first in general and then in the monopole context. Before doing this however it will be useful to make some remarks regarding the gauge ambiguities of the solution. The matrices A, M and C were initially defined up to constant gauge transformations. By choosing the form (2.12) these were reduced to constant diagonal gauge transformations, such preserving the normalization (2.19). At this stage then our matrix $Q_0(z)$ defined in terms of the Baker-Akhieser function is defined up to constant diagonal gauge transformations, $Q_0(z)_{ij} \sim d_i Q_0(z)_{ij} d_j^{-1}$ for $d_i \neq 0$ $(i, j = 1 \ldots n)$.

2.3.2. *Baker-Akhieser functions.* The functions Ψ_j of theorem 2.1 may be written explicitly in terms of θ-functions and the Abel map $\boldsymbol{\phi}$. Given the normalised differential $d\Omega^{[m]}$ of the second kind (2.17, 2.18) define the vector $U^{[m]}$ with coordinates

$$U_k^{[m]} = \frac{1}{2\pi i} \oint_{\mathfrak{b}_k} d\Omega^{[m]}.$$

Then the function $\Psi_j(z,P)$ may be expressed as

$$\Psi_j(z,P) = g_j(P) \frac{\theta\left(\boldsymbol{\phi}(P) - \boldsymbol{\mathcal{Z}}_j + z\,U^{[m]}\right)\theta\left(\boldsymbol{\phi}(P_j) - \boldsymbol{\mathcal{Z}}_j\right)}{\theta\left(\boldsymbol{\phi}(P_j) - \boldsymbol{\mathcal{Z}}_j + z\,U^{[m]}\right)\theta\left(\boldsymbol{\phi}(P) - \boldsymbol{\mathcal{Z}}_j\right)} e^{z \int_{P_0}^{P} d\Omega^{[m]}}. \tag{2.23}$$

Here

$$\boldsymbol{\mathcal{Z}}_j = \boldsymbol{\mathcal{Z}}_T + \boldsymbol{\phi}(P_j) \equiv \boldsymbol{\phi}(\Delta_j) + \boldsymbol{K}, \qquad \boldsymbol{\mathcal{Z}}_T = \sum_{s=1}^{g_{\mathcal{C}}+n-1} \boldsymbol{\phi}(\delta_s) - \sum_{j=1}^{n} \boldsymbol{\phi}(P_j) + \boldsymbol{K},$$

where $\boldsymbol{K}$ is the vector of Riemann constants (with base point P_0). Our conventions for theta functions are given in the Appendix. By Abel's theorem $\boldsymbol{\mathcal{Z}}_j$ is equivalent to a an effective divisor Δ_j of degree g. The function $g_j(P)$ is the unique meromorphic function with

$$g_j(P_l) = \delta_{jl}$$

and for $n \geq 2$ having poles from $\{\delta_1, \ldots, \delta_{g_{\mathcal{C}}+n-1}\}$. For the case $n = 1$ we have $g_j(P) = 1$. Again, this function may be explicitly constructed. Set

$$g_j(P) = \frac{f_j(P)}{f_j(P_j)},$$

where

$$f_j(P) = \theta(\boldsymbol{\phi}(P) - \boldsymbol{\mathcal{Z}}_j) \frac{\prod_{l \neq j} \theta(\boldsymbol{\phi}(P) - \boldsymbol{R}_l)}{\prod_{k=1}^{n} \theta(\boldsymbol{\phi}(P) - \boldsymbol{S}_k)}$$

and

$$\boldsymbol{R}_j = \sum_{s=1}^{g_{\mathcal{C}}-1} \boldsymbol{\phi}(\delta_s) + \boldsymbol{\phi}(P_j) + \boldsymbol{K}, \qquad \boldsymbol{S}_j = \sum_{s=1}^{g_{\mathcal{C}}-1} \boldsymbol{\phi}(\delta_s) + \boldsymbol{\phi}(\delta_{g_{\mathcal{C}}-1+j}) + \boldsymbol{K}.$$

Observe that for $n \geq 2$ the factors $\theta(\boldsymbol{\phi}(P) - \boldsymbol{\mathcal{Z}}_j)$ cancel between the term involving $g_j(P)$ and the theta function in the denominator of (2.23), and so no extraneous poles are added.

Now the function $\Psi_j(z,P)$, which depends on the choice of base point of the Abel map, has the requisite properties of theorem 2.1 aside from that of normalization. Set

$$\nu_j \equiv \nu_j(P_0) = \lim_{P \to \infty_j} \left[\int_{P_0}^{P} d\Omega^{[m]} - \frac{1}{\tilde{w}^m(P)} \right]. \tag{2.24}$$

Thus for the local coordinate $\tilde{w}$ the Baker-Akhieser function differs from the normalization of (2.16) by the exponential $\exp(z\nu_j)$. For this local coordinate the function $\exp(-z\nu_j)\,\Psi_j$ has the desired normalization. Alternately we may make a change of local coordinates

$$w = \frac{\tilde{w}}{1 + \frac{1}{m}\tilde{w}^m \nu(P_0)}$$

for which

$$\frac{1}{w^m} = \frac{(1 + \frac{1}{m}\tilde{w}^m c(P_0))^m}{\tilde{w}^m} = \frac{1}{\tilde{w}^m} + \nu(P_0) + O(\tilde{w}^m)$$

and we have a local coordinate for which the Baker-Akhieser function has the desired expansion.

Let us conclude with some discussion of the divisor $\delta \equiv \sum_{s=1}^{g_{\mathcal{C}}+n-1} \delta_s$ explaining what is meant by saying that it is "in general position". We may interpret the meromorphic functions $g_j(P)$ as follows. Let L_δ denote the line bundle on $\mathcal{C}$ determined by the divisor δ and denote by s_δ a (nonzero) meromorphic section of this line bundle. (We shall further identify L_δ in the monopole setting in due course.) Then the divisor of $g_j(P)$ is

$$\operatorname{Div} g_j(P) = \Delta_j + P_1 + \ldots + \hat{P}_j + \ldots + P_n - \delta. \tag{2.25}$$

Thus $g_j(P)s_\delta$ yields a holomorphic section of L_δ. Now by Riemann-Roch

$$\operatorname{Dim} H^0(\mathcal{C}, \mathcal{O}(L_\delta)) = \deg L_\delta + 1 - g_{\mathcal{C}} + \operatorname{Dim} H^1(\mathcal{C}, \mathcal{O}(L_\delta)) = n + \operatorname{Dim} H^1(\mathcal{C}, \mathcal{O}(L_\delta)) \geq n,$$

and L_δ has precisely n holomorphic sections when $\operatorname{Dim} H^1(\mathcal{C}, \mathcal{O}(L_\delta)) = 0$. This latter constraint means δ is a nonspecial divisor. This is a condition on the divisor. Now consideration of the short exact sequence

$$0 \to \mathcal{O}(L) \xrightarrow{s_p} \mathcal{O}(LL_p) \to \mathcal{O}_p(LL_p) \to 0$$

and the corresponding long exact sequence

$$0 \to H^0(\mathcal{C}, \mathcal{O}(L)) \to H^0(\mathcal{C}, \mathcal{O}(LL_p)) \to \mathbb{C} \to H^1(\mathcal{C}, \mathcal{O}(L)) \to H^1(\mathcal{C}, \mathcal{O}(LL_p)) \to 0$$

shows us that either of two possibilities arise,

(a) $\operatorname{Dim} H^0(\mathcal{C}, \mathcal{O}(LL_p)) = \operatorname{Dim} H^0(\mathcal{C}, \mathcal{O}(L)) + 1$, and $\operatorname{Dim} H^1(\mathcal{C}, \mathcal{O}(L)) = \operatorname{Dim} H^1(\mathcal{C}, \mathcal{O}(LL_p))$,

(b) $\operatorname{Dim} H^0(\mathcal{C}, \mathcal{O}(LL_p)) = \operatorname{Dim} H^0(\mathcal{C}, \mathcal{O}(L))$, and $\operatorname{Dim} H^1(\mathcal{C}, \mathcal{O}(L)) = \operatorname{Dim} H^1(\mathcal{C}, \mathcal{O}(LL_p)) + 1$.

In particular, if $\operatorname{Dim} H^1(\mathcal{C}, \mathcal{O}(L)) = 0$ then $\operatorname{Dim} H^0(\mathcal{C}, \mathcal{O}(LL_p)) = \operatorname{Dim} H^0(\mathcal{C}, \mathcal{O}(L)) + 1$. (When the divisor of LL_p is effective setting (a) is the generic situation, true for general p.) Using these results together with Riemann-Roch we find that (for each $j = 1, \ldots, n$)

$$\begin{aligned}
&\operatorname{Dim} H^0(\mathcal{C}, \mathcal{O}(L_{\delta - \sum_{k=1}^n P_k})) = \operatorname{Dim} H^0(\mathcal{C}, \mathcal{O}(L_{\Delta_j - P_j})) = 0 \\
\Longleftrightarrow &\operatorname{Dim} H^1(\mathcal{C}, \mathcal{O}(L_{\delta - \sum_{k=1}^n P_k})) = \operatorname{Dim} H^1(\mathcal{C}, \mathcal{O}(L_{\Delta_j - P_j})) = 0, \\
\Longrightarrow &\begin{cases} \operatorname{Dim} H^0(\mathcal{C}, \mathcal{O}(L_{\delta + P_j - \sum_{k=1}^n P_k})) = \operatorname{Dim} H^0(\mathcal{C}, \mathcal{O}(L_{\Delta_j})) = 1, \\ \operatorname{Dim} H^1(\mathcal{C}, \mathcal{O}(L_{\delta + P_j - \sum_{k=1}^n P_k})) = \operatorname{Dim} H^1(\mathcal{C}, \mathcal{O}(L_{\Delta_j})) = 0, \end{cases} \\
\Longrightarrow &\operatorname{Dim} H^0(\mathcal{C}, \mathcal{O}(L_\delta)) = n, \quad \operatorname{Dim} H^1(\mathcal{C}, \mathcal{O}(L_\delta)) = 0.
\end{aligned}$$

The condition $\operatorname{Dim} H^0(\mathcal{C}, \mathcal{O}(L_{\Delta_j})) = 1$ says that the divisors Δ_j are nonspecial and we have used this in our construction to assert the uniqueness of the functions $g_j(P)$ and correspondingly that of the Baker-Akhieser functions. Actually our requirement that $g_j(P_j) = 1$ means that we can say more here. Our analysis of the long exact sequence shows that either $\operatorname{Dim} H^0(\mathcal{C}, \mathcal{O}(L_{\Delta_j - P_j})) = 0$ or 1. If the latter then the divisor $\Delta_j - P_j$ is equivalent to an effective divisor, $\Delta_j - P_j \sim_l \sum_{k=1}^{g_{\mathcal{C}}-1} Q_k$, whence $\Delta_j \sim_l P_j + \sum_{k=1}^{g_{\mathcal{C}}-1} Q_k$. But then $g_j(P_j) = 0$, a contradiction. Thus $\operatorname{Dim} H^0(\mathcal{C}, \mathcal{O}(L_{\Delta_j - P_j})) = 0$ and we have established the necessary and sufficient condition for the construction of the Baker-Akhieser function (for each $j = 1, \ldots, n$),

$$\operatorname{Dim} H^0(\mathcal{C}, \mathcal{O}(L_{\delta - \sum_{k=1}^n P_k})) = 0 \Longleftrightarrow \operatorname{Dim} H^0(\mathcal{C}, \mathcal{O}(L_{\delta + P_j - \sum_{k=1}^n P_k})) = 1. \tag{2.26}$$

Condition (2.26) says that the degree $g_{\mathcal{C}} - 1$ divisor $\delta - \sum_{k=1}^n P_k$ is noneffective. In particular this means that $\delta - \sum_{k=1}^n P_k \in \operatorname{Jac}^{g_{\mathcal{C}}-1}(\mathcal{C}) \setminus \Theta$. Here the theta divisor Θ is precisely the image (up to a shift by the vector of Riemann constants) by the Abel map of degree $g_{\mathcal{C}} - 1$

effective divisors in the Jacobian $\mathrm{Jac}^{g_{\mathcal{C}}-1}(\mathcal{C})$. Now Θ is of codimension one in the Jacobian, and so (2.26) will hold for generic divisors. This is what we mean by δ being "in general position". Finally let us remark that just as the functions $g_j(P) = \Psi_j(0, P)$ yield sections of a line bundle L_δ, then similarly the functions $\Psi_j(z, P)$ yield sections of a line bundle which we will denote L_δ^z, but now the transition functions in the vicinity of P_l involve the exponential term $\exp(z w_l^{-i})$.

2.3.3. *The Ercolani-Sinha constraints.* It follows from the last paragraph that, upon setting

$$\nu_j = \lim_{P\to\infty_j}\left[\int_{P_0}^{P}\gamma_\infty(P) + \frac{\eta}{\zeta}\right], \tag{2.27}$$

we may write the Baker-Akhieser function of theorem 2.2 as

$$\Psi_j\,(z,P) = g_j(P)\frac{\theta\,(\boldsymbol{\phi}(P) - \boldsymbol{\mathcal{Z}}_j + z\,U)\,\theta\,(\boldsymbol{\phi}\,(P_j) - \boldsymbol{\mathcal{Z}}_j)}{\theta\,(\boldsymbol{\phi}\,(P_j) - \boldsymbol{\mathcal{Z}}_j + z\,U)\,\theta\,(\boldsymbol{\phi}(P) - \boldsymbol{\mathcal{Z}}_j)}e^{z\int_{P_0}^{P}\gamma_\infty - z\,\nu_j} \tag{2.28}$$

for a suitably generic divisor $\delta = \sum_{s=1}^{n(n-1)}\delta_s$. Thus from (2.22) we obtain the matrix Q_0 of (2.10) as

$$(Q_0)_{jl} = -(\rho_j - \rho_l)\,c_{jl}\,e^{z[\nu_l - \nu_j]}\,\frac{\theta\,(\boldsymbol{\phi}(P_l) - \boldsymbol{\mathcal{Z}}_j + z\,U)\,\theta\,(\boldsymbol{\phi}\,(P_j) - \boldsymbol{\mathcal{Z}}_j)}{\theta\,(\boldsymbol{\phi}\,(P_j) - \boldsymbol{\mathcal{Z}}_j + z\,U)\,\theta\,(\boldsymbol{\phi}(P_l) - \boldsymbol{\mathcal{Z}}_j)}, \tag{2.29}$$

where

$$c_{jl} = \lim_{P\to\infty_l}\zeta\,g_j(P), \qquad P = (\zeta,\eta)\in\mathcal{C}. \tag{2.30}$$

We note that the constants c_{ij} appearing in this solution depend on the divisor δ through the functions $g_j(P)$. A puzzle is what this dependence corresponds to in the physical setting. In due course we shall show that this corresponds to a gauge choice and give a simple form for these constants.

At this stage we have not imposed the Hitchin constraints A2, A3 on our curve $\mathcal{C}$. First let us identify the line bundles L_δ^z in the construction. From (2.27, 2.28) we have that $\Psi_j \sim \exp(-z\eta/\zeta)|_{\infty_j}$ and so leads to transition functions of this form in the neighbourhood of $\zeta = \infty$. Now from (2.25) the line bundle $L_\delta^{z=0} = L_\delta$ has divisor (for each j)

$$\delta \sim_l \Delta_j + \infty_1 + \ldots + \hat{\infty}_j + \ldots + \infty_n,$$

and consequently has a zero of order $n-1$ above infinity. In terms of the local coordinate $\tilde{\zeta} = 1/\zeta$ this corresponds to a section $s_1 = \sum_{k=0}^{n-1}\mu_k\tilde{\zeta}^k$ with transition function $g_{01} = z^{n-1}$. (Here $s_0(P) = \Psi_j(z,P)$ on $\hat{U}_0 \cap \mathcal{C} \cap \backslash\{\delta\}$ and $s_1(P) = 1$ on $\hat{U}_1 \cap \mathcal{C}$, while for patches $V_{j+1} \subset \hat{U}_0 \cap \hat{U}_1 \cap \mathcal{C}$, $\delta_j \in V_{j+1}$ and with $V_j \cap V_k = \emptyset$ $(j \neq k)$ we have for $P \in V_{j+1}$ that $s_{j+1}(P) = w_j$.) Thus we may identify our bundle L_δ^z with the bundle $L^{z+1}(n-1)$ of Hitchin. Further the bundle $L_{\delta - \sum_{k=1}^{n}P_k} = L_\delta \otimes \pi^*\mathcal{O}(-1) \equiv L_\delta(-1)$ is then identified with Hitchin's $L^1(n-2)$. Condition A3 for $\lambda = 1$ is then the constraint (2.26),

$$L_\delta(-1) \in \mathrm{Jac}^{g_{\mathcal{C}}-1}(\mathcal{C}) \setminus \Theta.$$

This condition means that the (push-forward) rank n vector bundle $E = \pi_* L_\delta$ on $\mathbb{P}^1$ is holomorphically trivial. We shall now discuss the constraints A2 and A3.

A2 With regards to A2 Ercolani and Sinha show that the functions $g_j(P)$ form a basis of the holomorphic sections of $L(n-1)$ and as a consequence $L(n-1)$ is real. Then they consider the logarithmic derivative of (2.5) representing the triviality of L^2 on $\mathcal{C}$,

$$\mathrm{dlog}\, f_0 = d\left(-2\frac{\eta}{\zeta}\right) + \mathrm{dlog} f_1. \tag{2.31}$$

(Hurtubise considered a similar construction in the $n=2$ case [Hur83].) Now in order to avoid essential singularities in $f_{0,1}$ we have from (2.13, 2.15) that

$$\mathrm{dlog}\, f_1(P) = \left(-\frac{2\eta_j(0)}{\zeta^2} + O(1)\right) d\zeta = \left(\frac{2\bar{\rho}_j(0)}{\zeta^2} + O(1)\right) d\zeta, \quad \text{at} \quad P \to 0_j, \tag{2.32}$$

$$\mathrm{dlog}\, f_0(P) = \left(\frac{2\rho_j}{t^2} + O(1)\right) dt, \quad \text{at} \quad P \to \infty_j. \tag{2.33}$$

Because f_0 is a function on $U_0 = \mathcal{C} \setminus \{P_j\}_{j=1}^n$, then

$$\exp \oint_\lambda \mathrm{dlog}\, f_0 = 1, \tag{2.34}$$

for all cycles λ from $H_1(\mathbb{Z}, \mathcal{C})$. A similar result follows for f_1 and upon noting (2.31) we may define

$$m_j = -\frac{1}{2\pi\imath} \oint_{\mathfrak{a}_j} \mathrm{dlog}\, f_0 = -\frac{1}{2\pi\imath} \oint_{\mathfrak{a}_j} \mathrm{dlog}\, f_1, \tag{2.35}$$

$$n_j = \frac{1}{2\pi\imath} \oint_{\mathfrak{b}_j} \mathrm{dlog}\, f_0 = \frac{1}{2\pi\imath} \oint_{\mathfrak{b}_j} \mathrm{dlog}\, f_1. \tag{2.36}$$

Further, in view of (2.20) and (2.33), we may write

$$\gamma_\infty = \frac{1}{2}\, \mathrm{dlog}\, f_0 + \imath\pi \sum_{j=1}^{g} m_j \int_{P_0}^{P} v_j, \tag{2.37}$$

where v_j are canonically $\mathfrak{a}$-normalized holomorphic differentials. Integrating γ_∞ around $\mathfrak{b}$-cycles leads to the Ercolani-Sinha constraints

$$\oint_{\mathfrak{b}_k} \gamma_\infty = \imath\pi n_k + \imath\pi \sum_{l=1}^{g} \tau_{kl} m_l,$$

which are necessary and sufficient conditions for L^2 to be trivial when restricted to $\mathcal{C}$. Thus the winding vector U appearing in the Baker-Akhieser function (2.28) takes the form

$$\boldsymbol{U} = \frac{1}{2}\boldsymbol{n} + \frac{1}{2}\tau \boldsymbol{m}. \tag{2.38}$$

Therefore the vector $2\boldsymbol{U} \in \Lambda$, the period lattice for the curve $\mathcal{C}$, and so the "winding-vector" vector is a half-period. Note that $\boldsymbol{U} \neq 0$ or otherwise γ_∞ would be holomorphic contrary to our choice.

Using the bilinear relations (and that $\oint_{\mathfrak{a}_l} \gamma_\infty(P) = 0$) we have that

$$U_k = \frac{1}{2\pi\imath} \oint_{\mathfrak{b}_k} \gamma_\infty(P)$$

$$(2.39)\qquad \begin{aligned} &= \frac{1}{2\pi\imath}\sum_{l=1}^{g}\left(\oint_{\mathfrak{a}_l} v_k(P)\oint_{\mathfrak{b}_l}\gamma_\infty(P') - \oint_{\mathfrak{b}_l} v_k(P)\oint_{\mathfrak{a}_l}\gamma_\infty(P')\right)\\ &= \frac{1}{2\pi\imath}\oint_{\partial\Gamma}\gamma_\infty(P)\int_{P_0}^{P} v_k(P')\\ &= \sum_{i=1}^{n}\mathrm{Res}_{P\to\infty_i}\gamma_\infty(P)\int_{P_0}^{P} v_k(P')\\ &= \sum_{i=1}^{n}\rho_j\, V_k^{(j)}. \end{aligned}$$

Here we have defined "winding vectors" $\boldsymbol{T}^{(i)}$, $\boldsymbol{V}^{(i)}$, $\boldsymbol{W}^{(i)}$, as the coefficients of the expansion in the vicinity of ∞_i of

$$(2.40)\qquad \left.\int\limits_{P_0}^{P}\boldsymbol{v}\right|_{P\to\infty_i} = \boldsymbol{T}^{(i)} + t\boldsymbol{V}^{(i)} + \frac{t^2}{2}\boldsymbol{W}^{(i)} + \ldots,$$

and so $\boldsymbol{T}^{(i)} = \int\limits_{P_0}^{\infty_i}\boldsymbol{v}$. More generally, for any holomorphic differential Ω

$$\sum_{l=1}^{g} U_l\oint_{\mathfrak{a}_l}\Omega = \frac{1}{2\pi\imath}\sum_{l=1}^{g}\left(\oint_{\mathfrak{a}_l}\Omega\oint_{\mathfrak{b}_l}\gamma_\infty(P) - \oint_{\mathfrak{b}_l}\Omega\oint_{\mathfrak{a}_l}\gamma_\infty(P)\right) = \sum_{i=1}^{n}\mathrm{Res}_{P\to\infty_i}\gamma_\infty(P)\int_{P_0}^{P}\Omega.$$

Houghton, Manton and Ramão utilise this expression to express a dual form of the Ercolani-Sinha constraints (2.38). Define the 1-cycle

$$(2.41)\qquad \mathfrak{c} = \sum_{l=1}^{g}(n_l\mathfrak{a}_l + m_l\mathfrak{b}_l).$$

Then (upon recalling that $\tau_{lk}\oint_{\mathfrak{a}_k}\Omega = \oint_{\mathfrak{b}_l}\Omega$, where τ is the period matrix) we have the equivalent constraint:

$$(2.42)\qquad \oint_{\mathfrak{c}}\Omega = 2\sum_{i=1}^{n}\mathrm{Res}_{P\to\infty_i}\gamma_\infty(P)\int_{P_0}^{P}\Omega.$$

The right-hand side of this equation is readily evaluated. We may express an arbitrary holomorphic differential Ω as,

$$(2.43)\qquad \begin{aligned} \Omega &= \frac{\beta_0\eta^{n-2} + \beta_1(\zeta)\eta^{n-3} + \ldots + \beta_{n-2}(\zeta)}{\frac{\partial\mathcal{P}}{\partial\eta}}\,d\zeta\\ &= \frac{\beta_0(\eta/\zeta^2)^{n-2} + \tilde\beta_1(1/\zeta)(\eta/\zeta^2)^{n-3} + \ldots + \tilde\beta_{n-2}(1/\zeta)}{\sum_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}(\eta/\zeta^2 - \mu_j(1/\zeta))}\,\frac{d\zeta}{\zeta^2}, \end{aligned}$$

where $\beta_j(\zeta)\equiv\zeta^{2j}\tilde\beta_j(1/\zeta)$ is a polynomial of degree at most $2j$ in ζ. Thus using (2.13) we obtain

$$\sum_{i=1}^{n}\mathrm{Res}_{P\to\infty_i}\gamma_\infty(P)\int_{P_0}^{P}\Omega = -\sum_{i=1}^{n}\frac{\beta_0\rho_i^{n-1} + \tilde\beta_1(0)\rho_i^{n-2} + \ldots + \tilde\beta_{n-2}(0)\rho_i}{\prod_{j\neq i}^{n}(\rho_i - \rho_j)} = -\beta_0,$$

upon using Lagrange interpolation. At this stage we have from the condition A2,

Lemma 2.3 (Ercolani-Sinha Constraints)**.** *The following are equivalent:*

(1) *L^2 is trivial on $\mathcal{C}$.*
(2) *There exists a 1-cycle $\mathfrak{c} = \boldsymbol{n}\cdot\mathfrak{a} + \boldsymbol{m}\cdot\mathfrak{b}$ such that for every holomorphic differential Ω (2.43),*

$$\oint_{\mathfrak{c}} \Omega = -2\beta_0, \tag{2.44}$$

(3) $2\boldsymbol{U} \in \Lambda \Longleftrightarrow$

$$\boldsymbol{U} = \frac{1}{2\pi\imath}\left(\oint_{\mathfrak{b}_1}\gamma_\infty, \ldots, \oint_{\mathfrak{b}_g}\gamma_\infty\right)^T = \frac{1}{2}\boldsymbol{n} + \frac{1}{2}\tau\boldsymbol{m}. \tag{2.45}$$

Here (2) is the dual form of the Ercolani-Sinha constraints given by Houghton, Manton and Ramão. Their 1-cycle generalises a similar constraint arising in the work of Corrigan and Goddard [CG81]. The only difference between (3) and that of Ercolani-Sinha Theorem II.2 is in the form of $\boldsymbol{U}$ in which we disagree. We also know that $\boldsymbol{U} \neq 0$.

The Ercolani-Sinha constraints impose g conditions on the period matrix of our curve. We have seen that the coefficients $a_r(\zeta)$ each give $2r+1$ (real) parameters, thus the moduli space of charge n centred $SU(2)$ monopoles is

$$\sum_{r=2}^{n}(2r+1) - g = (n+3)(n-1) - (n-1)^2 = 4(n-1)$$

(real) dimensional.

The 1-cycle appearing in the work of Houghton, Manton and Ramão further satisfies

Corollary 2.4 (Houghton, Manton and Ramão, 2000)**.** *$\tau_*\mathfrak{c} = -\mathfrak{c}$.*

This result is the dual of Hitchin's remark [Hit83, p164] that the triviality of L^2 together with the antiholomorphic isomorphism $L \cong L^*$ yields an imaginary lattice point with respect to $H^1(\mathcal{C},\mathbb{Z}) \subset H^1(\mathcal{C},\mathcal{O})$.

The Picard group of degree zero line bundles on $\mathcal{C}$ may be identified with the (principally polarized) Jacobian of $\mathcal{C}$ via the Abel map. We may identify the origin with the trivial bundle. The degree of the trivial bundle $L^2 = L_\delta^{z=1} \otimes \mathcal{O}(-n+1)$ is zero. Thus

$$0 = \boldsymbol{\phi}\left(\mathrm{Div}(L^2)\right) = \boldsymbol{\phi}\left(\mathrm{Div}(L_\delta^{z=1} \otimes \mathcal{O}(-n+1))\right) = \boldsymbol{\phi}\left(\mathrm{Div}(L_\delta^{z=1}) - (n-1)\sum_{k=1}^{n}\infty_k\right).$$

Further, consideration of the sections $\Psi_j(1,P)$ associated to $L_\delta^{z=1}$ gives us that

$$\boldsymbol{\phi}\left(\mathrm{Div}(L_\delta^{z=1})\right) = \boldsymbol{\phi}(\delta) - \boldsymbol{U}.$$

Together with (2.25) these yield that the winding vector $\boldsymbol{U}$ may be expressed in terms of the degree zero divisor (for each $j = 1, \ldots, n$)

$$\boldsymbol{U} = \boldsymbol{\phi}\left(\Delta_j - \infty_j - (n-2)\sum_{k=1}^{n}\infty_k\right). \tag{2.46}$$

A3 The full condition A3 is that $L_\delta^z(-1) \in \mathrm{Jac}^{g_{\mathcal{C}}-1}(\mathcal{C}) \setminus \Theta$ for $z \in (-1,1)$. This constraint must be checked using knowledge of the Θ divisor. The exact sequence $\mathcal{O}(L^s) \hookrightarrow \mathcal{O}(L^s(n-2))$ given by multiplication by a section of $\pi^*\mathcal{O}(n-2)|_{\mathcal{C}}$ does however give us the necessary condition

$$H^0\left(\mathcal{C}, \mathcal{O}(L^s(n-2))\right) = 0 \Longrightarrow H^0\left(\mathcal{C}, \mathcal{O}(L^s)\right) = 0, \qquad s \in (0,2). \tag{2.47}$$

If L^s were trivial we would have a section, contradicting this vanishing result. The same treatment given to the triviality of L^2 shows that if L^s were trivial then $s\,\boldsymbol{U} \in \Lambda$. Therefore (2.47) shows that $s\,\boldsymbol{U} \notin \Lambda$ for $s \in (0, 2)$. Thus $2\boldsymbol{U}$ is a primitive vector in Λ and we obtain the final part of the Ercolani-Sinha constraints,

$$2\boldsymbol{U} \text{ is a primitive vector in } \Lambda \Longleftrightarrow \mathfrak{c} \text{ is primitive in } H_1(\mathcal{C}, \mathbb{Z}). \tag{2.48}$$

The Ercolani-Sinha constraints (2.45) or (2.44) place g transcendental constraints on the spectral curve $\mathcal{C}$ and a major difficulty in implementing this construction has been in solving these, even in simple examples. Beyond these constraints several further constants have appeared in the construction (2.28) of the Baker-Akhieser function. To make the the Ercolani-Sinha construction effective these need to be calculated and to this we now turn.

3. Extensions to the Ercolani-Sinha Theory

We shall now both simplify and extend the formulae of Ercolani-Sinha. In particular the construction thus far has depended on the divisor δ through the constants c_{ij} in (2.29). We shall show that this divisor encodes a gauge choice, and show how the constants c_{ij} may be chosen in a particularly simple form independent of δ. Our form for the matrix $Q_0(z)$ (given in (3.8) below) is wholly in terms of quantities intrinsic to the curve. We will highlight the ingredients needed to calculate Q_0 and conclude by showing how the fundamental bi-differential may be employed in the construction.

3.1. **Vanishing and symmetry properties.** In addition to the Ercolani-Sinha vector, which from (2.46) may be written,

$$\boldsymbol{U} = \boldsymbol{\phi}\left(\Delta_j - \infty_j - (n-2)\sum_{k=1}^{n} \infty_k\right),$$

a further vector plays a special role in the monopole construction. Set

$$\widetilde{\boldsymbol{K}} = \boldsymbol{K} + \boldsymbol{\phi}\left((n-2)\sum_{k=1}^{n} \infty_k\right).$$

Here $\boldsymbol{K}$ is the vector of Riemann constants; our conventions regarding this are given in the Appendix. Let us observe several points about this vector. First is that $\widetilde{\boldsymbol{K}}$ is independent of the choice of base point of the Abel map, for

$$\widetilde{\boldsymbol{K}}_P = \boldsymbol{K}_P + \boldsymbol{\phi}_P\left((n-2)\sum_{k=1}^{n} \infty_k\right) = \boldsymbol{K}_Q + (g-1)\boldsymbol{\phi}_Q(P) + \boldsymbol{\phi}_P\left((n-2)\sum_{k=1}^{n} \infty_k\right) = \widetilde{\boldsymbol{K}}_Q,$$

using the fact that $g - 1 = (n-1)^2 - 1 = n(n-2)$. The same fact shows us that $\widetilde{\boldsymbol{K}} \in \boldsymbol{K} + \boldsymbol{\phi}\left(\mathcal{C}^{g-1}\right)$ and so secondly,

$$\theta(\widetilde{\boldsymbol{K}}) = 0. \tag{3.1}$$

We have already established that $\operatorname{Div} \pi^*\mathcal{O}(1) = \sum_{k=1}^{n} \infty_k$ whence $\operatorname{Div} \pi^*\mathcal{O}(2n-4) = 2(n-2)\sum_{k=1}^{n} \infty_k$. Now Hitchin has established that

$$K_{\mathcal{C}} = \pi^*\mathcal{O}(2n-4)$$

utilising the adjunction formula. Thus

$$2\widetilde{\boldsymbol{K}} = 2\widetilde{\boldsymbol{K}} + \boldsymbol{\phi}\left(2(n-2)\sum_{k=1}^{n} \infty_k\right) = 2\widetilde{\boldsymbol{K}} + \boldsymbol{\phi}(\operatorname{Div}(K_{\mathcal{C}})) = 0$$

upon using (A.7). Thus we have thirdly,

$$2\widetilde{\boldsymbol{K}} \in \Lambda. \tag{3.2}$$

Finally Riemann's vanishing theorem for a degree $g-1$ line bundle,

$$\text{multiplicity}_L\, \theta = \operatorname{Dim} H^0(\mathcal{C}, \mathcal{O}(L)),$$

together with the fact that each of the $n-1$ sections of $\mathcal{O}(n-2)$ on $\mathbb{P}^1$ yield sections of the pull-back, gives us that

$$\text{multiplicity}_{\tilde{K}}\, \theta = \operatorname{Dim} H^0(\mathcal{C}, \pi^*\mathcal{O}(n-2)) \geq n-1.$$

Thus for $n \geq 3$ we have fourthly, that

$$\widetilde{\boldsymbol{K}} \in \Theta_{\text{singular}}. \tag{3.3}$$

Indeed, from [Hit83, Prop. 4.5] we find that the index of speciality of $(n-2)\sum_{k=1}^n \infty_k$ is

$$s \equiv i\left((n-2)\sum_{k=1}^n \infty_k\right) = \operatorname{Dim} H^0(\mathcal{C}, \pi^*\mathcal{O}(n-2)) = \begin{cases} \frac{1}{4}n^2 & \text{if } n \text{ is even,} \\ \frac{1}{4}(n-1)^2 & \text{if } n \text{ is odd.}\end{cases} \tag{3.4}$$

This means that all partial derivatives of θ of order $s-1$ or less vanish at the point $\widetilde{\boldsymbol{K}}$. The point $\widetilde{\boldsymbol{K}}$ is the distinguished point Hitchin uses to identify degree $g-1$ line bundles with $\operatorname{Jac}(\mathcal{C})$. Finally we remark that $(n-2)\sum_{k=1}^n \infty_k$ and $\Delta_j - \infty_j$ (for each j) are theta characteristics (see the Appendix).

Using the point $\widetilde{\boldsymbol{K}}$ we may express the functions (2.28) and (2.29) in the form

$$\Psi_j\,(z,P) = g_j(P)\, \frac{\theta_{\frac{\boldsymbol{m}}{2},\frac{\boldsymbol{n}}{2}}\left(\boldsymbol{\phi}(P) - \boldsymbol{\phi}(\infty_j) + z\boldsymbol{U} - \widetilde{\boldsymbol{K}}\right)\theta_{\frac{\boldsymbol{m}}{2},\frac{\boldsymbol{n}}{2}}\left(-\widetilde{\boldsymbol{K}}\right)}{\theta_{\frac{\boldsymbol{m}}{2},\frac{\boldsymbol{n}}{2}}\left(\boldsymbol{\phi}(P) - \boldsymbol{\phi}(\infty_j) - \widetilde{\boldsymbol{K}}\right)\theta_{\frac{\boldsymbol{m}}{2},\frac{\boldsymbol{n}}{2}}\left(z\boldsymbol{U} - \widetilde{\boldsymbol{K}}\right)}\, e^{z\int_{P_0}^{P}\gamma_\infty - z\,\nu_j} \tag{3.5}$$

and

$$\left(Q_0(z)\right)_{jl} = -(\rho_j - \rho_l)\, c_{jl}\, e^{z[\nu_l - \nu_j]}\, \frac{\theta_{\frac{\boldsymbol{m}}{2},\frac{\boldsymbol{n}}{2}}\left(\boldsymbol{\phi}(\infty_l) - \boldsymbol{\phi}(\infty_j) + z\boldsymbol{U} - \widetilde{\boldsymbol{K}}\right)\theta_{\frac{\boldsymbol{m}}{2},\frac{\boldsymbol{n}}{2}}\left(-\widetilde{\boldsymbol{K}}\right)}{\theta_{\frac{\boldsymbol{m}}{2},\frac{\boldsymbol{n}}{2}}\left(\boldsymbol{\phi}(\infty_l) - \boldsymbol{\phi}(\infty_j) - \widetilde{\boldsymbol{K}}\right)\theta_{\frac{\boldsymbol{m}}{2},\frac{\boldsymbol{n}}{2}}\left(z\boldsymbol{U} - \widetilde{\boldsymbol{K}}\right)}, \tag{3.6}$$

where c_{jl} has been defined in (2.30).

The matrix Q_0 is to satisfy $Q_0(z) = Q_0(-z)^T$. We find that

$$Q_0(0) = Q_0(0)^T \Longleftrightarrow c_{jl} = -c_{lj}.$$

This, together with (3.2), yields that $Q_0(z) = Q_0(-z)^T$. Thus we must establish that $c_{jl} = -c_{lj}$.

Before doing this, let us consider the behaviour of (3.6). We require $z = 0$ to be a regular point. This is equivalent to our requirement that $\operatorname{Dim} H^0(\mathcal{C}, \mathcal{O}(L_{\Delta_j - \infty_j})) = 0$, for that means

$$0 \neq \theta\left(\boldsymbol{\phi}\left(\Delta_j - \infty_j\right) + \boldsymbol{K}\right) = \theta\left(\boldsymbol{U} + \boldsymbol{\phi}\left((n-2)\sum_{k=1}^n \infty_k\right) + \boldsymbol{K}\right) = \theta\left(\boldsymbol{U} + \widetilde{\boldsymbol{K}}\right), \tag{3.7}$$

and consequently that $\boldsymbol{U} \pm \widetilde{\boldsymbol{K}}$ is a non-singular even theta characteristic. Therefore we have the requirement of the Ercolani-Sinha vector

Lemma 3.1. $\boldsymbol{U} \pm \widetilde{\boldsymbol{K}}$ *is a non-singular even theta characteristic.*

Further the Nahm construction requires (3.6) to have a simple pole at $z=\pm1$. Because $2\widetilde{\boldsymbol{K}}\in\Lambda$ and $2\boldsymbol{U}\in\Lambda$ this means we wish the order of the vanishing of $\theta(\widetilde{\boldsymbol{K}})$ in the direction $\boldsymbol{U}$ to be one more than the order of the vanishing of $\theta(\boldsymbol{\phi}(\infty_l)-\boldsymbol{\phi}(\infty_j)+\widetilde{\boldsymbol{K}})$ in the direction $\boldsymbol{U}$ (for each $j\neq l$). In principle the order of vanishing can be higher than that given by the index of speciality and Riemann's vanishing theorem, for these only provide the minimal order to which all derivatives vanish and there may be some directions yielding higher order vanishing. The desired vanishing of (3.6) may be deduced from the following property of theta functions

$$\frac{\theta(\boldsymbol{\phi}(Q)-\boldsymbol{\phi}(P)+\boldsymbol{e})\theta(\boldsymbol{\phi}(Q)-\boldsymbol{\phi}(P)-\boldsymbol{e})}{\theta^2(\boldsymbol{e})E(P,Q)^2}=\Omega_{\mathbf{B}}(P,Q)+\sum_{i,k=1}^{g}\frac{\partial^2\ln\theta(e)}{\partial z_i\partial z_k}v_i(P)v_k(Q)$$

valid for all P, $Q\in\mathcal{C}$ and $e\in\mathbb{C}^g$ (see [Fay73, 2.12]). Here $E(P,Q)=\mathcal{E}(P,Q)/\sqrt{dx(P)dx(Q)}$ is the prime form, and $\Omega_{\mathbf{B}}(P,Q)$ a symmetric differential on $\mathcal{C}\times\mathcal{C}$ with poles only on the diagonal. Using that $\widetilde{\boldsymbol{K}}$ is a half period, $2\widetilde{\boldsymbol{K}}=\boldsymbol{p}+\tau\boldsymbol{q}$ for $\boldsymbol{p}$, $\boldsymbol{q}\in\mathbb{Z}^g$, we may use this identity to obtain an expression of the form,

$$F(s)F(-s)=G(s)e(s),$$

where

$$\begin{aligned}
F(s)&=\frac{\theta(\boldsymbol{\phi}(\infty_l)-\boldsymbol{\phi}(\infty_j)+s\boldsymbol{U}-\widetilde{\boldsymbol{K}})}{E(\infty_j,\infty_l)},\\
G(s)&=\theta^2(s\boldsymbol{U}-\tilde{K})\Omega_{\mathbf{B}}(\infty_j,\infty_l)\\
&\quad+\sum_{i,k=1}^{g}\left(\frac{\partial^2\theta(s\boldsymbol{U}-\widetilde{\boldsymbol{K}})}{\partial z_i\partial z_k}\theta(s\boldsymbol{U}-\widetilde{\boldsymbol{K}})-\frac{\partial\theta(s\boldsymbol{U}-\widetilde{\boldsymbol{K}})}{\partial z_i}\frac{\partial\theta(s\boldsymbol{U}-\widetilde{\boldsymbol{K}})}{\partial z_k}\right)v_i(\infty_j)v_k(\infty_l),\\
e(s)&=\exp(i\pi[q\cdot\tau\cdot q+2q\cdot(\boldsymbol{\phi}(\infty_l)-\boldsymbol{\phi}(\infty_j)+s\boldsymbol{U}-\widetilde{\boldsymbol{K}})]).
\end{aligned}$$

Comparison of the Taylor series in s shows that if the order of vanishing of $\theta(s\boldsymbol{U}-\widetilde{\boldsymbol{K}})$ is m and that of $\theta(\boldsymbol{\phi}(\infty_l)-\boldsymbol{\phi}(\infty_j)+s\boldsymbol{U}-\widetilde{\boldsymbol{K}})$ is k then $2k=2m-2$. Therefore the order of vanishing of the denominator of (3.6) is one more than the numerator, and consequently that we have a simple pole at $z=\pm1$. For $n\geq3$ the divisor $\infty_j-\infty_l+(n-2)\sum_{k=1}^{n}\infty_k$ is special.

3.2. **The matrix $Q_0(z)$.** It remains to discuss the constants c_{jl}. Thus far in the construction these depend on the divisor δ. In this subsection we now establish the following:

Theorem 3.2. *The matrix $Q_0(z)$ (which has poles of first order at $z=\pm1$) may be written*

(3.8)
$$Q_0(z)_{jl}=\epsilon_{jl}\,\frac{(\rho_j-\rho_l)}{\mathcal{E}(\infty_j,\infty_l)}\,e^{i\pi\tilde{\boldsymbol{q}}\cdot(\boldsymbol{\phi}(\infty_l)-\boldsymbol{\phi}(\infty_j))}\,\frac{\theta(\boldsymbol{\phi}(\infty_l)-\boldsymbol{\phi}(\infty_j)+[z+1]\boldsymbol{U}-\widetilde{\boldsymbol{K}})}{\theta([z+1]\boldsymbol{U}-\widetilde{\boldsymbol{K}})}\,e^{z(\nu_l-\nu_j)}.$$

Here $E(P,Q)=\mathcal{E}(P,Q)/\sqrt{dx(P)dx(Q)}$ is the Schottky-Klein prime form, $\boldsymbol{U}-\widetilde{\boldsymbol{K}}=\frac{1}{2}\tilde{\boldsymbol{p}}+\frac{1}{2}\tau\tilde{\boldsymbol{q}}$ ($\boldsymbol{p}$, $\boldsymbol{q}\in\mathbb{Z}^g$) is a non-singular even theta characteristic, and $\epsilon_{jl}=\epsilon_{lj}=\pm1$ is determined (for $j<l$) by $\epsilon_{jl}=\epsilon_{jj+1}\epsilon_{j+1j+2}\dots\epsilon_{l-1l}$. The $n-1$ signs $\epsilon_{jj+1}=\pm1$ are arbitrary.

We remark that the prime form may be defined in terms of theta functions with odd non-singular characteristic, the prime form itself being independent of this choice. The construction (3.8) depends on this choice via $\mathcal{E}(\infty_j,\infty_l)$, but any two choices lead to $Q_0(z)$ differing by a constant diagonal gauge transformation, the ambiguity noted earlier. We also note that the contours implicit in $\boldsymbol{\phi}(\infty_l)$ are taken to be the same for each term in (3.8), including that implicit in the limit (2.27). A change to this contour leads to a constant gauge transformation by a diagonal matrix with entries ± 1 which explains the signs ϵ_{jl} appearing in this theorem.

Proof. It will be convenient to introduce the following shorthand for a recurring combination of functions appearing in this work. For any divisor $\mathcal{A}$ set

$$< P - \mathcal{A} > \equiv \theta(\boldsymbol{\phi}(P) - \boldsymbol{\phi}(\mathcal{A}) - \boldsymbol{K}).$$

The function $g_j(P)$ has been specified by

$$g_j(P_j) = 1, \qquad \mathrm{Div}\, g_j(P) = \Delta_j - P_j + \sum_{k=1}^{n} P_k - \delta.$$

We may express $g_j(P)$ in several ways. First,

$$g_j(P) = \frac{f_j(P)}{f_j(P_j)},$$

where

$$f_j(P) = \frac{< P - \Delta_j >}{< P - \tilde{\delta} - \delta_{g-1+j} >} \prod_{t\neq j} \frac{< P - \tilde{\delta} - P_t >}{< P - \tilde{\delta} - \delta_{g-1+t} >}$$

and

$$\tilde{\delta} = \sum_{k=1}^{g-1} \delta_k, \qquad \delta = \sum_{k=1}^{g+n-1} \delta_k.$$

This was the form presented when we discussed the Baker-Akhiezer function in general. In the case of the monopole we have

$$g = (n-1)^2 \geq n - 1 \text{ for } \ n \geq 2,$$

and a second, more economical, representation exists. Now we could take

$$\begin{aligned} f_j(P) &= \frac{< P - \sum_{k\neq j} P_k - \sum_{s=1}^{g-(n-1)} \delta_s > < P - \Delta_j >}{< P - \sum_{t=1}^{n-1} \delta_{t+g-(n-1)} - \sum_{s=1}^{g-(n-1)} \delta_s > \quad < P - \sum_{t=1}^{n-1} \delta_{t+g} - \sum_{s=1}^{g-(n-1)} \delta_s >} \\ &= \frac{< P - \sum_{k\neq j} P_k - \hat{\delta} >< P - \Delta_j >}{< P - \delta^{(1)} - \hat{\delta} >< P - \delta^{(2)} - \hat{\delta} >}, \end{aligned}$$

where

$$\hat{\delta} = \sum_{k=1}^{g-(n-1)} \delta_k, \qquad \delta^{(1)} = \sum_{j=1}^{n-1} \delta_{g-(n-1)+j}, \qquad \delta^{(2)} = \sum_{j=1}^{n-1} \delta_{g+j}.$$

Then $\delta = \hat{\delta} + \delta^{(1)} + \delta^{(2)}$.

We have

$$\begin{aligned} \Psi_j(P) &= g_j(P)\, \frac{< P_j - \Delta_j >}{< P - \Delta_j >}\, \frac{\theta(\boldsymbol{\phi}(P) - \boldsymbol{\phi}(P_j) + (z+1)\boldsymbol{U} - \widetilde{\boldsymbol{K}})}{\theta((z+1)\boldsymbol{U} - \widetilde{\boldsymbol{K}})} e^{z[\int_{P_0}^{P} \gamma_\infty - \nu_j]}, \\ Q_0(z)_{jl} &= -(\rho_j - \rho_l)\, \hat{c}_{jl}\, \frac{\theta(\boldsymbol{\phi}(P_l - P_j) + (z+1)\boldsymbol{U} - \widetilde{\boldsymbol{K}})}{\theta((z+1)\boldsymbol{U} - \widetilde{\boldsymbol{K}})}\, e^{z(\nu_l - \nu_j)}, \\ \hat{c}_{jl} &= \lim_{P\to P_l} \zeta g_j(P) \frac{< P_j - \Delta_j >}{< P - \Delta_j >} \equiv c_{jl} \frac{< P_j - \Delta_j >}{< P_l - \Delta_j >} = c_{jl} \frac{\theta(\boldsymbol{U} - \widetilde{\boldsymbol{K}})}{\theta(\boldsymbol{\phi}(P_l - P_j) + \boldsymbol{U} - \widetilde{\boldsymbol{K}})} \end{aligned}$$

where $c_{jl} = \lim_{P\to P_l} \zeta g_j(P)$ is Ercolani-Sinha's constant. We note that by breaking up the grouping of the theta functions in our expression for Q_0 we need to be a little more careful regarding its quasi-periodicity properties. If $2(\boldsymbol{U} - \widetilde{\boldsymbol{K}}) = \tilde{\boldsymbol{p}} + \tau\tilde{\boldsymbol{q}}$ ($\boldsymbol{p}$, $\boldsymbol{q} \in \mathbb{Z}^g$) then

$$\begin{aligned} c_{jl} = -c_{lj} &\Longleftrightarrow \hat{c}_{jl} = -\hat{c}_{lj} \exp(2\pi i[\tilde{\boldsymbol{q}} \cdot (\boldsymbol{\phi}(P_l - P_j) + \boldsymbol{U} - \widetilde{\boldsymbol{K}}) - \frac{1}{2}\tilde{\boldsymbol{q}} \cdot \tau\tilde{\boldsymbol{q}}]) \\ &\Longleftrightarrow \hat{c}_{jl} = -\hat{c}_{lj} \exp(2\pi i \tilde{\boldsymbol{q}} \cdot \boldsymbol{\phi}(P_l - P_j)). \end{aligned}$$

Here we have used that $\boldsymbol{U} - \widetilde{\boldsymbol{K}}$ is an even theta characteristic.

Using the second representation we wish to evaluate

$$\begin{aligned} \hat{c}_{jl} &= \lim_{P\to P_l} \zeta \frac{f_j(P)}{f_j(P_j)} \frac{< P_j - \Delta_j >}{< P - \Delta_j >} \\ &= \lim_{P\to P_l} \zeta \frac{< P - \sum_{k\neq j} P_k - \hat{\delta} >}{< P - \delta^{(1)} - \hat{\delta} >< P - \delta^{(2)} - \hat{\delta} >} \cdot \frac{< P_j - \delta^{(1)} - \hat{\delta} >< P_j - \delta^{(2)} - \hat{\delta} >}{< P_j - \sum_{k\neq j} P_k - \hat{\delta} >}. \end{aligned}$$

Now we use [Fay73, 2.17]

$$< P - \sum_{i=1}^{g} x_i > = c \, \frac{\det(v_i(x_j))}{\prod_{i<j} E(x_i, x_j)} \cdot \frac{\sigma(P)}{\prod_{i=1}^{g} \sigma(x_i)} \prod_{i=1}^{g} E(x_i, P)$$

with

$$\sigma(P) = \exp\left(-\sum_{s=1}^{g} \int_{\mathfrak{a}_s} v_s(y) \ln E(y, P)\right).$$

Taking $\{x_i^{(0)}\} = \{P_k\}_{k\neq j} \cup \hat{\delta}$, $\{x_i^{(1)}\} = \delta^{(1)} \cup \hat{\delta}$ and $\{x_i^{(2)}\} = \delta^{(2)} \cup \hat{\delta}$ this yields

$$\begin{aligned} \hat{c}_{jl} &= \lim_{P\to P_l} \zeta \frac{\sigma(P)}{\sigma(P_j)} \prod_{i=1}^{g} \frac{E(x_i^{(0)}, P)}{E(x_i^{(0)}, P_j)} \left[\frac{\sigma(P_j)}{\sigma(P)}\right]^2 \prod_{i=1}^{g} \left[\frac{E(x_i^{(1)}, P_j)}{E(x_i^{(1)}, P)} \frac{E(x_i^{(2)}, P_j)}{E(x_i^{(2)}, P)}\right] \\ &= \lim_{P\to P_l} \zeta \frac{\sigma(P_j)}{\sigma(P)} \cdot \left[\prod_{k\neq j} \frac{E(P_k, P)}{E(P_k, P_j)} \prod_{\substack{\delta_i \in \delta^{(1)} \\ \delta_r \in \hat{\delta},\, \delta_s \in \delta^{(2)}}} \frac{E(\delta_i, P_j) E(\delta_r, P_j) E(\delta_s, P_j)}{E(\delta_i, P) E(\delta_r, P) E(\delta_s, P)}\right] \\ &= \lim_{P\to P_l} \zeta \frac{\sigma(P_j)}{\sigma(P)} \cdot \frac{E(P_l, P)}{E(P_l, P_j)} \cdot \prod_{k\neq l,j} \frac{E(P_k, P)}{E(P_k, P_j)} \cdot \prod_{i=1}^{g+n-1} \frac{E(\delta_i, P_j)}{E(\delta_i, P_l)} \\ &= \left[\lim_{P\to P_l} \zeta \frac{E(P_l, P)}{E(P_l, P_j)}\right] \frac{\sigma(P_j)}{\sigma(P_l)} \prod_{k\neq l,j} \frac{E(P_k, P_l)}{E(P_k, P_j)} \cdot \prod_{i=1}^{g+n-1} \frac{E(\delta_i, P_j)}{E(\delta_i, P_l)}. \end{aligned}$$

The prime form $E(P, Q)$ that appears here is a differential of weight $(-\frac{1}{2}, -\frac{1}{2})$ on $\mathcal{C} \times \mathcal{C}$. If $(\boldsymbol{a}, \boldsymbol{b})$ is a non-singular odd theta characteristic then we may write

$$E(P, Q) = \frac{\theta_{\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{\phi}(P) - \boldsymbol{\phi}(Q))}{h_{\boldsymbol{a},\boldsymbol{b}}(P) h_{\boldsymbol{a},\boldsymbol{b}}(Q)}, \qquad h_{\boldsymbol{a},\boldsymbol{b}}^2(P) = \sum_{r=1}^{g} \frac{\partial \theta_{\boldsymbol{a},\boldsymbol{b}}}{\partial z_r}(0)\, v_r(P). \tag{3.9}$$

The prime form is independent of the choice of α. We remark that in our expressions above the half-differentials $h_{\boldsymbol{a},\boldsymbol{b}}$ cancel exactly between numerator and denominator upon noting that the P_k each are pre-images of the same point $P \in \mathbb{P}^1$. It will be convenient to write

$$E(P,Q) = \frac{\mathcal{E}(P,Q)}{\sqrt{dx(P)dx(Q)}}. \tag{3.10}$$

Now $\lim_{P\to\infty_l} \zeta\,\mathcal{E}(\infty_l, P) = 1$ and we have

$$\hat{c}_{jl}\,\mathcal{E}(P_l,P_j) = \frac{\sigma(P_j)}{\sigma(P_l)} \prod_{k\neq l,j} \frac{E(P_k,P_l)}{E(P_k,P_j)} \prod_{i=1}^{g+n-1} \frac{E(\delta_i,P_j)}{E(\delta_i,P_l)}.$$

Using the anti-symmetry of the prime form,$E(P_l,P_j) = -E(P_j,P_l)$, then

$$c_{jl} = -c_{lj} \Longleftrightarrow \left[\sigma(P_l)\frac{\prod_{i=1}^{g+n-1} E(\delta_i,P_l)}{\prod_{k\neq l,j} E(P_k,P_l)}\, e^{i\pi\tilde{q}\cdot\boldsymbol{\phi}(P_l)}\right]^2 = \left[\sigma(P_j)\frac{\prod_{i=1}^{g+n-1} E(\delta_i,P_j)}{\prod_{k\neq l,j} E(P_k,P_j)}\, e^{i\pi\tilde{q}\cdot\boldsymbol{\phi}(P_j)}\right]^2.$$

But this is to be true for all $j \neq l$, thus

$$c_{jl} = -c_{lj} \Longleftrightarrow \left[\sigma(P_j)\frac{\prod_{i=1}^{g+n-1} E(\delta_i,P_j)}{\prod_{k\neq j} E(P_k,P_j)}\, e^{i\pi\tilde{q}\cdot\boldsymbol{\phi}(P_j)}\right]^2 = c^2$$

for some constant c, independent of j, whence we require

$$\frac{\sigma(P_j)}{\sigma(P_l)} \prod_{k\neq l,j} \frac{E(P_k,P_l)}{E(P_k,P_j)} \prod_{i=1}^{g+n-1} \frac{E(\delta_i,P_j)}{E(\delta_i,P_l)} = \epsilon_{jl}\, e^{i\pi\tilde{q}\cdot(\boldsymbol{\phi}(P_l)-\boldsymbol{\phi}(P_j))} = \pm\, e^{i\pi\tilde{q}\cdot(\boldsymbol{\phi}(P_l)-\boldsymbol{\phi}(P_j))}. \tag{3.11}$$

Because ϵ_{jl} is of the form s_j/s_l (where $s_j = \pm c$) then there are $n-1$ constraints here, which may be specified by choosing $\epsilon_{j\,j+1}$. There are thus 2^{n-1} choices of signs for the c_{jl}. Thus

$$\hat{c}_{jl}\,\mathcal{E}(P_l,P_j) = \epsilon_{jl}\, e^{i\pi\tilde{q}\cdot(\boldsymbol{\phi}(P_l)-\boldsymbol{\phi}(P_j))}. \tag{3.12}$$

Lets consider the various constraints involved. Having determined $\boldsymbol{U}$ then, as $\boldsymbol{\phi}(P_j)$ are specified once a choice of the Abel map has been made, we have that

$$\boldsymbol{\phi}(\Delta_j) = \boldsymbol{\phi}\left(P_j + (n-2)\sum_{k=1}^{n} P_k\right) + \boldsymbol{U}.$$

This then determines the degree g effective divisor Δ_j. Thus the degree $g+n-1$ divisor δ is constrained by

$$\delta \sim \Delta_j - P_j + \sum_{k=1}^{n} P_k, \quad \boldsymbol{\phi}(\delta) = \boldsymbol{\phi}\left(\Delta_j - P_j + \sum_{k=1}^{n} P_k\right) = \boldsymbol{U} + (n-1)\boldsymbol{\phi}\left(\sum_{k=1}^{n} P_k\right).$$

This yields a further g constraints on the divisor δ in addition to the $n-1$ constraints of (3.11). Thus we have $g+n-1$ constraints on the degree $g+n-1$ nonspecial divisor δ. We remark that the $n-1$ constraints of (3.11) which are of the form s_j/s_l correspond to the constant diagonal gauge freedom that exists for the matrix $Q_0(z)$. Our solving the constraints (3.11) is equivalent to choosing a gauge.

For the moment let us suppose we may find a divisor δ satisfying the required constraints. If this is the case, then bringing the above results together establishes the theorem, and that we have

$$Q_0(z)_{jl} = \epsilon_{jl}\,\frac{(\rho_j-\rho_l)}{\mathcal{E}(P_j,P_l)}\, e^{i\pi\tilde{q}\cdot(\boldsymbol{\phi}(P_l)-\boldsymbol{\phi}(P_j))}\, \frac{\theta(\boldsymbol{\phi}(P_l-P_j)+[z+1]\boldsymbol{U}-\widetilde{\boldsymbol{K}})}{\theta([z+1]\boldsymbol{U}-\widetilde{\boldsymbol{K}})}\, e^{z(\nu_l-\nu_j)},$$

where $\epsilon_{jl} = \epsilon_{lj} = \pm1$ is determined (for $j<l$) by $\epsilon_{jl} = \epsilon_{jj+1}\epsilon_{j+1j+2}\ldots\epsilon_{l-1l}$ and the $n-1$ signs $\epsilon_{jj+1} = \pm1$ are arbitrary.

Alternate Calculation Our proof used the second parameterization of the functions $f_j(P)$. The same constraints arise if we use our first parameterization,

$$g_j(P) = \frac{f_j(P)}{f_j(P_j)}, \qquad f_j(P) = \frac{< P - \Delta_j >}{< P_j - \tilde{\delta} - \delta_{g-1+j} >} \prod_{t \neq j} \frac{< P - P_t - \tilde{\delta} >}{< P - \delta_{g-1+t} - \tilde{\delta} >},$$

where $\tilde{\delta} = \sum_{k=1}^{g-1} \delta_k$. Let $\{\tilde{x}^{(l)}\} = \{P_l, \tilde{\delta}\}$, $\tilde{y}^{(l)} = \{\delta_{g-1+l}, \tilde{\delta}\}$. Then

$$\begin{aligned}
&\frac{< P - P_t - \tilde{\delta} >}{< P_j - P_t - \tilde{\delta} >} \frac{< P_j - \delta_{g-1+t} - \tilde{\delta} >}{< P - \delta_{g-1+t} - \tilde{\delta} >} \\
&\qquad = \frac{\sigma(P)}{\sigma(P_j)} \cdot \left(\prod_{k \neq t} \frac{E(\tilde{x}^{(k)}, P)}{E(\tilde{x}^{(k)}, P_j)} \cdot \frac{E(\tilde{y}^{(k)}, P_j)}{E(\tilde{y}^{(k)}, P)} \right) \cdot \frac{\sigma(P_j)}{\sigma(P)} \\
&\qquad = \frac{E(P_t, P)}{E(P_t, P_j)} \cdot \frac{E(\delta_{g-1+t}, P_j)}{E(\delta_{g-1+t}, P)}.
\end{aligned}$$

We again wish to evaluate

$$\begin{aligned}
\hat{c}_{jl} &= \lim_{P \to P_l} \zeta \frac{f_j(P)}{f_j(P_j)} \cdot \frac{< P_j - \Delta_j >}{< P - \Delta_j >} \\
&= \lim_{P \to P_l} \zeta \frac{< P_j - \tilde{\delta} - \delta_{g-1+j} >}{< P - \tilde{\delta} - \delta_{g-1+j} >} \prod_{t \neq j} \frac{E(P_t, P)}{E(P_t, P_j)} \cdot \frac{E(\delta_{g-1+t}, P_j)}{E(\delta_{g-1+t}, P)} \\
&= \lim_{P \to P_l} \zeta \frac{\sigma(P_j)}{\sigma(P)} \cdot \left(\prod_{k=1}^{g-1} \frac{E(\delta_k, P_j)}{E(\delta_k, P)} \right) \frac{E(\delta_{g-1+j}, P_j)}{E(\delta_{g-1+j}, P)} \prod_{t \neq j} \frac{E(P_t, P) E(\delta_{g-1+t}, P_j)}{E(P_t, P_j) E(\delta_{g-1+t}, P)},
\end{aligned}$$

Therefore

$$\hat{c}_{jl}\, E(P_l, P_j) = \left[\lim_{P \to P_l} \zeta E(P_l, P) \right] \frac{\sigma(P_j)}{\sigma(P_l)} \cdot \prod_{k \neq l,j} \frac{E(P_k, P_l)}{E(P_k, P_j)} \cdot \prod_{i=1}^{g+n-1} \frac{E(\delta_i, P_j)}{E(\delta_i, P_l)},$$

which is the same expression as our previous method of calculating. □

We have established our new expression for $Q_0(z)$ once the following is established:

Proposition 3.3. *Given n (generic) constants $\alpha_j \neq 0$ there exists a degree $g + n - 1$ nonspecial divisor $\delta = \sum_{s=1}^{g+n-1} \delta_s$ such that*

$$\boldsymbol{\phi} \left(\delta - (n-1) \sum_{k=1}^{n} \infty_k \right) = \boldsymbol{U} \tag{3.13}$$

and satisfying the $n - 1$ (equivalent) constraints

$$C \qquad \alpha_1 \prod_{s=1}^{g+n-1} \theta_{\boldsymbol{a},\boldsymbol{b}} \left(\int_{\infty_1}^{\delta_s} \boldsymbol{v} \right) = \alpha_2 \prod_{s=1}^{g+n-1} \theta_{\boldsymbol{a},\boldsymbol{b}} \left(\int_{\infty_2}^{\delta_s} \boldsymbol{v} \right) = \cdots = \alpha_n \prod_{s=1}^{g+n-1} \theta_{\boldsymbol{a},\boldsymbol{b}} \left(\int_{\infty_n}^{\delta_s} \boldsymbol{v} \right),$$

$$C' \qquad \prod_{i=1}^{g+n-1} \frac{E(\delta_i, \infty_s)}{E(\delta_i, \infty_n)} = \tilde{\alpha}_s, \qquad s = 1, \ldots, n-1,$$

$$C'' \qquad \sum_{i=1}^{g+n-1} \int_{P_0}^{\delta_i} \omega_{\infty_s, \infty_n}(P) = \hat{\alpha}_s, \qquad s = 1, \ldots, n-1.$$

Here $\tilde{\alpha}_s = \alpha_n/\alpha_s$ and $\hat{\alpha}_s = \ln\tilde{\alpha}_s - (g+n-1)\ln\left[E(P_0,\infty_s)/E(P_0,\infty_n)\right]$, $\boldsymbol{\phi}$ is the Abel map, $\boldsymbol{v}$ the $\mathfrak{a}$-normalized holomorphic differentials, $(\boldsymbol{a},\boldsymbol{b})$ is a non-singular odd (half) theta characteristic and $\boldsymbol{U}$ a further specified theta (half) characteristic. The equivalence of the constraints C and C$'$ follows upon writing the prime form in terms of theta functions (3.9) and observing that the half-differentials $h_{\boldsymbol{a},\boldsymbol{b}}$ in the definition of the primeform cancel exactly between numerator and denominator upon noting that the ∞_k each are pre-images of the same point $\infty\in\mathbb{P}^1$. The final equivalence with C$''$ is obtained upon using the expression

$$\omega_{\mathcal{P},\mathcal{Q}}(P) = d\,\ln\,\prod_{j=1}^{m}\frac{E(P,P_j)}{E(P,Q_j)}, \tag{3.14}$$

where $\mathcal{P}=P_1+\ldots,P_m$ and $\mathcal{Q}=Q_1+\ldots+Q_m$ are divisors of the same degree m. We have for example that

$$\omega_{\infty_s,\infty_n}(P) = d\ln\frac{E(P,\infty_s)}{E(P,\infty_n)}$$

and so

$$\int_{P_0}^{\delta_i}\omega_{\infty_s,\infty_n}(P) = \ln\frac{E(\delta_i,\infty_s)}{E(\delta_i,\infty_n)} - \ln\frac{E(P_0,\infty_s)}{E(P_0,\infty_n)}.$$

The proposition then is an extension of the usual Abel-Jacobi inversion: (3.13) is the usual Abel-Jacobi map and C$''$ adds to the usual holomorphic differentials abelian differentials of the third kind. Such generalized Abel-Jacobi maps frequently arise when considering integrable systems. Clebsch and Gordan [ClG66] considered the situation when the holomorphic differentials are supplemented by n abelian differentials $\omega_{X_iY_i}$ of the third kind for distinct pairs (X_i,Y_i). (This is enough to solve the $n=2$ genus 1 case.) More recently Fedorov [Fed99] has developed this theory. Though our form C$''$ in which there is a point common to each of the abelian differentials appears new, the approach of [Fed99] can deal with this case and a proof of the proposition that will appear elsewhere with Fedorov.

3.3. **Ingredients for the construction.** It is perhaps worthwhile recording the elements needed to make effective the construction Q_0, the second step in the Ercolani-Sinha construction of the Nahm data, the roots ρ_j of (2.13) having been calculated in the first step.

The whole construction is predicated on the theta functions built from the spectral curve. Thus we need

(1) To construct the period matrix τ associated to $\mathcal{C}$.
(2) To determine the half-period $\widetilde{\boldsymbol{K}}$.
(3) To determine the Ercolani-Sinha vector $\boldsymbol{U}$.
(4) For normalised holomorphic differentials $\boldsymbol{v}$ to calculate $\int\limits_{\infty_i}^{\infty_j}\boldsymbol{v} = \boldsymbol{\phi}(\infty_j)-\boldsymbol{\phi}(\infty_i)$.
(5) To determine $E(\infty_j,\infty_l)$.
(6) To determine $\gamma_\infty(P)$ and $\nu_i = \lim_{P\to\infty_i}\left(\int_{P_0}^{P}\gamma_\infty(P') + \frac{\eta}{\zeta}(P)\right)$.

3.4. **The fundamental bi-differential.** We shall now describe how to calculate the meromorphic differential γ_∞ and the constants appearing in the Ercolani-Sinha construction. This will be in terms of the fundamental bi-differential.

Let $\boldsymbol{v}$ be the vector of $\mathfrak{a}$-normalised holomorphic differentials with expansion (2.40). Introduce directional derivatives along the vectors fields $\boldsymbol{V}^{(i)}$, $\boldsymbol{W}^{(i)}$ etc,

$$\partial_{\boldsymbol{V}}f(\boldsymbol{v}) = \sum_{k=1}^{g}V_k\frac{\partial}{\partial v_k}f(\boldsymbol{v}), \qquad \partial_{\boldsymbol{V},\boldsymbol{W}}f(\boldsymbol{v}) = \sum_{k=1}^{g}\sum_{l=1}^{g}V_kW_l\frac{\partial^2}{\partial v_k\partial v_l}f(\boldsymbol{v}).$$

Recall (see [Fay73]) that the fundamental bi-differential $\Omega_{\mathbf{B}}$ is the symmetric 2-differential of the second kind on $\mathcal{C}\times\mathcal{C}$ is defined by,

$$\Omega_{\mathbf{B}}(P,Q)=\frac{\partial^2}{\partial x\partial y}\ln\theta_{\boldsymbol{a},\boldsymbol{b}}\left(\int_Q^P\boldsymbol{v}\right)dxdy=\left(\frac{1}{(x(P)-y(Q))^2}+\text{nonsingular}\right)dxdy, \tag{3.15}$$

where P, Q are two different points of the curve $\mathcal{C}$ with local coordinates $x(P)$, $y(Q)$, and $[\boldsymbol{a},\boldsymbol{b}]$ is an odd, nonsingular, half-integer theta characteristic. The kernel has a second order pole along the diagonal. For fixed $Q\in\mathcal{C}$ we have that

$$\oint_{\mathfrak{a}_j}\Omega_{\mathbf{B}}(P,Q)=0,\qquad \oint_{\mathfrak{b}_j}\Omega_{\mathbf{B}}(P,Q)=2\pi i\,v_j(Q). \tag{3.16}$$

Consider for each infinity ∞_i ($i=1,\ldots,n$) the differential of the second kind,

$$\Omega_{\mathbf{B}}^{(i)}(P)=\left.\frac{\Omega_{\mathbf{B}}(P,Q)}{dt(Q)}\right|_{Q=\infty_i}=-\sum_{k,l=1}^{g}\frac{\partial^2}{\partial z^k\partial z^l}\ln\theta_{\boldsymbol{a},\boldsymbol{b}}\left(\int_{\infty_i}^P\boldsymbol{v}\right)v_k(P)\left.\frac{v_l(Q)}{dt(Q)}\right|_{Q=\infty_i},$$

where $t(Q)$ is a local coordinate in the vicinity of Q. Then noting the expansion (2.40) we have $v_l(Q)/dt(Q)\to V_l^{(i)}$ as $Q\to\infty_i$. Using (3.16) we see that

$$\oint_{\mathfrak{a}_l}\Omega_{\mathbf{B}}^{(i)}(P)=0,\quad \oint_{\mathfrak{b}_l}\Omega_{\mathbf{B}}^{(i)}(P)=2\imath\pi V_l^{(i)}\quad l=1,\ldots,g. \tag{3.17}$$

The quantities $\int_{P_0}^P\Omega_{\mathbf{B}}^{(i)}(P)$ are then Abelian integrals of the second kind with unique pole of the first order at ∞_i. Further, from (3.15), we know that if $t(P)$ is a local coordinate in the vicinity of ∞_i that $\Omega_{\mathbf{B}}^{(i)}(P)=(1/t^2+\text{regular})dt$. This together with (3.17) shows that

$$\gamma_\infty(P)=\sum_{i=1}^n\rho_i\,\Omega_{\mathbf{B}}^{(i)}(P)=-\frac{\partial}{\partial x(P)}\sum_{i=1}^n\rho_i\,\partial_{\boldsymbol{V}^{(i)}}\ln\theta_{\boldsymbol{a},\boldsymbol{b}}\left(\int_{\infty_i}^P\boldsymbol{v}\right)dx(P) \tag{3.18}$$

and that $\boldsymbol{U}=\sum_{i=1}^g\rho_i\boldsymbol{V}^{(i)}$. Now

$$\int_{P_0}^P\Omega_{\mathbf{B}}^{(i)}(P')=\int_{P_0}^P-\frac{\partial}{\partial x(P')}\partial_{\boldsymbol{V}^{(i)}}\ln\theta_{\boldsymbol{a},\boldsymbol{b}}\left(\int_{\infty_i}^{P'}\boldsymbol{v}\right)dx(P')=-\partial_{\boldsymbol{V}^{(i)}}\ln\left[\frac{\theta_{\boldsymbol{a},\boldsymbol{b}}\left(\int_{\infty_i}^{P}\boldsymbol{v}\right)}{\theta_{\boldsymbol{a},\boldsymbol{b}}\left(\int_{\infty_i}^{P_0}\boldsymbol{v}\right)}\right].$$

Combining these with (3.18) yields

$$\begin{aligned}\nu_i&=\lim_{P\to\infty_i}\left(\int_{P_0}^P\gamma_\infty(P')+\frac{\eta}{\zeta}(P)\right)\\&=\lim_{P\to\infty_i}\rho_i\left[\frac{1}{t(P)}-\partial_{\boldsymbol{V}^{(i)}}\ln\theta_{\boldsymbol{a},\boldsymbol{b}}\left(\int_{\infty_i}^P\boldsymbol{v}\right)\right]\\&\quad-\sum_{j\neq i}\rho_j\,\partial_{\boldsymbol{V}^{(j)}}\ln\theta_{\boldsymbol{a},\boldsymbol{b}}\left(\int_{\infty_j}^{\infty_i}\boldsymbol{v}\right)+\operatorname{Res}_{\infty_i}\frac{\eta}{\zeta^2}+c(P_0),\end{aligned}$$

where $c(P_0) = \sum_{i=1}^{g} \rho_i \, \partial_{\boldsymbol{V}^{(i)}} \ln \, \theta_{\boldsymbol{a},\boldsymbol{b}} \left(\int\limits_{\infty_i}^{P_0} \boldsymbol{v} \right)$ is a constant independent of the index i. The first limit may be calculated directly. Assuming for the sake of exposition a first order vanishing we find in terms of the local coordinate t that

$$\lim_{P\to\infty_i} \partial_{\boldsymbol{V}^{(i)}} \ln \theta_{\boldsymbol{a},\boldsymbol{b}} \left(\int\limits_{\infty_i}^{P} \boldsymbol{v} \right) = \frac{1}{t} + \frac{1}{2} \frac{\partial^2_{\boldsymbol{V}^{(i)},\boldsymbol{V}^{(i)}} \theta_{\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{0})}{\partial_{\boldsymbol{V}^{(i)}} \theta_{\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{0})} - \frac{1}{2} \frac{\partial_{\boldsymbol{W}^{(i)}} \theta_{\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{0})}{\partial_{\boldsymbol{V}^{(i)}} \theta_{\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{0})} + \dots$$

Now because $[\boldsymbol{a}, \boldsymbol{b}]$ is an odd theta characteristic then $\partial^2_{\boldsymbol{V}^{(i)},\boldsymbol{V}^{(i)}} \theta_{\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{0}) = 0$. Therefore

$$\nu_i = \frac{\rho_i}{2} \frac{\partial_{\boldsymbol{W}^{(i)}} \theta_{\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{0})}{\partial_{\boldsymbol{V}^{(i)}} \theta_{\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{0})} - \sum_{j\neq i} \rho_j \, \partial_{\boldsymbol{V}^{(j)}} \ln \, \theta_{\boldsymbol{a},\boldsymbol{b}} \left(\int\limits_{\infty_j}^{\infty_i} \boldsymbol{v} \right) + \mathrm{Res}_{\infty_i} \frac{\eta}{\zeta^2} + c(P_0). \tag{3.19}$$

Because we are only interested in calculating the differences $\nu_i - \nu_j$ there exists a further representation making use of normalised differentials of the third kind. Let $\omega_{\infty_i,\infty_j}$ be the meromorphic differential of the third kind, with simple pole of residue $+1$ at ∞_i and -1 at ∞_j with vanishing $\mathfrak{a}$-periods. Now consider the integral

$$\mathfrak{I} = \int_{\partial\Gamma} \int_{P_0}^{P} \left[\gamma_\infty(P') + d\left(\frac{\eta}{\zeta}\right)(P') \right] \omega_{\infty_i,\infty_j}(P)$$

taken over the boundary $\partial\Gamma$ of the fundamental domain and compute it in two ways: as sum of residues and as a contour integral. Because of the normalisation of the differentials γ_∞ and $\omega_{\infty_i,\infty_j}$ the contour integral vanishes. Therefore the sum of residues vanishes too, which upon using

$$\mathrm{Res}_{P=0_k} \frac{\eta}{\zeta}(P) = \eta(0_k),$$

leads to the equality

$$\nu_i - \nu_j = -\sum_{k=1}^{n} \eta(0_k) \omega_{\infty_i,\infty_j}(0_k). \tag{3.20}$$

Using (3.14) this formula can be written in terms of the prime form and we obtain the formula

$$\nu_i - \nu_j = -\sum_{k=1}^{n} \eta(0_k) \, \frac{\partial}{\partial z} \ln \frac{\mathcal{E}(P,\infty_i)}{\mathcal{E}(P,\infty_j)} \Bigg|_{P=0_k} = -\sum_{k=1}^{n} \eta(0_k) \, \frac{\partial}{\partial z} \ln \frac{\theta_{\boldsymbol{a},\boldsymbol{b}}\left(\int_{\infty_i}^{P} \boldsymbol{v}\right)}{\theta_{\boldsymbol{a},\boldsymbol{b}}\left(\int_{\infty_j}^{P} \boldsymbol{v}\right)} \Bigg|_{P=0_k} \tag{3.21}$$

We remark that more explicit results we can be found upon utilising the Klein-Weierstrass realisation of third kind differentials [Bak95].

With the general construction now at hand we shall turn to some explicit examples. The case of $n = 2$ has been treated by several authors and by way of illustration we too treat this example using the formulae just described. Our formulae, though different in detail, lead to the known results. Going beyond these results we consider the case of $n = 3$ in the second part of this work.

4. An illustration: the charge 2 monopole

We shall consider the well-studied case of $n = 2$ to enable comparison with other authors. Our first step will be to assemble the ingredients for the construction, noted above, and so to determine the matrix $Q_0(z)$. For completeness we will also perform the remaining steps needed to reconstruct the Nahm data.

We will work with the (centred) spectral curve in the form chosen by Ercolani-Sinha,

$$0 = \eta^2 + \frac{\kappa^2}{4}(\zeta^4 + 2(k^2 - k'^2)\zeta^2 + 1) \tag{4.1}$$

$$= \eta^2 + \frac{\kappa^2}{4}(\zeta - k' - ik)(\zeta - k' + ik)(\zeta + k' - ik)(\zeta + k' + ik) \tag{4.2}$$

where $k'^2 = 1 - k^2$. With $k' = \cos\alpha$, $k = \sin\alpha$, then the roots may be written as $\pm e^{\pm i\alpha}$ and these lie on the unit circle. We may take $0 \le \alpha \le \pi/4$. We choose cuts between $-k' + ik = -e^{-i\alpha}$ and $k' + ik = e^{i\alpha}$ as well as $-k' - ik$ and $k' - ik$. Let $\mathfrak{b}$ encircle $-k' + ik$ and $k' + ik$ with $\mathfrak{a}$ encircling $k' + ik$ and $-k' + ik$ on the two sheets as on the diagram. We take as our assignment of sheets ($j = 1$, 2, with analytic continuation from $\zeta = 0$ avoiding the cuts) to be

$$\eta_j = (-1)^j\, i\frac{\kappa}{2}\sqrt{\zeta^4 + 2(k^2 - k'^2)\zeta^2 + 1}.$$

Then, upon using the substitutions $\zeta = e^{i\theta}$ and $k \sin u = \sin\theta$ on sheet 1,

$$\frac{d\zeta}{\eta} = i\frac{2}{\kappa}\frac{d\zeta}{\sqrt{(\zeta^2 - e^{2i\alpha})(\zeta^2 - e^{-2i\alpha})}} = \frac{-1}{k\kappa}\frac{d\theta}{\sqrt{1 - \frac{1}{k^2}\sin^2\theta}} = \frac{-1}{\kappa}\frac{du}{\sqrt{1 - k^2\sin^2 u}}.$$

Thus

$$\oint_{\mathfrak{a}} \frac{d\zeta}{\eta} = \frac{-2}{\kappa}\int_{\alpha}^{-\alpha} \frac{d\theta}{\sqrt{k^2 - \sin^2\theta}} = \frac{4}{\kappa}\int_0^{\pi/2} \frac{du}{\sqrt{1 - k^2\sin^2 u}} = \frac{4}{\kappa}\,\mathbf{K}(k),$$

where $\mathbf{K} = \mathbf{K}(k)$ is the complete elliptic integral of the first kind. Similarly (with $\zeta = \exp i(w + \pi/2)$)

$$\oint_{\mathfrak{b}} \frac{d\zeta}{\eta} = \frac{2i}{\kappa}\int_{\alpha-\pi/2}^{\pi/2-\alpha} \frac{dw}{\sqrt{k'^2 - \sin^2 w}} = \frac{4i}{\kappa}\int_0^{\pi/2} \frac{du}{\sqrt{1 - k'^2\sin^2 u}} = \frac{4i}{\kappa}\,\mathbf{K}'(k).$$

For the curve (4.1) and this choice of homology basis the normalized holomorphic differential is then $\boldsymbol{v} = \kappa d\zeta/(4\mathbf{K}\eta)$. Comparison with (2.44) shows that the Ercolani-Sinha constraint is satisfied for $\kappa = \mathbf{K}(k)$ and $\mathfrak{c} = -\mathfrak{a}$. Thus $\boldsymbol{U} = -1/2$. The period matrix for the curve is then $\tau = i\mathbf{K}'/\mathbf{K}$. Symmetry now enables us to evaluate various Abel-maps (with base point $P_0 = k' + ik$):

$$\boldsymbol{\phi}(\infty_1) = \frac{1+\tau}{4} = -\boldsymbol{\phi}(\infty_2), \qquad \boldsymbol{\phi}(0_1) = \frac{1-\tau}{4} = -\boldsymbol{\phi}(0_2).$$

From (2.13) and our assignment of sheets we have that

$$\rho_1 = -\frac{i}{2}\,\mathbf{K}, \qquad \rho_2 = \frac{i}{2}\,\mathbf{K}.$$

Many features of this example can be determined without calculation. For example, $\widetilde{\boldsymbol{K}}$ is a theta (half-) characteristic such that θ vanishes to order $s = \frac{1}{4}2^2 = 1$ at $\widetilde{\boldsymbol{K}}$. This identifies $\tilde{K}$ as the unique odd theta characteristic $\widetilde{\boldsymbol{K}} = (1+\tau)/2$. Further $\tau^*(d\zeta/\eta) = -\overline{(d\zeta/\eta)}$. The property $\tau_*(\mathfrak{c}) = -\mathfrak{c}$ then fixes $\mathfrak{c} = \pm\mathfrak{a}$ and consequently $\boldsymbol{U} = \pm 1/2$, the relevant sign being selected by (2.44). The non-singular even theta characteristic $\boldsymbol{U} + \widetilde{\boldsymbol{K}} = \tau/2$. Then

$$\boldsymbol{U} - \widetilde{\boldsymbol{K}} = -1 - \tau/2 = \frac{1}{2}\tilde{p} + \frac{1}{2}\tau\tilde{q} \Rightarrow \tilde{p} = -2,\ \tilde{q} = -1.$$

Substitution of the quantities collected thus far into (3.8) yields

$$Q_0(z)_{12} = \epsilon_{12}\, \frac{-i\mathbf{K}}{\mathcal{E}(\infty_1, \infty_2)}\, e^{i\pi(1+\tau)/2}\, \frac{\theta(-[z+1]/2 - 1 - \tau)}{\theta(-[z+1]/2 - (1+\tau)/2)}\, e^{z(\nu_2 - \nu_1)}. \tag{4.3}$$

Upon using the identification of $\theta(z)$ with the Jacobi theta function $\theta_3(z)$ and the periodicities of the Jacobi theta functions $\theta_*(z)$ then

$$\theta(-[z+1]/2 - 1 - \tau) = -e^{-i\pi(z+\tau)}\, \theta_4(z/2),$$
$$\theta(-[z+1]/2 - (1+\tau)/2) = e^{-i\pi(z/2+\tau/4)}\, \theta_2(z/2),$$

giving

$$Q_0(z)_{12} = -\epsilon_{12}\, \frac{\mathbf{K}}{\mathcal{E}(\infty_1, \infty_2)}\, e^{-i\pi\tau/4}\, \frac{\theta_4(z/2)}{\theta_2(z/2)}\, e^{z(\nu_2 - \nu_1 - i\pi/2)}.$$

The prime form. Let us now evaluate the prime form. We have from (3.9) that

$$E(P,Q) = \frac{\theta\begin{bmatrix}1/2\\1/2\end{bmatrix}(\boldsymbol{\phi}(P) - \boldsymbol{\phi}(Q))}{h(P)h(Q)}, \qquad h^2(P) = \frac{\partial\theta\begin{bmatrix}1/2\\1/2\end{bmatrix}(0)}{\partial z}\, v(P).$$

If $\zeta = 1/t$ is a local parameter at ∞_j then $v(\infty_j) = d\zeta/4\eta|_{\infty_j} = -dt/(4\rho_j)$. Identifying $\theta\begin{bmatrix}1/2\\1/2\end{bmatrix}(z)$ with the Jacobi theta function $-\theta_1(z)$ then gives

$$\begin{aligned}\mathcal{E}(\infty_1, \infty_2) = E(\infty_1, \infty_2)dt = -4\sqrt{\rho_1\rho_2}\, \frac{\theta_1\left(\int_{\infty_2}^{\infty_1} \boldsymbol{v}\right)}{\theta_1'} &= -2\mathbf{K}\frac{\theta_1\left(\int_{\infty_2}^{\infty_1} \boldsymbol{v}\right)}{\theta_1'}\\ = -2\mathbf{K}\, e^{-i\pi\tau/4}\, \frac{\theta_3}{\theta_1'}.\end{aligned}$$

Upon noting that $\theta_1' = \pi\theta_2\theta_3\theta_4$, we then have that

$$Q_0(z)_{12} = \epsilon_{12}\, \frac{\pi\theta_2\theta_4}{2}\, \frac{\theta_4(z/2)}{\theta_2(z/2)}\, e^{z(\nu_2 - \nu_1 - i\pi/2)}. \tag{4.4}$$

Alternate calculation. Let us check this calculation proceeding from the form (3.6). Then

$$\begin{aligned}Q_0(z)_{12} &= -(\rho_1 - \rho_2)\, c_{12}\, \frac{\theta(\boldsymbol{\phi}(\infty_2) - \boldsymbol{\phi}(\infty_1) + (z+1)\boldsymbol{U} - \widetilde{\boldsymbol{K}})\theta(\boldsymbol{U} - \widetilde{\boldsymbol{K}})}{\theta(\boldsymbol{\phi}(\infty_2) - \boldsymbol{\phi}(\infty_1) + \boldsymbol{U} - \widetilde{\boldsymbol{K}})\theta((z+1)\boldsymbol{U} - \widetilde{\boldsymbol{K}})}\, e^{z(\nu_2 - \nu_1)}\\ &= c_{12}\, i\mathbf{K}\, \frac{\theta_2\, \theta_4(z/2)}{\theta_4\, \theta_2(z/2)}\, e^{z(\nu_2 - \nu_1 - i\pi/2)}.\end{aligned}$$

where $c_{12} = \lim_{P \to \infty_2} \zeta g_1(P)$ is Ercolani-Sinha's constant. To evaluate this we need the function $g_1(P) = f_1(P)/f_1(\infty_1)$, with

$$f_1(P) = \frac{\theta(\boldsymbol{\phi}(P) - \boldsymbol{\phi}(\Delta_1) - \boldsymbol{K})\, \theta(\boldsymbol{\phi}(P) - \boldsymbol{\phi}(\infty_2) - \boldsymbol{K})}{\theta(\boldsymbol{\phi}(P) - \boldsymbol{\phi}(\delta_1) - \boldsymbol{K})\, \theta(\boldsymbol{\phi}(P) - \boldsymbol{\phi}(\delta_2) - \boldsymbol{K})}$$

where here $\delta_1+\delta_2 \sim \Delta_1 - \infty_1$ and $\boldsymbol{\phi}(\Delta_1-\infty_1)=\boldsymbol{U}=-1/2$. Using the values of the Abel map this last equality yields $\boldsymbol{\phi}(\Delta_1)=(-1+\tau)/4$ and we may identify $\Delta_1 = 0_2$. (Similarly $\Delta_2=0_1$.) With these properties then

$$g_1(P)=\frac{\theta_1\left(\int_{\infty_2}^{P}\boldsymbol{v}\right)\,\theta_1\left(\int_{0_2}^{P}\boldsymbol{v}\right)\,\theta_1\left(\int_{\delta_1}^{\infty_1}\boldsymbol{v}\right)\,\theta_1\left(\int_{\delta_2}^{\infty_1}\boldsymbol{v}\right)}{\theta_1\left(\int_{\infty_2}^{\infty_1}\boldsymbol{v}\right)\,\theta_1\left(\int_{0_2}^{\infty_1}\boldsymbol{v}\right)\,\theta_1\left(\int_{\delta_1}^{P}\boldsymbol{v}\right)\,\theta_1\left(\int_{\delta_2}^{P}\boldsymbol{v}\right)}.$$

This function has poles in δ and vanishes at ∞_2 and $\Delta_1=0_2$.[3] Now

$$\lim_{P\to\infty_2}\zeta(P)\,\theta_1\left(\int_{\infty_2}^{P}\boldsymbol{v}\right)=-\frac{\theta_1'}{4\rho_2}$$

and

$$c_{12}=\frac{\pi\theta_4^2}{2\mathbf{K}}\,\frac{\theta_1\left(\int_{\delta_1}^{\infty_1}\boldsymbol{v}\right)\,\theta_1\left(\int_{\delta_2}^{\infty_1}\boldsymbol{v}\right)}{\theta_1\left(\int_{\delta_1}^{\infty_2}\boldsymbol{v}\right)\,\theta_1\left(\int_{\delta_2}^{\infty_2}\boldsymbol{v}\right)}.$$

A similar calculation shows also that

$$c_{21}=\frac{\pi\theta_4^2}{2\mathbf{K}}\,\frac{\theta_1\left(\int_{\delta_1}^{\infty_2}\boldsymbol{v}\right)\,\theta_1\left(\int_{\delta_2}^{\infty_2}\boldsymbol{v}\right)}{\theta_1\left(\int_{\delta_1}^{\infty_1}\boldsymbol{v}\right)\,\theta_1\left(\int_{\delta_2}^{\infty_1}\boldsymbol{v}\right)}.$$

Thus

$$c_{12}=-c_{21}\Longleftrightarrow\frac{\theta_1\left(\int_{\delta_1}^{\infty_2}\boldsymbol{v}\right)\,\theta_1\left(\int_{\delta_2}^{\infty_2}\boldsymbol{v}\right)}{\theta_1\left(\int_{\delta_1}^{\infty_1}\boldsymbol{v}\right)\,\theta_1\left(\int_{\delta_2}^{\infty_1}\boldsymbol{v}\right)}=\pm i,$$

which again yields the solution (4.4). To solve for the divisor δ we note that $\boldsymbol{\phi}(\delta_1)+\boldsymbol{\phi}(\delta_2)=-1/2$, whence we wish to solve for $x=\boldsymbol{\phi}(\delta_1)$,

$$\frac{\theta_1\left(\frac{1+\tau}{4}-x\right)\,\theta_1\left(\frac{3+\tau}{4}+x\right)}{\theta_1\left(\frac{1+\tau}{4}+x\right)\,\theta_1\left(\frac{3+\tau}{4}-x\right)}=\pm i.$$

The left-hand side is an elliptic function (with periods 1 and x) in x and so has solutions. For example $x=\boldsymbol{\phi}(\delta_1)=3\tau/4$ and $1/2+\tau/4$ yield the plus sign, while $x=\boldsymbol{\phi}(\delta_1)=1/2+3\tau/4$ and $\tau/4$ yield the minus sign. The two solutions for a fixed sign correspond to the interchange of δ_1 and δ_2, thus up to equivalence there are the two solutions arising from the different choices of sign.

The fundamental bi-differential. Finally let us calculate the fundamental bi-differential and use our formulae to show that

$$\nu_2-\nu_1=\frac{i\pi}{2}.$$

The evenness of the curve (4.1) means that for P near ∞_i we have

$$\int_{P_0}^{P}\boldsymbol{v}=\int_{P_0}^{\infty_i}\boldsymbol{v}+\int_{\infty_i}^{P_0}\boldsymbol{v}=\int_{P_0}^{\infty_i}\boldsymbol{v}-\frac{t}{4\rho_i}+O(t^3),$$

$$\mathbf{V}^{(i)}=-\frac{1}{4\rho_i},\ \ \mathbf{W}^{(i)}=0,$$

$$\mathrm{Res}_{\infty_1,\infty_2}\frac{\eta}{\zeta^2}=0,$$

[3]We disagree with the formulae of Ercolani and Sinha at this stage: their function (IV.26a) does not have poles where stated.

$$\Omega_{\mathbf{B}}^{(i)}(P) = \frac{1}{4\rho_i}\partial_x \left[\frac{\theta_1'\left(\int_{\infty_i}^P \boldsymbol{v}\right)}{\theta_1\left(\int_{\infty_i}^P \boldsymbol{v}\right)} \right] dx(P).$$

Further, $\theta_1(x)$ is an odd function and so $\theta''(0) = 0$. Thus

$$\nu_2 - \nu_1 = \frac{1}{4}\,\partial_x \ln \left[\frac{\theta_1\left(x + \int_{\infty_1}^{\infty_2} \boldsymbol{v}\right)}{\theta_1\left(x + \int_{\infty_2}^{\infty_1} \boldsymbol{v}\right)} \right]_{x=0} = \frac{1}{4}\,\partial_x \ln \left[\frac{\theta_1\left(x - \frac{1+\tau}{2}\right)}{\theta_1\left(x + \frac{1+\tau}{2}\right)} \right]_{x=0} = \frac{i\pi}{2},$$

upon using $\theta_1(x + \frac{1+\tau}{2}) = B(x)\,\theta_3(x)$, with $B(x) = \exp -i\pi(x + \tau/4)$. The same result ensues from (3.21),

$$\begin{aligned}
\nu_1 - \nu_2 &= -\eta(0_1)\,\frac{\partial}{\partial z}\ln \frac{\theta_1\left(\int_{\infty_1}^P \boldsymbol{v}\right)}{\theta_1\left(\int_{\infty_2}^P \boldsymbol{v}\right)}\Bigg|_{P=0_1} - \eta(0_2)\,\frac{\partial}{\partial z}\ln \frac{\theta_1\left(\int_{\infty_1}^P \boldsymbol{v}\right)}{\theta_1\left(\int_{\infty_2}^P \boldsymbol{v}\right)}\Bigg|_{P=0_2} \\
&= -\frac{1}{4}\partial_x \ln \frac{\theta_1\left(x + \int_{\infty_1}^{0_1} \boldsymbol{v}\right)}{\theta_1\left(x + \int_{\infty_2}^{0_2} \boldsymbol{v}\right)}\Bigg|_{x=0} - \frac{1}{4}\partial_x \ln \frac{\theta_1\left(x + \int_{\infty_2}^{0_1} \boldsymbol{v}\right)}{\theta_1\left(x + \int_{\infty_1}^{0_2} \boldsymbol{v}\right)}\Bigg|_{x=0} \\
&= -\frac{1}{4}\partial_x \ln \frac{\theta_1(x - \frac{\tau}{2})}{\theta_1(x + \frac{\tau}{2})}\Bigg|_{x=0} + \frac{1}{4}\partial_x \ln \frac{\theta_1(x + \frac{1}{2})}{\theta_1(x - \frac{1}{2})}\Bigg|_{x=0}.
\end{aligned}$$

Here we have used our expressions for $\boldsymbol{\phi}(\infty_{1,2})$ and $\boldsymbol{\phi}(0_{1,2})$. Now upon using $\theta_1(x+\tau/2) = \imath B(x)\theta_4(x)$, $B(x) = \exp\{-\imath\pi(x + \tau/4)\}$ we again conclude that $\nu_1 - \nu_2 = \frac{\imath\pi}{2}$. For completeness we record that

$$\omega_{\infty_1,\infty_2} = -\frac{\imath\mathbf{K}(k)}{2}\frac{\zeta}{\eta}d\zeta + c\,\boldsymbol{v}(P),$$

where the normalisation constant is given by

$$c = \frac{\imath\mathbf{K}(k)}{8}\oint_{\mathfrak{a}} \frac{\zeta d\zeta}{\eta} = \frac{\imath\pi}{4}.$$

One may simply work from this and (3.20) to obtain the same result.

Determining the Nahm data. At this stage we have established that

$$Q_0(z)_{12} = \epsilon_{12}\,\frac{\pi\theta_2\theta_4}{2}\,\frac{\theta_4(z/2)}{\theta_2(z/2)} = \epsilon_{12}\mathbf{K}k'\,\frac{1}{\operatorname{cn}\mathbf{K}z},$$

and that we have the matrix

$$Q_0(z) = \begin{pmatrix} 0 & Q_0(z)_{12} \\ Q_0(z)_{12} & 0 \end{pmatrix}.$$

Without loss of generality we may take $\epsilon_{12} = 1$. We conclude by deriving the well known elliptic solution of the Nahm equations given by

$$T_j(z) = \frac{\sigma_j}{2\imath}\,f_j(z), \quad j = 1, 2, 3, \tag{4.5}$$

where σ_j are Pauli matrices and the functions $f_j(z)$ are expressible in terms of Jacobian elliptic functions

$$\begin{aligned}
&f_1(z) = \mathbf{K}\,\frac{\operatorname{dn}\mathbf{K}z}{\operatorname{cn}\mathbf{K}z} = \frac{\pi\theta_2\theta_3}{2}\,\frac{\theta_3(z/2)}{\theta_2(z/2)}, \quad f_2(z) = \mathbf{K}k'\,\frac{\operatorname{sn}\mathbf{K}z}{\operatorname{cn}\mathbf{K}z} = \frac{\pi\theta_3\theta_4}{2}\,\frac{\theta_1(z/2)}{\theta_2(z/2)}, \\
&f_3(z) = \mathbf{K}k'\,\frac{1}{\operatorname{cn}\mathbf{K}z} = \frac{\pi\theta_2\theta_4}{2}\,\frac{\theta_4(z/2)}{\theta_2(z/2)}.
\end{aligned} \tag{4.6}$$

We shall further use

$$\int \frac{du}{\operatorname{cn} u} = \frac{1}{k'}\ln\frac{\operatorname{dn}u + k'\operatorname{sn}u}{\operatorname{cn}u}. \tag{4.7}$$

Following theorem (2.2) we have outlined the steps involved in determining the Nahm data one $Q_0(z)$ is known. First we find the matrix $C(z)$, subject to initial condition $C(0) = \mathrm{Id}_2$ and satisfying the differential equation

$$\frac{dC(z)}{dz} = \frac{1}{2}\, C(z)Q_0(z).$$

The solution of this satisfying our initial condition is simply

$$C(z) = \begin{pmatrix} F(z) & G(z) \\ G(z) & F(z) \end{pmatrix}$$

with

$$F(z) = \operatorname{ch}\left(\frac{1}{2}\int_0^z f_3(u)du\right), \quad G(z) = \operatorname{sh}\left(\frac{1}{2}\int_0^z f_3(u)du\right).$$

Therefore we have that

$$A_0(z) = C(z)Q_0(z)C^{-1}(z) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} Q_0(z)_{12} = \frac{1}{2\imath}\sigma_3 f_3(z). \tag{4.8}$$

Step (b) of our procedure then says that

$$\begin{aligned} A_1(z) &= C(z)A_1(0)C^{-1}(z) = C(z)\begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix} C^{-1}(z) \\ &= \frac{\mathbf{K}}{2\imath}\begin{pmatrix} 1+2G(z)^2 & -2F(z)G(z) \\ 2F(z)G(z) & -1-2G(z)^2 \end{pmatrix}. \end{aligned} \tag{4.9}$$

Straightforward calculations (where we now employ (4.7)) now give that

$$G(z)^2 = \frac{1}{2}\frac{\operatorname{dn}\mathbf{K}z}{\operatorname{cn}\mathbf{K}z} - \frac{1}{2}, \qquad F(z)G(z) = k'\frac{\operatorname{sn}\mathbf{K}z}{\operatorname{cn}\mathbf{K}z}.$$

Therefore

$$A_1(z) = \frac{\mathbf{K}}{2\imath}\begin{pmatrix} \dfrac{\operatorname{dn}\mathbf{K}z}{\operatorname{cn}\mathbf{K}z} & -k'\dfrac{\operatorname{sn}\mathbf{K}z}{\operatorname{cn}\mathbf{K}z} \\ k'\dfrac{\operatorname{sn}\mathbf{K}z}{\operatorname{cn}\mathbf{K}z} & -\dfrac{\operatorname{dn}\mathbf{K}z}{\operatorname{cn}\mathbf{K}z} \end{pmatrix}$$

and

$$A_{-1}(z) = -A^\dagger(z) = -\frac{\mathbf{K}}{2\imath}\begin{pmatrix} -\dfrac{\operatorname{dn}\mathbf{K}z}{\operatorname{cn}\mathbf{K}z} & -k'\dfrac{\operatorname{sn}\mathbf{K}z}{\operatorname{cn}\mathbf{K}z} \\ k'\dfrac{\operatorname{sn}\mathbf{K}z}{\operatorname{cn}\mathbf{K}z} & \dfrac{\operatorname{dn}\mathbf{K}z}{\operatorname{cn}\mathbf{K}z} \end{pmatrix}.$$

The final step in obtaining the Nahm data is then

$$\begin{aligned} T_1(z) &= \frac{1}{2}(A_1(z)+A_{-1}(z)) = \frac{\mathbf{K}}{2\imath}\begin{pmatrix} \frac{\operatorname{dn}\mathbf{K}z}{\operatorname{cn}\mathbf{K}z} & 0 \\ 0 & -\frac{\operatorname{dn}\mathbf{K}z}{\operatorname{cn}\mathbf{K}z} \end{pmatrix} = \frac{1}{2\imath}\sigma_1 f_1(z), \\ T_2(z) &= \frac{1}{2\imath}(A_{-1}(z)-A_1(z)) = \frac{\mathbf{K}}{2\imath}\begin{pmatrix} 0 & -\imath k'\frac{\operatorname{sn}\mathbf{K}z}{\operatorname{cn}\mathbf{K}z} \\ \imath k'\frac{\operatorname{sn}\mathbf{K}z}{\operatorname{cn}\mathbf{K}z} & 0 \end{pmatrix} = \frac{1}{2\imath}\sigma_2 f_2(z), \end{aligned}$$

and our procedure leads to the known solution (4.6).

Part 2. Charge Three Monopole Constructions

5. The trigonal curve

We shall now introduce the class of curves that will be the focus of our attention. These are

$$\eta^3 + \hat{\chi}(\zeta-\lambda_1)(\zeta-\lambda_2)(\zeta-\lambda_3)(\zeta-\lambda_4)(\zeta-\lambda_5)(\zeta-\lambda_6) = 0. \tag{5.1}$$

For suitable λ_i they correspond[4] to centred charge three monopoles restricted by $a_2(\zeta) = 0$. Thus the eight dimensional moduli space of centred monopoles has been reduced to three dimensions. The asymptotic behaviour of the curve gives us

$$\rho_k = -\hat{\chi}^{\frac{1}{3}}\, e^{2\imath k\pi/3}. \tag{5.2}$$

For notational convenience we will study (5.1) in the form ($w = -\hat{\chi}^{-\frac{1}{3}}\eta$, $z = \zeta$)

$$w^3 = \prod_{i=1}^{6}(z-\lambda_i). \tag{5.3}$$

The moduli space of such curves with an homology marking can be regarded as the configuration space of six distinct points on $\mathbb{P}^1$. This class of curves has been studied by Picard [Pic83], Wellstein [Wel99], Shiga [Shi88] and more recently by Matsumoto [Mat01]; we shall recall some of their results. To make concrete the θ-functions arising in the Ercolani-Sinha construction we need to have the period matrix for the curve, the vector of Riemann constants, and to understand the special divisors. We shall now make these things explicit, beginning first with our choice of homology basis.

5.1. **The curve and homologies.** Let $\mathcal{C}$ denote the curve (5.3) of genus four where the six points $\lambda_i \in \mathbb{C}$ are assumed distinct and ordered according to the rule $\arg(\lambda_1) < \arg(\lambda_2) < \ldots < \arg(\lambda_6)$. Let $\mathcal{R}$ be the automorphism of $\mathcal{C}$ defined by

$$\mathcal{R} : (z, w) \to (z, \rho w), \quad \rho = \exp\{2\imath\pi/3\}. \tag{5.4}$$

The bilinear transformation $(z, w) \leftrightarrow (Z, W)$

$$\begin{aligned} Z &= \frac{(\lambda_2-\lambda_1)(z-\lambda_4)}{(\lambda_2-\lambda_4)(z-\lambda_1)},\\ W &= -\frac{w}{(z-\lambda_1)^2}\left(\prod_{k=2}^{6}(\lambda_1-\lambda_k)\right)^{-\frac{1}{3}}\left(\frac{(\lambda_1-\lambda_4)(\lambda_1-\lambda_2)}{\lambda_2-\lambda_4}\right)^{\frac{5}{3}} \end{aligned} \tag{5.5}$$

and its inverse

(5.6)

$$\begin{aligned} z &= \frac{Z\lambda_1(\lambda_2-\lambda_4)+\lambda_4(\lambda_1-\lambda_2)}{Z(\lambda_2-\lambda_4)-(\lambda_2-\lambda_1)}\\ w &= -\frac{W}{(Z(\lambda_2-\lambda_4)-(\lambda_2-\lambda_1))^2}\left(\prod_{k=2}^{6}(\lambda_1-\lambda_k)\right)^{\frac{1}{3}}(\lambda_1-\lambda_2)^{\frac{1}{3}}(\lambda_1-\lambda_4)^{\frac{1}{3}}(\lambda_2-\lambda_4)^{\frac{5}{3}} \end{aligned}$$

leads to the following normalization of the curve (5.14)

$$W^3 = Z(Z-1)(Z-\Lambda_1)(Z-\Lambda_2)(Z-\Lambda_3), \tag{5.7}$$

---

[4]Here $\{\lambda_i\}_{i=1}^6 = \{\alpha_j, -1/\overline{\alpha}_j\}_{j=1}^3$ and $\hat{\chi} = \chi_3\left[\prod_{l=1}^3\left(\frac{\overline{\alpha}_l}{\alpha_l}\right)^{1/2}\right]$.

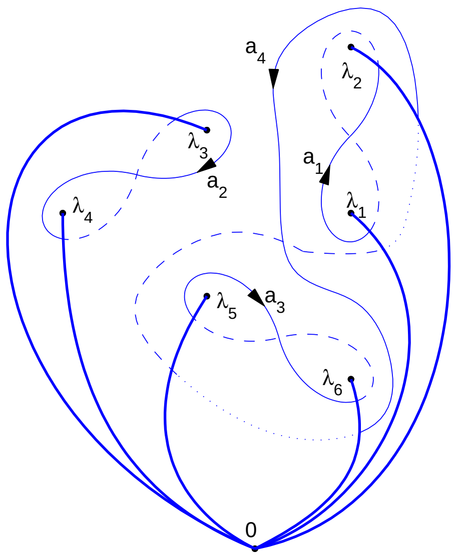


FIGURE 1. Homology basis: $\mathfrak{a}$-cycles

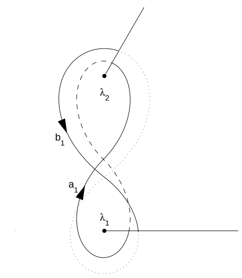


FIGURE 2. Cycles $\mathfrak{a}_1$ and $\mathfrak{b}_1$

where

$$\Lambda_i = \frac{\lambda_2 - \lambda_1}{\lambda_2 - \lambda_4}\frac{\lambda_{2+j(i)} - \lambda_4}{\lambda_{2+j(i)} - \lambda_1}, \quad i = 1, 2, 3;\; j(1) = 1,\; j(2) = 3,\; j(3) = 4. \tag{5.8}$$

Fix the following lexicographical ordering of independent canonical holomorphic differentials of $\mathcal{C}$,

$$du_1 = \frac{dz}{w}, \quad du_2 = \frac{dz}{w^2}, \quad du_3 = \frac{zdz}{w^2}, \quad du_4 = \frac{z^2dz}{w^2}. \tag{5.9}$$

To construct the symplectic basis $(\mathfrak{a}_1, \ldots, \mathfrak{a}_4; \mathfrak{b}_1, \ldots, \mathfrak{b}_4)$ of $H_1(\mathcal{C}, \mathbb{Z})$ we introduce oriented paths $\gamma_k(z_i, z_j)$ going from $P_i = (z_i, w_i)$ to $P_j = (z_j, w_j)$ in the k-th sheet. Define 1-cycles $\mathfrak{a}_i, \mathfrak{b}_i$ on $\mathcal{C}$ as follows[5]

$$\begin{aligned}
&\mathfrak{a}_1 = \gamma_1(\lambda_1, \lambda_2) + \gamma_2(\lambda_2, \lambda_1), \qquad \mathfrak{b}_1 = \gamma_1(\lambda_2, \lambda_1) + \gamma_3(\lambda_1, \lambda_2),\\
&\mathfrak{a}_2 = \gamma_1(\lambda_3, \lambda_4) + \gamma_2(\lambda_4, \lambda_3), \qquad \mathfrak{b}_2 = \gamma_1(\lambda_4, \lambda_3) + \gamma_3(\lambda_3, \lambda_4),\\
&\mathfrak{a}_3 = \gamma_1(\lambda_5, \lambda_6) + \gamma_2(\lambda_6, \lambda_5), \qquad \mathfrak{b}_3 = \gamma_1(\lambda_6, \lambda_5) + \gamma_3(\lambda_5, \lambda_6),\\
&\mathfrak{a}_4 = \gamma_3(\lambda_1, \lambda_2) + \gamma_1(\lambda_2, \lambda_6) + \gamma_3(\lambda_6, \lambda_5) + \gamma_2(\lambda_5, \lambda_1),\\
&\mathfrak{b}_4 = \gamma_2(\lambda_2, \lambda_1) + \gamma_3(\lambda_6, \lambda_2) + \gamma_2(\lambda_5, \lambda_6) + \gamma_1(\lambda_1, \lambda_5).
\end{aligned} \tag{5.10}$$

The $\mathfrak{a}$-cycles of the homology basis are given in Figure 1, with the $\mathfrak{b}$-cycles shifted by one sheet. We have the pairings $\mathfrak{a}_k \circ \mathfrak{a}_l = \mathfrak{b}_k \circ \mathfrak{b}_l = 0$, $\mathfrak{a}_k \circ \mathfrak{b}_l = -\mathfrak{b}_k \circ \mathfrak{a}_l = \delta_{k,l}$ and therefore $(\mathfrak{a}_1, \ldots, \mathfrak{a}_4; \mathfrak{b}_1, \ldots, \mathfrak{b}_4)$ is a symplectic basis of $H_1(\mathcal{C}, \mathbb{Z})$. In the homology basis introduced we have

$$\mathcal{R}(\mathfrak{b}_i) = \mathfrak{a}_i, \quad i = 1, 2, 3, \quad \mathcal{R}(\mathfrak{b}_4) = -\mathfrak{a}_4. \tag{5.11}$$

As $(1 + \mathcal{R} + \mathcal{R}^2)\mathfrak{c} = 0$ for any cycle $\mathfrak{c}$ we have, for example, that $\mathcal{R}(\mathfrak{a}_i) = -\mathfrak{a}_i - \mathcal{R}^2(\mathfrak{a}_i) = -\mathfrak{a}_i - \mathfrak{b}_i$ for $i = 1, 2, 3$ and $\mathcal{R}(\mathfrak{a}_4) = -\mathfrak{a}_4 + \mathfrak{b}_4$, so completing the $\mathcal{R}$ action on the homology basis.

[5]This is the basis from [Mat01]; another but equivalent basis can be found in [Wel99].

5.2. **The Riemann period matrix.** Denote vectors

$$\begin{aligned}
\boldsymbol{x} &= (x_1,x_2,x_3,x_4)^T = \left(\oint_{\mathfrak{a}_1} du_1,\ldots,\oint_{\mathfrak{a}_4} du_1\right)^T,\\
\boldsymbol{b} &= (b_1,b_2,b_3,b_4)^T = \left(\oint_{\mathfrak{a}_1} du_2,\ldots,\oint_{\mathfrak{a}_4} du_2\right)^T,\\
\boldsymbol{c} &= (c_1,c_2,c_3,c_4)^T = \left(\oint_{\mathfrak{a}_1} du_3,\ldots,\oint_{\mathfrak{a}_4} du_3\right)^T,\\
\boldsymbol{d} &= (d_1,d_2,d_3,d_4)^T = \left(\oint_{\mathfrak{a}_1} du_4,\ldots,\oint_{\mathfrak{a}_4} du_4\right)^T.
\end{aligned}$$

Crucial for us is the fact that the symmetry (5.4) allows us to relate the matrices of $\mathfrak{a}$ and $\mathfrak{b}$-periods. For any contour Γ and one form ω we have that $\oint\limits_{\mathcal{R}(\Gamma)} \omega = \oint\limits_{\Gamma} \mathcal{R}^*\omega$. If $(\tilde{z},\tilde{w}) = (z,\rho w) = R(z,w)$ then, for example,

$$\mathcal{R}^*\left(du_2\right) = \mathcal{R}^*\left(\frac{d\tilde{z}}{\tilde{w}^2}\right) = \frac{dz}{\tilde{w}^2} = \frac{dz}{\rho^2 w^2} = \rho\frac{dz}{w^2}$$

leading to

$$\oint_{\mathfrak{a}_1} du_2 = \oint_{\mathcal{R}(\mathfrak{b}_1)} du_2 = \oint_{\mathfrak{b}_1} \mathcal{R}^*(du_2) = \oint_{\mathfrak{b}_1} \frac{dz}{\rho^2 w^2} = \rho\oint_{\mathfrak{b}_1} du_2.$$

We find that

$$\begin{aligned}
\mathcal{A} &= (\mathcal{A}_{ki}) = \left(\oint_{\mathfrak{a}_k} du_i\right)_{i,k=1,\ldots,4} = (\boldsymbol{x},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})\\
\mathcal{B} &= (\mathcal{B}_{ki}) = \left(\oint_{\mathfrak{b}_k} du_i\right)_{i,k=1,\ldots,4} = (\rho H\boldsymbol{x},\rho^2 H\boldsymbol{b},\rho^2 H\boldsymbol{c},\rho^2 H\boldsymbol{d}) = H\mathcal{A}\Lambda,
\end{aligned} \tag{5.12}$$

where $H = \mathrm{diag}(1,1,1,-1)$ and $\Lambda = \mathrm{diag}(\rho,\rho^2,\rho^2,\rho^2)$. This relationship between the $\mathfrak{a}$ and $\mathfrak{b}$-periods leads to various simplifications of the Riemann identities,

$$\sum_i \left(\oint_{\mathfrak{a}_i} du_k \oint_{\mathfrak{b}_i} du_l - \oint_{\mathfrak{b}_i} du_k \oint_{\mathfrak{a}_i} du_l\right) = 0.$$

For $k=1$ and $l=2,3,4$ we obtain (respectively) that

$$\boldsymbol{x}^T H\boldsymbol{b} = \boldsymbol{x}^T H\boldsymbol{c} = \boldsymbol{x}^T H\boldsymbol{d} = 0, \tag{5.13}$$

relations we shall employ throughout the paper.

Given $\mathcal{A}$ and $\mathcal{B}$ we now construct the Riemann period matrix which belongs to the Siegel upper half-space $\mathbb{S}^4$ of degree 4. If one works with canonically $\mathfrak{a}$-normalized differentials the period matrix (in our conventions) is $\tau_{\mathfrak{a}} = \mathcal{B}\mathcal{A}^{-1}$ while for canonically $\mathfrak{b}$-normalized differentials it is $\tau_{\mathfrak{b}} = \mathcal{A}\mathcal{B}^{-1}$. Clearly $\tau_{\mathfrak{b}} = \tau_{\mathfrak{a}}^{-1}$ and we shall simply denote the period matrix by τ if neither normalization is necessary.

Proposition 5.1 (Wellstein, 1899; Matsumoto, 2000). *Let $\mathcal{C}$ be the triple covering of $\mathbb{P}^1$ with six distinct point $\lambda_1, \ldots, \lambda_6$,*

$$w^3 = \prod_{i=1}^{6}(z - \lambda_i). \tag{5.14}$$

Then the Riemann period matrix is of the form

$$\tau_{\mathfrak{b}} = \rho\left(H - (1-\rho)\frac{\boldsymbol{x}\boldsymbol{x}^T}{\boldsymbol{x}^T H \boldsymbol{x}}\right), \tag{5.15}$$

where $H = \mathrm{diag}(1,1,1,-1)$. Then $\tau_{\mathfrak{b}}$ is positive definite if and only if

$$\bar{\boldsymbol{x}}^T H \boldsymbol{x} < 0. \tag{5.16}$$

Both Wellstein and Matsumoto give broadly similar proofs of (5.15) and we shall present another variant as we need to use an identity established in the proof later in the text.

Proof. From (5.13) we see that we have

$$\mathcal{A}^T H \boldsymbol{x} = (\Delta, 0, 0, 0)^T, \qquad \Delta := \boldsymbol{x}^T H \boldsymbol{x}.$$

We know that $\mathcal{A}$ is nonsingular and consequently $\boldsymbol{x} \neq 0$ and $\Delta \neq 0$. Now $H\boldsymbol{x} = \mathcal{A}^{T\,-1}(\Delta, 0, 0, 0)^T$ which gives

$$(H\boldsymbol{x})_\mu = \mathcal{A}^{-1}_{1\mu}\Delta. \tag{5.17}$$

Now from (5.12) we see that

$$\mathcal{B}\mathcal{A}^{-1} = \rho^2 H + (\rho - \rho^2) H(\boldsymbol{x}, 0, 0, 0)\mathcal{A}^{-1}.$$

From (5.17) we obtain

$$H(\boldsymbol{x}, 0, 0, 0)\mathcal{A}^{-1} = \frac{1}{\Delta} H\boldsymbol{x}\boldsymbol{x}^T H$$

and therefore

$$\mathcal{B}\mathcal{A}^{-1} = \rho^2 H + \frac{(\rho - \rho^2)}{\Delta} H\boldsymbol{x}\boldsymbol{x}^T H.$$

Finally one sees that

$$\left[\rho^2 H + \frac{(\rho-\rho^2)}{\Delta} H\boldsymbol{x}\boldsymbol{x}^T H\right]\left[\rho H - \frac{(\rho-\rho^2)}{\Delta}\boldsymbol{x}\boldsymbol{x}^T\right] = 1,$$

whence the result (5.15) follows for $\tau_{\mathfrak{b}} = \mathcal{A}\mathcal{B}^{-1}$. The remaining constraint arises by requiring $\mathrm{Im}\,\tau$ to be positive definite. We note that (5.16) ensures that both $\boldsymbol{x} \neq 0$ and $\Delta \neq 0$. □

The branch points can be expressed in terms of θ-constants. Following Matsumoto [Mat01] we introduce the set of characteristics

$$(\boldsymbol{a}, \boldsymbol{b}), \qquad \boldsymbol{b} = -\boldsymbol{a}H, \quad a_i \in \left\{\frac{1}{6}, \frac{3}{6}, \frac{5}{6}\right\} \tag{5.18}$$

and denote $\theta_{\boldsymbol{a}, -H\boldsymbol{b}}(\tau) = \theta\{6\boldsymbol{a}\}(\tau)$ (see Appendix A for out theta function conventions). The characteristics (5.18) are classified in [Mat01] by the representations of the braid group. Further, the period matrix determines the branch points as follows.

Proposition 5.2 (Diez 1991, Matsumoto 2000). *Let $\tau_{\mathfrak{b}}$ be the period matrix of (5.7) given in Proposition 5.1. Then*

$$\Lambda_1 = \left(\frac{\theta\{3,3,3,5\}}{\theta\{1,1,3,3\}}\right)^3, \quad \Lambda_2 = -\left(\frac{\theta\{1,5,3,3\}}{\theta\{1,1,5,5\}}\right)^3, \quad \Lambda_3 = -\left(\frac{\theta\{1,1,3,3\}}{\theta\{5,1,1,1\}}\right)^3. \tag{5.19}$$

These results have the following significance for our construction of monopoles. First we observe that the period matrix is invariant under $\mathbf{x} \to \lambda \mathbf{x}$. Thus to our surface we may associate a point $[x_1 : x_2 : x_3 : x_4] \in \mathbb{B}^3 = \{\mathbf{x} \in \mathbb{P}^3 \,|\, \boldsymbol{x}^T H \boldsymbol{x} < 0\} \subset \mathbb{P}^3$ and from this point we may obtain the normalized curve (5.7). It is known that a dense open subset of $\mathbb{B}^3$ arises in this way from curves with distinct roots with the complement corresponding to curves with multiple roots. Correspondingly, if we choose a point $[x_1 : x_2 : x_3 : x_4] \in \mathbb{B}^3$ we may construct a period matrix and corresponding normalized curve.

We note that with $d\boldsymbol{u} = (du_1, \ldots, du_4)$ then

$$\tau^*(d\boldsymbol{u}) = \overline{d\boldsymbol{u}} \cdot T, \qquad T = \begin{pmatrix} -\kappa & 0 & 0 & 0 \\ 0 & 0 & 0 & \kappa^2 \\ 0 & 0 & -\kappa^2 & 0 \\ 0 & \kappa^2 & 0 & 0 \end{pmatrix}, \qquad \kappa = \frac{\hat{\chi}^{\frac{1}{3}}}{\bar{\hat{\chi}}^{\frac{1}{3}}},$$

and so we obtain

$$\oint_{\tau_*\mathfrak{c}} d\boldsymbol{u} = \oint_{\mathfrak{c}} \tau^*(d\boldsymbol{u}) = \oint_{\mathfrak{c}} \overline{d\boldsymbol{u}} \cdot T = \overline{\left(\oint_{\mathfrak{c}} d\boldsymbol{u}\right)} \cdot T = \overline{\left(6\hat{\chi}^{\frac{1}{3}}\,(1\ 0\ 0\ 0)\right)} \cdot T = -6\hat{\chi}^{\frac{1}{3}}\,(1\ 0\ 0\ 0) = \oint_{-\mathfrak{c}} d\boldsymbol{u}$$

and as a consequence corollary (2.4) of Houghton, Manton and Romão, $\tau_*\mathfrak{c} = -\mathfrak{c}$. More generally, let us write for an arbitrary cycle

$$\gamma = \mathbf{p} \cdot \mathfrak{a} + \mathbf{q} \cdot \mathfrak{b} = \begin{pmatrix} \mathbf{p} & \mathbf{q} \end{pmatrix} \begin{pmatrix} \mathfrak{a} \\ \mathfrak{b} \end{pmatrix}, \qquad \tau(\gamma) = \begin{pmatrix} \mathbf{p} & \mathbf{q} \end{pmatrix} \mathcal{M} \begin{pmatrix} \mathfrak{a} \\ \mathfrak{b} \end{pmatrix}.$$

Then the equality

$$\oint_{\tau(\gamma)} d\boldsymbol{u} = \oint_{\gamma} \tau^*(d\boldsymbol{u}) = \oint_{\gamma} \overline{d\boldsymbol{u}} \cdot T = \overline{\left(\oint_{\gamma} d\boldsymbol{u}\right)} \cdot T$$

leads to the equation

$$\begin{pmatrix} \mathbf{p} & \mathbf{q} \end{pmatrix} \mathcal{M} \begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix} = \begin{pmatrix} \mathbf{p} & \mathbf{q} \end{pmatrix} \overline{\begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix}} \cdot T.$$

We have then that the matrix $\mathcal{M}$ representing the involution τ on homology and Ercolani-Sinha vector satisfy

$$\mathcal{M}^2 = \mathrm{Id}, \quad \mathcal{M} \begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix} = \overline{\begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix}} \cdot \mathrm{T}, \quad \mathbf{U}\mathcal{M} = \begin{pmatrix} \mathbf{n} & \mathbf{m} \end{pmatrix} \mathcal{M} = -\begin{pmatrix} \mathbf{n} & \mathbf{m} \end{pmatrix}. \tag{5.20}$$

A calculation employing the algorithm of Tretkoff and Tretkoff [TT84] to describe the homology basis generators and relations, together with some analytic continuation of the paths associated to our chosen homology cycles (with the sheet conventions described later in the text), yields that for our curve

$$\mathcal{M} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & -1 & 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 1 & -1 & 0 & -1 \\ 1 & -1 & 0 & 2 & 0 & -1 & 1 & -2 \end{pmatrix}.$$

The matrix $\mathcal{M}$ is not symplectic but satisfies

$$\mathcal{M}J\mathcal{M}^T = -J,$$

where J is the standard symplectic form. (The minus sign appears here because of the reversal of orientation under the antiholomorphic involution.)

5.3. **The vector $\widetilde{\boldsymbol{K}}$ and $\int_{\infty_i}^{\infty_j} \boldsymbol{v}$.** We shall now describe the vector $\widetilde{\boldsymbol{K}}$ and various related results, including the quantity $\boldsymbol{\phi}(\infty_i) - \boldsymbol{\phi}(\infty_j)$.

First let us record some elementary facts about our curve. For ease in defining various divisors of the curve (5.3) let $\infty_{1,2,3}$ be the three points over infinity and $Q_i = (\lambda_i, 0)$ $(i = 1, \dots 6)$ be the branch points. Then

$$\mathrm{Div}(z-\lambda_i) = \frac{Q_i^3}{\infty_1\infty_2\infty_3}, \quad \mathrm{Div}(w) = \frac{\prod_{i=1}^6 Q_i}{(\infty_1\infty_2\infty_3)^2}, \qquad \mathrm{Div}(dz) = \frac{(\prod_{i=1}^6 Q_i)^2}{(\infty_1\infty_2\infty_3)^2},$$

$$\mathrm{Div}\left(\frac{dz}{w}\right) = \textstyle\prod_{i=1}^6 Q_i, \qquad \mathrm{Div}\left(\frac{dz}{w^2}\right) = (\infty_1\infty_2\infty_3)^2, \quad \mathrm{Div}\left(\frac{(z-\lambda_i)dz}{w^2}\right) = Q_i^3\infty_1\infty_2\infty_3,$$

$$\mathrm{Div}\left(\frac{(z-\lambda_i)^2dz}{w^2}\right) = Q_i^6.$$

Consideration of the function $(z-\lambda_i)/(z-\lambda_j)$ shows that $3\int_{Q_j}^{Q_i} \boldsymbol{v} \in \Lambda$. The order of vanishing of the differentials $d(z-\lambda_i)/w^2$, $d(z-\lambda_i)/w$, $(z-\lambda_i)d(z-\lambda_i)/w^2$ and $(z-\lambda_i)d(z-\lambda_i)/w^2$ at the point Q_i are found to be 0, 1, 3 and 6 respectively, which means that the gap sequence at Q_i is 1, 2, 4 and 7. From this we deduce that the index of speciality of the divisor Q_i^3 is $i(Q_i^3) = 2$. Because the genus four curve $\mathcal{C}$ has the function w of degree 3 then $\mathcal{C}$ is not hyperelliptic. The function $1/(z-\lambda_i)$ has divisor $\mathcal{U}/D$, with $\mathcal{U} = \infty_1\infty_2\infty_3$ and $D = Q_i^3$ such that D^2 is canonical. This means that any other function of degree 3 on $\mathcal{C}$ is a fractional linear transformation of w and that Θ_{singular} consists of precisely one point which is of order 2 in $\mathrm{Jac}(\mathcal{C})$ [FK80, III.8.7, VII.1.6]. The vector of Riemann constants $\boldsymbol{K}_{Q_i}$ is a point of order 2 in $\mathrm{Jac}(\mathcal{C})$ because Q_i^6 is canonical [FK80, VI.3.6]. Let us fix Q_1 to be our base point. Then as $\boldsymbol{K}_{Q_1} = \boldsymbol{\phi}_{Q_1}(Q_1^3) + \boldsymbol{K}_{Q_1}$ we have that $\boldsymbol{K}_{Q_1} \in \Theta$. Because $i(Q_1^3) = 2$ we may identify $\boldsymbol{K}_{Q_1}$ as the unique point in Θ_{singular}. We may further identify $\boldsymbol{K}_{Q_1}$ as the unique even theta characteristic belonging to Θ.

With Q_1 as our base point $\boldsymbol{\phi}\left(\sum_k \infty_k\right)$ corresponds to the image under the Abel map of the divisor of the function $1/(z-\lambda_1)$, and so vanishes (modulo the period lattice). Thus for our curve $\widetilde{\boldsymbol{K}} = \boldsymbol{K}_{Q_1} + \boldsymbol{\phi}\left(\sum_k \infty_k\right) = \boldsymbol{K}_{Q_1} = \Theta_{\text{singular}}$ is the unique even theta characteristic. The point $\boldsymbol{K}_{Q_1}$ may be constructed several ways: directly, using the formula (A.6) of the Appendix (the evaluation of the integrals of normalised holomorphic differentials between branch points is described in Appendix B); by enumeration we may find which of the 136 even theta characteristics $\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}$ leads to the vanishing of $\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(z;\tau)$; using a monodromy argument of Matsumoto [Mat01]. One finds that the relevant half period is $\frac{1}{2}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\end{bmatrix}$.

The analysis of the previous paragraph, together with (3.7), tells us that $\boldsymbol{U}$ must also be an even theta characteristic.

Again using that $\sum_k \infty_k \sim_l 0$ we have that $\infty_i - \infty_j \sim_l 2\infty_i + \infty_k$ (with i, j, k distinct) and so $\theta(\boldsymbol{\phi}(\infty_j) - \boldsymbol{\phi}(\infty_i) - \widetilde{\boldsymbol{K}}) = \theta(\boldsymbol{\phi}(2\infty_i + \infty_k) + \boldsymbol{K}) = 0$. One sees from the above divisors (in particular $\mathrm{Div}(dz/w^2)$) that $\mathrm{Dim}\, H^0(\mathcal{C}, L_{2\infty_i+\infty_k}) = i(2\infty_i + \infty_k) = 1$. Thus $\theta(w+\boldsymbol{\phi}(\infty_j)-\boldsymbol{\phi}(\infty_i)-\widetilde{\boldsymbol{K}}$ and $\theta(w-\widetilde{\boldsymbol{K}})$ have order of vanishing differing by one for (generic) $w \to 0$.

5.4. **Calculating $\nu_i - \nu_j$.** From the results of the previous section we see that

$$\mathrm{Div}\left(\frac{z^4 dz}{w^2}\right) = \frac{(0_1 0_2 0_3)^4}{(\infty_1 \infty_2 \infty_3)^2}.$$

This has precisely the same divisor of poles as γ_∞ and we will use this to represent γ_∞. It is convenient to introduce the (meromorphic) differential

$$dr_1(P) = \frac{z^4 dz}{3w^2},$$

the factor of three here being introduced to give the pairing

$$\sum_{s=1}^{3} \mathrm{Res}_{P=\infty_s} dr_1(P) \int_{P_0}^{P} du_1(P') = 1.$$

We may therefore write

$$\gamma_\infty(P) = -3dr_1(P) + \sum_{i=1}^{4} c_i v_i(P). \tag{5.21}$$

The constants c_i are found from the condition of normalisation

$$\oint_{\mathfrak{a}_k} \gamma_\infty(P) = 0 \quad \Longleftrightarrow c_k = 3 \oint_{\mathfrak{a}_k} dr_1(P) \equiv 3y_k, \quad k = 1, \ldots, 4,$$

where we have defined the vector of $\mathfrak{a}$-periods $\boldsymbol{y}^T = \left(\oint_{\mathfrak{a}_1} dr_1(P), \ldots, \oint_{\mathfrak{a}_4} dr_1(P)\right)$. The vector of $\mathfrak{b}$-periods of dr_1 is found to be $\rho^2 H\boldsymbol{y}$. The pairing with du_1 then yields the Legendre relation

$$\boldsymbol{y}\cdot H\boldsymbol{x} = -\frac{2\pi}{\sqrt{3}}. \tag{5.22}$$

Now the $\mathfrak{b}$-periods of the differential γ_∞ give the Ercolani-Sinha vector. Using (5.21) we then obtain the equality

$$-3(\rho^2 H - \tau_{\mathfrak{a}})\boldsymbol{y} = \pi\imath\boldsymbol{n} + \pi\imath\tau\boldsymbol{m}. \tag{5.23}$$

Finally, using (5.21), we may write

$$\nu_i - \nu_j = 3\boldsymbol{y} \cdot \int_{\infty_j}^{\infty_i} \boldsymbol{v} + \int_{\infty_j}^{\infty_i} \left[d\left(\frac{w}{z}\right) - 3dr_1\right]. \tag{5.24}$$

6. Solving the Ercolani-Sinha constraints

We shall now describe how to solve the Ercolani-Sinha constraints for the spectral curve (5.1). This reduces to constraints just on the four periods $\boldsymbol{x}$. Later we shall restrict attention to the curves (1.2), which has the effect of reducing the number of integrals to be evaluated to two and consequently simplifies our present analysis.

We shall work with the Ercolani-Sinha constraints in the form (2.44). Let the holomorphic differentials be ordered as in (5.9). Then there exist two integer 4-vectors $\boldsymbol{n}, \boldsymbol{m} \in \mathbb{Z}^4$ and values of the parameters $\lambda_1, \ldots, \lambda_6$ and χ such that

$$\boldsymbol{n}^T \mathcal{A} + \boldsymbol{m}^T \mathcal{B} = \nu(1, 0, 0, 0). \tag{6.1}$$

Here ν depends on normalizations. For us this will be

$$\nu = 6\hat{\chi}^{\frac{1}{3}}.$$

To see this observe that (2.44) requires that

$$-2\delta_{1k} = \oint_{\boldsymbol{n}\cdot\mathfrak{a}+\boldsymbol{m}\cdot\mathfrak{b}} \Omega^{(k)} \quad \text{for the differentials } \Omega^{(1)} = \frac{\eta^{n-2}d\zeta}{\frac{\partial P}{\partial \eta}} = \frac{d\zeta}{n\eta}, \Omega^{(2)} = \frac{\eta^{n-3}d\zeta}{\frac{\partial P}{\partial \eta}}, \ldots.$$

In the parameterisation (5.3) we are using we have that

$$x_i = \oint_{\mathfrak{a}_i} \frac{dz}{w} = \oint_{\mathfrak{a}_i} \frac{d\zeta}{-\hat{\chi}^{-\frac{1}{3}}\eta} = -3\hat{\chi}^{\frac{1}{3}} \oint_{\mathfrak{a}_i} \Omega^{(1)}.$$

We wish

$$-2 = \oint_{\boldsymbol{n}\cdot\mathfrak{a}+\boldsymbol{m}\cdot\mathfrak{b}} \Omega^{(1)} = \frac{-1}{3\hat{\chi}^{\frac{1}{3}}}(\boldsymbol{n}.\boldsymbol{x} + \rho\boldsymbol{m}.H.\boldsymbol{x})$$

and so

$$\boldsymbol{n}.\boldsymbol{x} + \rho\boldsymbol{m}.H.\boldsymbol{x} = \nu, \tag{6.2}$$

with the value of ν stated. Consideration of the other differentials then yields (6.1), transcendental constraints on the curve $\mathcal{C}$. These constraints may be solved using the following result.

Proposition 6.1. *The Ercolani-Sinha constraints (6.1) are satisfied for the curve (5.1) if and only if*

$$\boldsymbol{x} = \xi(H\boldsymbol{n} + \rho^2\boldsymbol{m}), \tag{6.3}$$

where

$$\xi = \frac{\nu}{[\boldsymbol{n}.H\boldsymbol{n} - \boldsymbol{m}.\boldsymbol{n} + \boldsymbol{m}.H\boldsymbol{m}]} = \frac{6\hat{\chi}^{\frac{1}{3}}}{[\boldsymbol{n}.H\boldsymbol{n} - \boldsymbol{m}.\boldsymbol{n} + \boldsymbol{m}.H\boldsymbol{m}]}. \tag{6.4}$$

Proof. Rewriting (6.1) we have that

$$\boldsymbol{n}^T + \boldsymbol{m}^T\mathcal{B}\mathcal{A}^{-1} = \nu(1,0,0,0)\mathcal{A}^{-1} = \nu\mathcal{A}^{-1}_{1\mu}.$$

Upon using (5.17) we obtain

$$\boldsymbol{n}^T + \boldsymbol{m}^T\mathcal{B}\mathcal{A}^{-1} = \frac{\nu}{\Delta}\boldsymbol{x}^T H.$$

Therefore

$$\begin{aligned}\boldsymbol{x} &= \frac{\Delta}{\nu}(H\boldsymbol{n} + H(\mathcal{B}\mathcal{A}^{-1})^T\boldsymbol{m}) \\ &= \frac{\Delta}{\nu}(H\boldsymbol{n} + \rho^2\boldsymbol{m} + (\frac{\rho-\rho^2}{\Delta})\boldsymbol{x}\,\boldsymbol{x}^T H\boldsymbol{m})\end{aligned}$$

upon using that the period matrix is symmetric and our earlier expression for $\mathcal{B}\mathcal{A}^{-1}$. Rearranging now gives us that

$$(1 + \frac{\rho^2-\rho}{\nu}\boldsymbol{x}^T H\boldsymbol{m})\boldsymbol{x} = \frac{\Delta}{\nu}(H\boldsymbol{n} + \rho^2\boldsymbol{m}) \tag{6.5}$$

and so we have established (6.3) where

$$\xi = \frac{\Delta}{\nu}(1 + \frac{\rho^2-\rho}{\nu}\,\boldsymbol{x}.H\boldsymbol{m})^{-1}. \tag{6.6}$$

There are several constraints. First, the Ercolani-Sinha condition (6.2) is that

$$[\boldsymbol{n}^T + \rho \boldsymbol{m}^T H]\xi[H\boldsymbol{n} + \rho^2 \boldsymbol{m}] = \nu$$

and consequently

$$[\boldsymbol{n}.H\boldsymbol{n} - \boldsymbol{m}.\boldsymbol{n} + \boldsymbol{m}.H\boldsymbol{m}]\xi = \nu = 6\hat{\chi}^{\frac{1}{3}}, \tag{6.7}$$

thus establishing (6.4). We remark that if $\hat{\chi}$ is real, then $\hat{\chi}^{\frac{1}{3}}$ may be chosen real and hence ξ is real. We observe that (6.4) and (6.6) are consistent with

$$\Delta = \boldsymbol{x}^T H\boldsymbol{x} = \xi^2(\boldsymbol{n}^T H + \rho^2 \boldsymbol{m}^T)H(H\boldsymbol{n} + \rho^2\boldsymbol{m}) = \xi^2[\boldsymbol{n}.H\boldsymbol{n} + 2\rho^2\boldsymbol{m}.\boldsymbol{n} + \rho\boldsymbol{m}.H\boldsymbol{m}].$$

A further consistency check is given by (5.23). Using the form of the period matrix, the Legendre relation (5.22) and the proposition (with $\nu = -6$) we obtain (5.23). □

At this stage we have reduced the Ercolani-Sinha constraints to one of imposing the four constraints (6.3) on the periods x_k. In particular this means we must solve

$$\frac{x_1}{n_1 + \rho^2 m_1} = \frac{x_2}{n_2 + \rho^2 m_2} = \frac{x_3}{n_3 + \rho^2 m_3} = \frac{x_4}{-n_4 + \rho^2 m_4} = \xi, \tag{6.8}$$

which means $x_i/x_j \in \mathbb{Q}[\rho]$. Further we have from the conditions (5.16) that

$$\frac{\bar{\boldsymbol{x}}^T H\boldsymbol{x}}{|\xi|^2} = [\boldsymbol{n}.H\boldsymbol{n} - \boldsymbol{m}.\boldsymbol{n} + \boldsymbol{m}.H\boldsymbol{m}] = \sum_{i=1}^{3}(n_i^2 - n_i m_i + m_i^2) - n_4^2 - m_4^2 - m_4 n_4 < 0. \tag{6.9}$$

Our result admits another interpretation. Thus far we have assumed we have been given an appropriate curve and sought to satisfy the Ercolani-Sinha constraints. Alternatively we may start with a curve satisfying (most of) the Ercolani-Sinha constraints and seek one satisfying the reality constraints (and any remaining Ercolani-Sinha constraints). How does this progress? First note that the period matrix (5.15) for a curve satisfying (6.8) is independent of ξ: it is determined wholly in terms of the Ercolani-Sinha vector. Let us then start with a primitive vector $\mathbf{U} = (\mathbf{n}, \mathbf{m})$ satisfying the hyperboloid condition (6.9) and lemma 3.1. From this we construct a period matrix and then, via Proposition 5.2, a normalized curve (5.7). Now we must address whether the curve has the correct reality properties. For this we must show that there exists a Möbius transformation of the set $S = \{0, 1, \infty, \Lambda_1, \Lambda_2, \Lambda_3\}$ to one of the form $H = \{\alpha_j, -1/\overline{\alpha}_j\}_{j=1}^3$. We will show below that this question may be answered, with the roots α_i being determined up to an overall rotation. At this stage we have (using the rotational freedom) a curve of the form

$$W^3 = Z(Z-a)Z(Z+\frac{1}{a})(Z-w)(Z+\frac{1}{\overline{w}}), \qquad a \in \mathbb{R},\ w \in \mathbb{C}.$$

To reconstruct a monopole curve we need a normalization $\hat{\chi} = \chi_3 \left[\frac{\overline{w}}{w}\right]^{1/2}$. This is encoded in ξ, which has not appeared thus far. To calculate the normalization we must calculate a period. Then using (6.8) and (6.4) we determine $\hat{\chi}$. This is a constraint. For a consistent monopole curve we require

$$\arg(\xi) = \arg\left[\frac{\overline{w}}{w}\right]^{1/6}.$$

Of course, to complete the construction we need to check there are no roots of the theta function in $[-1, 1]$. Although the procedure outlined involves several transcendental calculations it is numerically feasible and gives a means of constructing putative monopole curves.

To conclude we state when there exists a Möbius transformation of the set $S = \{0, 1, \infty, \Lambda_1, \Lambda_2, \Lambda_3\}$ to one of the form $H = \{\alpha_j, -1/\overline{\alpha}_j\}_{j=1}^3$. For simplicity we give the case of distinct roots:

Theorem 6.2. *The roots* $S = \{0, 1, \infty, \Lambda_1, \Lambda_2, \Lambda_3\}$ *are Möbius equivalent to* $H = \{\alpha_j, -1/\overline{\alpha}_j\}_{j=1}^3$ *if and only if*

(1) *If just one of the roots, say* Λ_1*, is real and*
- $\Lambda_1 < 0$ *then* $\Lambda_2\overline{\Lambda}_3 = \Lambda_1$,
- $0 < \Lambda_1 < 1$ *then* $\frac{\Lambda_2}{\Lambda_2-1}\overline{\frac{\Lambda_3}{\Lambda_3-1}} = \frac{\Lambda_1}{\Lambda_1-1}$,
- $1 < \Lambda_1$ *then* $(1-\Lambda_2)\overline{(1-\Lambda_3)} = 1 - \Lambda_1$.

If all three roots are real then, up to relabelling, one of the above must hold.

(2) *All three roots are complex and, up to relabelling,*

$$0 < \Lambda_1\overline{\Lambda}_2 \in \mathbb{R}, \qquad 1 < \frac{\Lambda_1}{\Lambda_2}, \qquad \Lambda_3 = \Lambda_2\,\frac{1-\overline{\Lambda}_1}{1-\Lambda_2}.$$

7. Symmetric 3-monopoles

In this section we shall consider the curve $\mathcal{C}$ specialized to the form

$$\eta^3 + \chi(\zeta^6 + b\zeta^3 - 1) = 0, \tag{7.1}$$

where b is a real parameter. In this case branch points are

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) = (\alpha,\ \rho^2\beta,\ \rho\alpha,\ \beta,\ \rho^2\alpha,\ \rho\beta),$$

where α and β are real,

$$\alpha = \sqrt[3]{\frac{-b+\sqrt{b^2+4}}{2}} > 0, \quad \beta = \sqrt[3]{\frac{-b-\sqrt{b^2+4}}{2}} < 0, \quad \alpha^3\beta^3 = -1.$$

Here $\chi = \hat{\chi}$ is real and we choose our branches so that $\hat{\chi}^{\frac{1}{3}}$ is also real.

The effect of choosing such a symmetric curve will be to reduce the four period integrals x_i to two independent integrals. The tetrahedrally symmetric monopole is in the class (7.1). We note that a general rotation will alter the form of $a_3(\zeta)$. Thus the dimension of the moduli space is reduced from three by the 3 degrees of freedom of the rotations yielding a discrete space of solutions. We are seeking then a discrete family of spectral curves.

We shall begin by calculating the period integrals, and then imposing the Ercolani-Sinha constraints. We shall also consider the geometry of the curves (7.1).

7.1. The period integrals.

In terms of our Wellstein parameterization we are working with

$$w^3 = z^6 + bz^3 - 1 = (z^3 - \alpha^3)(z^3 + \frac{1}{\alpha^3})$$

$(1/\alpha^3 = -\beta^3 = (b+\sqrt{b^2+4})/2)$. We choose the first sheet so that $w = \sqrt[3]{(z^3-\alpha^3)(z^3+1/\alpha^3)}$ is negative and real on the real z-axis between the branch points $(-1/\alpha, \alpha)$.

Introduce integrals computed on the first sheet

$$\begin{aligned}\mathcal{I}_1(\alpha) &= \int_0^\alpha \frac{dz}{w} = -\frac{2\pi\sqrt{3}\alpha}{9}\,{}_2F_1\left(\frac{1}{3}, \frac{1}{3}; 1; -\alpha^6\right),\\ \mathcal{J}_1(\alpha) &= \int_0^\beta \frac{dz}{w} = \frac{2\pi\sqrt{3}}{9\alpha}\,{}_2F_1\left(\frac{1}{3}, \frac{1}{3}; 1; -\alpha^{-6}\right).\end{aligned} \tag{7.2}$$

Here ${}_2F_1(a,b;c;z)$ is the standard Gauss hypergeometric function and we have, for example, evaluated the first integral using the substitution $z=\alpha t^{1/3}$ and our specification of the first sheet. We also have that

$$\int\limits_0^{\rho^k\alpha}\frac{dz}{w}=\rho^k\mathcal{I}_1(\alpha),\qquad \int\limits_0^{\rho^k\beta}\frac{dz}{w}=\rho^k\mathcal{J}_1(\alpha),\quad k=1,2.$$

Our aim is to express the periods for our homology basis (5.10) in terms of the integrals $\mathcal{I}_1(\alpha)$ and $\mathcal{J}_1(\alpha)$. Consider for example

$$\begin{aligned}
x_1&=\oint_{\mathfrak{a}_1}du_1=\int_{\gamma_1(\lambda_1,\lambda_2)}\frac{dz}{w}+\int_{\gamma_2(\lambda_2,\lambda_1)}\frac{dz}{w}=\int_{\lambda_1}^{\lambda_2}\frac{dz}{w}-\rho^2\int_{\lambda_1}^{\lambda_2}\frac{dz}{w}\\
&=(1-\rho^2)\int_{\alpha}^{\rho^2\beta}\frac{dz}{w}=(1-\rho^2)\left[-\mathcal{I}(\alpha)+\int\limits_0^{\rho^2\beta}\frac{dz}{w}\right]=(1-\rho^2)\left[-\mathcal{I}_1(\alpha)+\rho^2\mathcal{J}_1(\alpha)\right]\\
&=-2\mathcal{I}_1(\alpha)-\mathcal{J}_1(\alpha)-\rho\left[\mathcal{I}_1(\alpha)+2\mathcal{J}_1(\alpha)\right].
\end{aligned}$$

Here we have used that on the second sheet $w_2=\rho w_1$ to obtain the last expression of the first line, and also that $1+\rho+\rho^2=0$ to obtain the final expression. Similarly we find (upon dropping the α dependence from $\mathcal{I}_1$ and $\mathcal{J}_1$ when no confusion arises) that

$$\begin{array}{llll}
x_1&=-(2\mathcal{J}_1+\mathcal{I}_1)\rho-2\mathcal{I}_1-\mathcal{J}_1, & x_2&=(\mathcal{J}_1-\mathcal{I}_1)\rho+\mathcal{I}_1+2\mathcal{J}_1,\\
x_3&=(\mathcal{J}_1+2\mathcal{I}_1)\rho-\mathcal{J}_1+\mathcal{I}_1, & x_4&=3(\mathcal{J}_1-\mathcal{I}_1)\rho+3\mathcal{J}_1.
\end{array}\tag{7.3}$$

Note that

$$x_2=\rho x_1,\quad x_3=\rho^2x_1.\tag{7.4}$$

7.2. **The Ercolani-Sinha constraints.** We next reduce the Ercolani-Sinha constraints to a number theoretic one. Using (6.3) and (7.3) we may rewrite the constraints as

$$x_i=\xi(\epsilon_in_i+\rho^2m_i)=(\alpha_i\mathcal{I}_1+\beta_i\mathcal{J}_1)+(\gamma_i\mathcal{I}_1+\delta_i\mathcal{J}_1)\rho.\tag{7.5}$$

We may solve for the various n_i,m_i in terms of n_1,m_1 as follows. Set

$$C_i=\begin{pmatrix}\epsilon_i&-1\\0&-1\end{pmatrix},\ D_i=\begin{pmatrix}\alpha_i&\beta_i\\\gamma_i&\delta_i\end{pmatrix},\ \hat{\mathcal{I}}=\mathcal{I}_1/\xi,\ \hat{\mathcal{J}}=\mathcal{J}_1/\xi.$$

Then (7.5) may be rewritten as

$$C_i\begin{pmatrix}n_i\\m_i\end{pmatrix}=D_i\begin{pmatrix}\hat{\mathcal{I}}\\\hat{\mathcal{J}}\end{pmatrix}$$

giving

$$\begin{pmatrix}n_i\\m_i\end{pmatrix}=C_i^{-1}D_i\begin{pmatrix}\hat{\mathcal{I}}\\\hat{\mathcal{J}}\end{pmatrix}=C_i^{-1}D_iD_1^{-1}C_1\begin{pmatrix}n_1\\m_1\end{pmatrix}.$$

This yields that the vectors $\boldsymbol{n}$, $\boldsymbol{m}$ are of the form

$$\boldsymbol{n}=\begin{pmatrix}n_1\\n_2\\n_3\\n_4\end{pmatrix}=\begin{pmatrix}n_1\\m_1-n_1\\-m_1\\2n_1-m_1\end{pmatrix},\qquad \boldsymbol{m}=\begin{pmatrix}m_1\\m_2\\m_3\\m_4\end{pmatrix}=\begin{pmatrix}m_1\\-n_1\\n_1-m_1\\-3n_1\end{pmatrix}.\tag{7.6}$$

One may verify that for vectors of this form then $(\boldsymbol{n},\boldsymbol{m})\mathcal{M}=-(\boldsymbol{n},\boldsymbol{m})$ as required by (5.20). Recall further that $(\boldsymbol{n},\boldsymbol{m})$ is to be a primitive vector: that is one for which the greatest common divisor of the components is 1, and hence a generator of $\mathbb{Z}^8$. We see that $(\boldsymbol{n}, \boldsymbol{m})$ is primitive if and only if

$$(n_1, m_1) = 1. \tag{7.7}$$

From

$$\begin{pmatrix} \hat{\mathcal{I}} \\ \hat{\mathcal{J}} \end{pmatrix} = D_i^{-1} C_i \begin{pmatrix} n_i \\ m_i \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} n_1 \\ m_1 \end{pmatrix}$$

we obtain

$$\begin{aligned} \frac{\hat{\mathcal{I}}}{\hat{\mathcal{J}}} &= \frac{\mathcal{I}}{\mathcal{J}} = \frac{m_1 - 2n_1}{m_1 + n_1}, \\ \mathcal{I}_1 &= \frac{m_1 - 2n_1}{3}\,\xi &&= -\frac{2\pi}{3\sqrt{3}}\,\alpha\ {}_2F_1(\tfrac{1}{3}, \tfrac{1}{3}; 1, -\alpha^6), \\ \mathcal{J}_1 &= \frac{m_1 + n_1}{3}\,\xi &&= \frac{2\pi}{3\sqrt{3}}\,\frac{1}{\alpha}\ {}_2F_1(\tfrac{1}{3}, \tfrac{1}{3}; 1, -\alpha^{-6}). \end{aligned}$$

Now given (7.6) we find that

$$\boldsymbol{n}.H\boldsymbol{n} - \boldsymbol{m}.\boldsymbol{n} + \boldsymbol{m}.H\boldsymbol{m} = 2(m_1 + n_1)(m_1 - 2n_1)$$

and so the constraint (5.16) is satisfied if

$$\bar{\boldsymbol{x}}^T H\boldsymbol{x} = \xi^2[\boldsymbol{n}.H\boldsymbol{n} - \boldsymbol{m}.\boldsymbol{n} + \boldsymbol{m}.H\boldsymbol{m}] = 2\xi^2(m_1 + n_1)(m_1 - 2n_1) < 0.$$

This requires

$$(m_1 + n_1)(m_1 - 2n_1) < 0. \tag{7.8}$$

In particular we have from (6.7) that

$$\xi = \frac{3\chi^{\frac{1}{3}}}{(n_1 + m_1)(m_1 - 2n_1)}.$$

Thus we have to solve

$$\begin{aligned} \mathcal{I}_1 &= \frac{\chi^{\frac{1}{3}}}{n_1 + m_1} = -\frac{2\pi}{3\sqrt{3}}\,\alpha\ {}_2F_1(\frac{1}{3}, \frac{1}{3}; 1, -\alpha^6), \\ \mathcal{J}_1 &= \frac{\chi^{\frac{1}{3}}}{m_1 - 2n_1} = \frac{2\pi}{3\sqrt{3}}\,\frac{1}{\alpha}\ {}_2F_1(\frac{1}{3}, \frac{1}{3}; 1, -\alpha^{-6}). \end{aligned} \tag{7.9}$$

Using the identity

$${}_2F_1(\frac{1}{3}, \frac{1}{3}; 1, x) = (1-x)^{-1/3}\,{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1, \frac{x}{x-1})$$

we then seek solutions of

$$\frac{\mathcal{I}_1}{\mathcal{J}_1} = \frac{m_1 - 2n_1}{m_1 + n_1} = -\frac{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1, t)}{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1, 1-t)}, \qquad t = \frac{\alpha^6}{1 + \alpha^6} = \frac{-b + \sqrt{b^2 + 4}}{2\sqrt{b^2 + 4}}.$$

From (7.8) the ratio of $\mathcal{I}_1/\mathcal{J}_1$ is negative. Consideration of the function

$$f(t) = \frac{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; t\right)}{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-t\right)}.$$

(see Figure 2 for its plot) shows that there exists unique root $t \in (0,1)$ for each value $f(t) \in (0, \infty)$ and correspondingly a unique real positive $\alpha = \sqrt[6]{t/(1-t)}$.

Bringing these results together we have established:

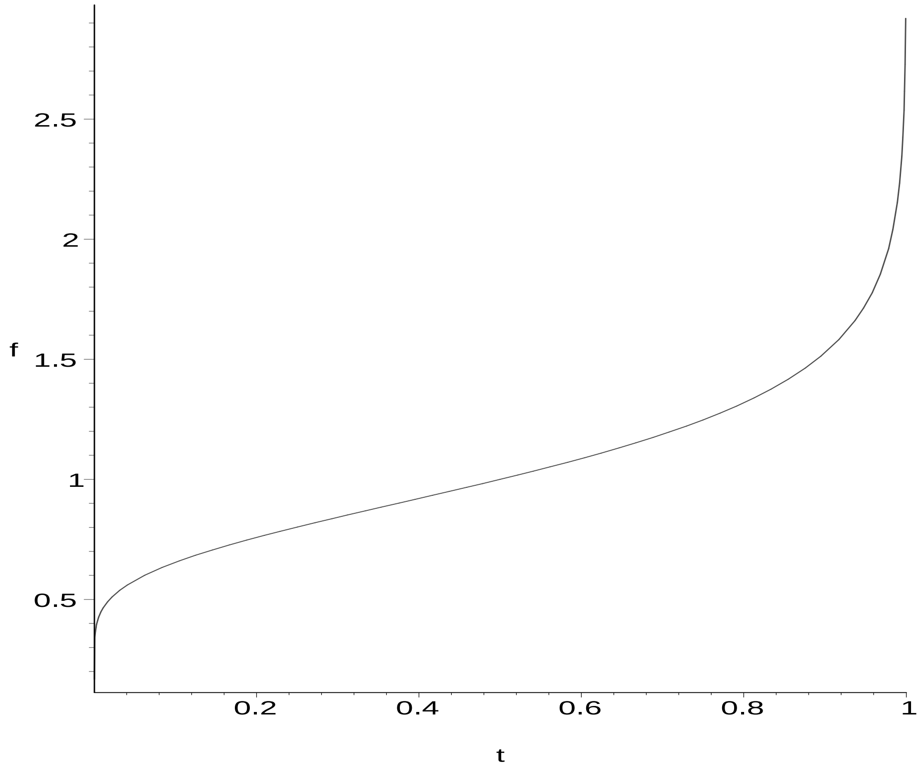


FIGURE 3. The function $f(t) = \frac{{}_2F_1(\frac{1}{3},\frac{2}{3};1,t)}{{}_2F_1(\frac{1}{3},\frac{2}{3};1,1-t)}$.

Proposition 7.1. *To each pair of relatively prime integers $(n_1, m_1) = 1$ for which*

$$(m_1 + n_1)(m_1 - 2n_1) < 0$$

we obtain a solution to the Ercolani-Sinha constraints for a curve of the form (7.1) as follows. First we solve for t, where

$$\frac{2n_1 - m_1}{m_1 + n_1} = \frac{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1, t)}{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1, 1-t)}. \tag{7.10}$$

Then

$$b = \frac{1 - 2t}{\sqrt{t(1-t)}}, \qquad t = \frac{-b + \sqrt{b^2 + 4}}{2\sqrt{b^2 + 4}}, \tag{7.11}$$

and we obtain χ from

$$\chi^{\frac{1}{3}} = -(n_1 + m_1)\,\frac{2\pi}{3\sqrt{3}}\;\frac{\alpha}{(1 + \alpha^6)^{\frac{1}{3}}}\;{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1, t) \tag{7.12}$$

with $\alpha^6 = t/(1-t)$.

7.3. **Ramanujan.** Thus far we have reduced the problem of finding an appropriate monopole curve within the class (7.1) to that of solving the transcendental equation (7.10) for which a unique solution exists. Can this ever be solved apart from numerically? Here we shall recount how a (recently proved) result of Ramanujan enables us to find solutions.

Let n be a natural number. A modular equation of degree n and signature r ($r = 2, 3, 4, 6$) is a relation between α, β of the form

$$n\,\frac{{}_2F_1(\frac{1}{r},\frac{r-1}{r};1;1-\alpha)}{{}_2F_1(\frac{1}{r},\frac{r-1}{r};1;\alpha)} = \frac{{}_2F_1(\frac{1}{r},\frac{r-1}{r};1;1-\beta)}{{}_2F_1(\frac{1}{r},\frac{r-1}{r};1;\beta)}. \tag{7.13}$$

When $r = 2$ we have the complete elliptic integral $\mathbf{K}(k) = \frac{\pi}{2}\,{}_2F_1(\frac{1}{2},\frac{1}{2};1;k^2)$ and (7.13) yields the usual modular relations. By interchanging $\alpha \leftrightarrow \beta$ we may interchange $n \leftrightarrow 1/n$. This, together with iteration of these modular equations, means we may obtain relations with n being an arbitrary rational number. Our equation (7.10) is precisely of this form for signature $r = 3$ and starting with say $\alpha = 1/2$.

Ramanujan in his second notebook presents results pertaining to these generalised modular equations and various theta function identities. For example, if $n = 2$ in signature $r = 3$ then α and β are related by

$$(\alpha\beta)^{\frac{1}{3}} + ((1-\alpha)(1-\beta))^{\frac{1}{3}} = 1. \tag{7.14}$$

He also states that (for $0 \le p < 1$)

$$(1+p+p^2)\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1,\frac{p^3(2+p)}{1+2p}\right) = \sqrt{1+2p}\,{}_2F_1\left(\frac{1}{3},\frac{2}{3};1,\frac{27p^2(1+p)^2}{4(1+p+p^2)^3}\right). \tag{7.15}$$

Ramanujan's results were derived in [BBG95] (see also [Cha98]), though some related to expansions of $1/\pi$ had been obtained earlier by J.M. and P.B. Borwein [BB87]. An account of the history and the associated theory of these equations may be found in the last volume dedicated to Ramanujan's notebooks [Ber98]. The associated theory of these modular equations presented in the accounts just cited is largely based on direct verification that appropriate expressions of hypergeometric functions satisfy the same differential equations and initial conditions and so are equal: we shall present a more geometric picture in due course.

Analogous expressions to (7.14) are known for $n = 3$, 5, 7 and 11 [Ber98, 7.13, 7.17, 7.24, 2.28 respectively]. Thus by iteration we may solve (7.10) for rational numbers whose numerator and denominator have these as their only factors. We include some examples of these in the table below. Thus to get the value 2 for the ratio $(2n_1 - m_1)/(m_1 + n_1)$ we set $\alpha = \frac{1}{2}$ in (7.14) and solve for

$$t^{\frac{1}{3}} + (1-t)^{\frac{1}{3}} = 2^{\frac{1}{3}},$$

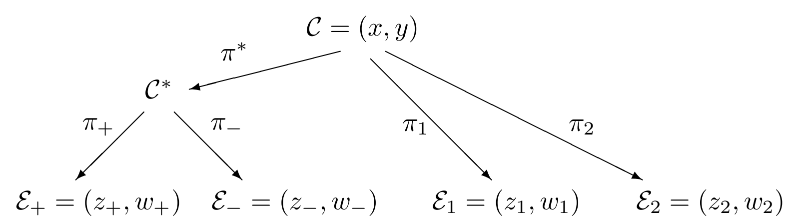
taking the larger value $t = \frac{1}{2} + \frac{5\sqrt{3}}{18}$ (the smaller value yielding the ratio $\frac{1}{2}$).

n_1	m_1	$(2n_1 - m_1)/(m_1 + n_1)$	t	b
2	1	1	$\frac{1}{2}$	0
1	0	2	$\frac{1}{2} + \frac{5\sqrt{3}}{18}$	$5\sqrt{2}$
1	1	$\frac{1}{2}$	$\frac{1}{2} - \frac{5\sqrt{3}}{18}$	$5\sqrt{2}$
4	-1	3	$(63 + 171\sqrt[3]{2} - 18\sqrt[3]{4})/250$	$(44 + 38\sqrt[3]{2} + 26\sqrt[3]{4})/3$
5	-2	4	$\frac{1}{2} + \frac{153\sqrt{3} - 99\sqrt{2}}{250}$	$9\sqrt{458 + 187\sqrt{6}}$

A theory exists then for solving (7.13) and this has been worked out for various low primes. These results enable us to reduce the Ercolani-Sinha conditions (7.10) to solving an algebraic equation.

7.4. **Covers of the sextic.** We shall now describe some geometry underlying our curves (7.1) which will lead to an understanding of the results of the last section. We shall first present a more computational approach, useful in actual calculations, and then follow this with a more invariant discussion. We begin with the observation that our curves each cover four elliptic curves.

Lemma 7.2. *The curve* $\mathcal{C} := \{(x,y)|y^3 + x^6 + bx^3 - 1 = 0\}$ *with arbitrary value of the parameter* b *is a simultaneous covering of the four elliptic curves* $\mathcal{E}_\pm$, $\mathcal{E}_{1,2}$ *as indicated in the diagram, where* $\mathcal{C}^*$ *is an intermediate genus two curve:*



The equations of the elliptic curves are

$$\mathcal{E}_\pm = \{(z_\pm, w_\pm)|w_\pm^2 = z_\pm(1-z_\pm)(1-k_\pm^2 z_\pm)\}, \tag{7.16}$$

$$\mathcal{E}_1 = \{(z_1, w_1)|z_1^3 + {w_1}^3 + 3z_1 + b = 0\}, \tag{7.17}$$

$$\mathcal{E}_2 = \{(z_2, w_2)|{w_2}^3 + {z_2}^2 + bz_2 - 1 = 0\}, \tag{7.18}$$

where the Jacobi moduli, $k_\pm$ *are given by*

$$k_\pm^2 = -\frac{\rho(\rho M \pm 1)(\rho M \mp 1)^3}{(M \pm 1)(M \mp 1)^3} \tag{7.19}$$

with

$$M = \frac{K}{L}, \quad K = (2\imath - b)^{\frac{1}{3}}, \quad L = (b^2 + 4)^{\frac{1}{6}}. \tag{7.20}$$

The covers $\pi_\pm, \pi_{1,2}$ *are given by*

$$\pi_\pm: \quad \begin{aligned} z_\pm &= -\frac{K^2 - L^2}{K^2 - \rho L^2}\,\frac{Kx - y}{\rho Kx - y}\,\frac{L^2 x - Ky}{L^2 x - K\rho y}, \\ w_\pm &= \imath\sqrt{2+\rho}\sqrt{\frac{L \pm K}{L \mp K}\frac{K^2}{L}}\,\frac{L^2 - \rho K^2}{\rho L^2 - K^2}\,\frac{(Lx \mp y)(x^6+1)}{(\rho Kx - y)^2 (L^2 x - \rho K y)^2} \end{aligned} \tag{7.21}$$

and

$$\begin{aligned} \pi_1: &\quad z_1 = x - \frac{1}{x}, \quad w_1 = \frac{y}{x}, \\ \pi_2: &\quad z_2 = x^3, \quad w_2 = y. \end{aligned}$$

The elliptic curves $\mathcal{E}_{1,2}$ *are equianharmonic* ($g_2 = 0$) *and consequently have vanishing* j*-invariant,* $j\,(\mathcal{E}_{1,2}) = 0$.

Proof. The derivation of the covers $\pi_{1,2}$ and the underlying curves is straightforward. The pullbacks $\pi_{1,2}^{-1}$ of these covers are

$$\pi_1^{-1} := \begin{cases} x = (z_1 \pm \sqrt{{z_1}^2+4})/2 \\ y = w_1(z_1 \pm \sqrt{{z_1}^2+4})/2 \end{cases} \qquad \pi_2^{-1} := \begin{cases} x = \rho\sqrt[3]{z_2} \\ y = w_2 \end{cases}$$

showing that the degrees of the cover are 2 and 3 respectively. A direct calculation putting these elliptic curves into Weierstrass form shows $g_2 = 0$ and hence the elliptic curves $\mathcal{E}_{1,2}$ are equianharmonic. Their j-invariants are therefore vanishing and $\mathcal{E}_{1,2}$ are birationally equivalent.

To derive the covers $\pi_\pm$ we first note that the curve $\mathcal{C}$ is a covering of the hyperelliptic curve $\mathcal{C}^*$ of genus two,

$$\mathcal{C}^* = \{(\mu,\nu)|\nu^2 = (\mu^3+b)^2+4\}. \tag{7.22}$$

The cover of this curve is given by the formulae

$$\pi^*: \quad \mu = \frac{y}{x}, \quad \nu = -x^3 - \frac{1}{x^3}. \tag{7.23}$$

The curve $\mathcal{C}^*$ covers two-sheetedly the two elliptic curves $\mathcal{E}_\pm$ given in (7.16)

$$\begin{aligned} z_\pm &= \frac{K^2-L^2}{K^2-\rho L^2}\,\frac{K-\mu}{\rho K-\mu}\,\frac{L^2-K\mu}{L^2-K\rho\mu}, \\ w_\pm &= -\imath\sqrt{2+\rho}\sqrt{\frac{L\pm K}{L\mp K}}\,\frac{K^2}{L}\,\frac{L^2-\rho K^2}{\rho L^2-K^2}\,\frac{\nu(L\mp\mu)}{(\mu-\rho K)^2(L^2-\rho K\mu)^2}. \end{aligned} \tag{7.24}$$

Composition of (7.23) and (7.24) leads to (7.21). □

Using these formulae direct calculation then yields

Corollary 7.3. *The holomorphic differentials of $\mathcal{C}$ are mapped to holomorphic differentials of $\mathcal{E}_\pm$, $\mathcal{E}_{1,2}$ as follows*

$$\begin{aligned} \frac{dz_\pm}{w_\pm} &= \sqrt{1+2\rho}\frac{L}{K}\sqrt{(L\pm K)(L\mp K)^3}\,\frac{Lx\pm y}{y^2}\,dx, \\ &= \sqrt{1+2\rho}\frac{L}{K}\sqrt{(L\pm K)(L\mp K)^3}\,(L\pm\mu)\,\frac{d\mu}{\nu} \end{aligned} \tag{7.25}$$

$$\frac{dz_1}{w_1^2} = \frac{x^2+1}{y^2}dx, \tag{7.26}$$

$$\frac{dz_2}{w_2^2} = \frac{3x^2}{y^2}dx, \tag{7.27}$$

where L, K are given in (7.20).

The absolute invariants $j_\pm$ of the curves $\mathcal{E}_\pm$ are

$$j_\pm = 108\,\frac{L^3\left(5\,L^3\mp 4\,b\right)^3}{\left(L^3\pm b\right)^2}. \tag{7.28}$$

Evidently $j_\pm \neq 0$ in general, as well $j_+ \neq j_-$; therefore these elliptic curves are not birationally equivalent to that one appearing in Hitchin's theory of the tetrahedral monopole which is equianharmonic [HMM95]. We observe that the substitution

$$M = \frac{1+2\rho+p}{1+2\rho-p}$$

leads to the parameterisation of Jacobi moduli being

$$k_+^2 = \frac{(p+1)^3(3-p)}{16p}, \quad k_-^2 = \frac{(p+1)(3-p)^3}{16p^3}, \tag{7.29}$$

which Ramanujan used in his hypergeometric relations of signature 3, see e.g. [BBG95]. The θ-functional representation of the moduli $k_\pm$ and parameter p can be found in [Law89, Section 9.7],

$$k_+ = \frac{\vartheta_2^2(0|\tau)}{\vartheta_3^2(0|\tau)}, \quad k_- = \frac{\vartheta_2^2(0|3\tau)}{\vartheta_3^2(0|3\tau)}, \quad p = \frac{3\vartheta_3^2(0|3\tau)}{\vartheta_3^2(0|\tau)}.$$

We shall now describe the geometry of the covers we have just presented explicitly. Our curve has several explicit symmetries which lie behind the covers described. We will first describe these symmetries acting on the field of functions $\mathfrak{k}$ of our curve as this field does not depend on whether we have a singular or nonsingular model of the curve; we will subsequently give a projective model for these, typically working in weighted projective spaces where the curves will be nonsingular.

Viewing $\bar{y} = y/x$ and x as functions on $\mathcal{C}$ we see that

$$\bar{y}^3 = x^3 + b - \frac{1}{x^3}$$

has symmetries ($\rho = e^{2\imath\pi/3}$)

$$\begin{aligned} \mathrm{a} &: x \to x, \quad \bar{y} \to \rho\bar{y}, \\ \mathrm{b} &: x \to \rho x, \quad \bar{y} \to \bar{y}, \\ \mathrm{c} &: x \to -1/x, \quad \bar{y} \to \bar{y}. \end{aligned}$$

Together these yield the group $G = C_3 \times S_3$, with $C_3 =< \mathrm{a}|\,\mathrm{a}^3 = 1 >$ and $S_3 =< \mathrm{b},\mathrm{c}|\,\mathrm{b}^3 = 1, \mathrm{c}^2 = 1, \mathrm{cbc} = \mathrm{b}^2 >$. When $b = 5\sqrt{2}$, the dihedral symmetry S_3 is enlarged to tetrahedral symmetry by

$$\mathrm{t} : x \to \frac{\sqrt{2}-x}{1+\sqrt{2}x}, \quad \bar{y} \to \frac{3x\bar{y}}{(1+\sqrt{2}x)(x-\sqrt{2})}, \qquad \mathrm{t}^2 = 1,$$

with A_4 being generated by b and t. Now to each subgroup $H \le G$ we have the fixed field $\mathfrak{k}^H$ associated to the quotient curve $\mathcal{C}/H$.

The canonical curve of a non-hyperelliptic curve of genus 4 is given by the intersection of an irreducible quadric and cubic surface in $\mathbb{P}^3$. In our case the quadric is in fact a cone and we may represent our curve $\mathcal{C}$ as the nonsingular curve[6] in the weighted projective space $\mathbb{P}^{1,1,2} = \{[z,t,w]\,|\,[z,t,w] \sim [\lambda z, \lambda t, \lambda^2 w]\}$ given by the vanishing of

$$f(z,t,w) = z^6 + b\,z^3t^3 - t^6 - w^3.$$

The group G acts on this as ($x = z/t$, $\bar{y} = w/(zt)$)

$$\begin{aligned} \mathrm{a} &: [z,t,w] \to [z,t,\rho w] \sim [\rho z, \rho t, w], \\ \mathrm{b} &: [z,t,w] \to [\rho z, t, \rho w] \sim [\rho^2 z, \rho t, w], \\ \mathrm{c} &: [z,t,w] \to [t,-z,-w] \sim [\imath t, -\imath z, w]. \end{aligned}$$

The fixed points of these actions on $\mathcal{C}$ and quotient curves are as follows:

---

[6] Had we represented $\mathcal{C} \subset \mathbb{P}^2$ as the plane curve given by the vanishing of $z^6 + b\,z^3t^3 - t^6 - w^3t^3$ the curve is singular. When b is real the point $[z,t,w] = [0,0,1]$ is the only singular point of $\mathcal{C}$ with delta invariant 6 and multiplicity 3 yielding $g_{\mathcal{C}} = 4$.

- a**:** There are 6 fixed points, $[1, \rho^k\,\alpha_\pm, 0]$, where $\alpha_\pm$ are the two roots of $\alpha^2 - b\alpha - 1 = 0$. For other points we have a $3:1$ map $\mathcal{C} \to \mathcal{C}/<\mathrm{a}>$. An application of the Riemann-Hurwitz theorem shows the genus of $\mathcal{C}/<\mathrm{a}>$ to be $g_{\mathcal{C}/<\mathrm{a}>} = 0$.
- b**:** The has no fixed points and an application of the Riemann-Hurwitz theorem shows the genus of $\mathcal{C}/<\mathrm{b}>$ to be $g_{\mathcal{C}/<\mathrm{b}>} = 2$.
- c**:** There are 6 fixed points, $[1, \pm\imath, \rho^k\,\beta_\pm]$, where $\beta_\pm$ is a root of $\beta_\pm^3 = 2 \pm \imath b$. Here the Riemann-Hurwitz theorem shows the genus of $\mathcal{C}/<\mathrm{c}>$ to be $g_{\mathcal{C}/<\mathrm{c}>} = 1$.

By using the invariants of H we may obtain nonsingular projective models of $\mathfrak{k}^H$. Take for example $H =<\mathrm{c}>$ with invariants $u = zt$, $v = z^2 - t^2$ and w (in degree 2). Then we obtain the quotient curve $w^3 = v^3 + 3u^2v + bu^3$ in $\mathbb{P}^{2,2,2} \sim \mathbb{P}^{1,1,1} = \{[u, v, w]\}$. The genus of the quotient is seen to be 1. We recognize this as the curve $\mathcal{E}_1$. One verifies that

$$\mathrm{c}^*\left(\frac{x^2+1}{y^2}\,dx\right) = \frac{x^2+1}{y^2}\,dx$$

giving us the invariant differential (7.26). Similarly, by taking $H =<\mathrm{bc}>$ and $H =<\mathrm{b}^2\mathrm{c}>$, we also obtain equianharmonic elliptic curves. The invariants of the involution $\mathrm{b}^2\mathrm{c}$ are again all in degree 2 and now are $u = zt$, $v = \rho^{1/2}z^2 - \rho^{-1/2}\,t^2$ and w.

By taking $H =<\mathrm{a}^2\mathrm{b}>$ we may identify $\mathcal{E}_2$. The invariant of $<\mathrm{a}^2\mathrm{b}>$: $[z, t, w] \to [\rho z, t, w]$ is $u = z^3$ and the curve $w^3 = u^2 - but^3 + t^6$ in $\mathbb{P}^{3,1,2} = \{[u, t, w]\}$.Using the formula for the genus of a smooth curve of degree d in $\mathbb{P}^{a_0,a_1,a_2}$,

$$g = \frac{1}{2}\left(\frac{d^2}{a_0a_1a_2} - d\sum_{i<j}\frac{\gcd(a_i, a_j)}{a_ia_j} + \sum_{i=0}^{2}\frac{\gcd(a_i, d)}{a_i} - 1\right),$$

the genus is seen to be 1. Now (7.27) is the invariant differential for this action. If we had taken $H =<\mathrm{a}>$ with invariants $u = z^3$, $v = t^3$ and w we obtain the curve $w^3 = u^2 + buv - v^2$ in $\mathbb{P}^{3,3,2}$ (which is equivalent to $W = u^2 + buv - v^2$ in $\mathbb{P}^{1,1,2}$). The genus of this quotient is seen to be 0.

We obtain the genus 2 curve $\mathcal{C}^*$ as follows. The invariants of $H =<\mathrm{b}>$ are $U = zt$, $V = z^3$, $T = t^3$ and w, subject to the relation $U^3 = VT$. The curve $\mathcal{C}$ may be written $T^2 = -w^3 + bU^3 + V^2$, and hence $U^6 = V^2T^2 = V^2(-w^3 + bU^3 + V^2)$. This curve has genus 2 in $\mathbb{P}^{2,3,2} = \{[U, V, w]\}$ and may be identified with $\mathcal{C}^*$. Be setting $\nu = 2V^2 - (w^3 - bU^3)$ this curve takes the form

$$\nu^2 = (w^3 - bU^3)^2 + 4U^6$$

in $\mathbb{P}^{1,3,1} = \{[U, \nu, w]\}$ and the identification with $\mathcal{C}^*$ in the affine chart of earlier is given by $\mu = -w$, $U = 1$. In this latter form we find that the action of c is given by $[U, \nu, w] \to [-U, \nu, -w] \sim [U, -\nu, w]$ which is the hyperelliptic involution; further quotienting yields a genus 0 curve.

The remaining genus 1 curves $\mathcal{E}_\pm$ are identified with the quotients of $\mathcal{C}^*$ by $U \to \pm w/\sqrt[6]{4+b^2}$, $w \to \pm\sqrt[6]{4+b^2}U$, $\nu \to \nu$. This action has invariants $A = Uw$ (in degree 2), $B = w \pm \sqrt[6]{4+b^2}U$ (in degree 1), and ν (in degree 3). The resulting degree 6 curve is

$$\nu^2 = B^6 \mp 6LAB^4 + 9L^2A^2B^2 \mp 2L^3A^3 - 2bA^3,$$

where, as previously, $L = \sqrt[6]{4+b^2}$. These curves have genus 1 in $\mathbb{P}^{2,1,3} = \{[A, B, \nu]\}$. To complete the identification with $\mathcal{E}_\pm$ we compute the j-invariants of these curves. In the affine patch with $B \neq 0$ which looks like $\mathbb{C}^2$ (the other affine patches have orbifold singularities and hence this choice) the curve takes the form

$$Y^2 = 1 \mp 6LX + 9L^2X^2 - 2(b \pm L^3)X^3.$$

The j-invariants of these curves agree with (7.28) and hence the identifications as stated.

Both the differentials dx/y and $x\,dx/y^2$ are invariant under b. These may be obtained by linear combinations of $dz_\pm/w_\pm$ (7.25). The latter differentials are those invariant under the symmetry of (7.22)

$$\mu \to \frac{L^2}{\mu}, \qquad \nu \to \pm\frac{L^3\nu}{\mu^3},$$

which yield the quotients $\mathcal{E}_\pm$. A birational transformation makes this symmetry more manifest[7]. Let

$$T = \frac{L+\mu}{L-\mu},\ S = \frac{8\nu}{(L-\mu)^3}, \qquad \mu = L\,\frac{T-1}{T+1},\ \nu = \frac{L^3 S}{(T+1)^3}.$$

Then (7.22) transforms to

$$S^2 = (T-1)^6 + 2\frac{b}{L^3}(T^2-1)^3 + (T+1)^6$$

which is manifestly invariant under $T \to -T$, $S \to \mp S$. The substitution $W = T^2$ reduces the canonical differentials dT/S and $T\,dT/S^2$ to the canonical differentials the elliptic curves

$$\begin{aligned}
\mathcal{E}_+ :\quad & S^2 = 2(1+\frac{b}{L^3})W^3 + 6(5-\frac{b}{L^3})W^2 + 6(5+\frac{b}{L^3})W + 2(1-\frac{b}{L^3}),\\
\mathcal{E}_- :\quad & S^2 = 2(1+\frac{b}{L^3})W^4 + 6(5-\frac{b}{L^3})W^3 + 6(5+\frac{b}{L^3})W^2 + 2(1-\frac{b}{L^3})W,
\end{aligned}$$

which correspond to our earlier parameterizations.

7.5. **Role of the higher Goursat hypergeometric identities.** We have seen that complete Abelian integrals of the curve $\mathcal{C}$ (1.2) are given by hypergeometric functions. The same is true for the various curves given in lemma 7.2 covered by $\mathcal{C}$. Relating the periods of $\mathcal{C}$ and the curves it covers leads to various relations between hypergeometric functions, and this underlies the higher hypergeometric identities of Goursat [Gou81]. Goursat gave detailed tables of transformations of hypergeometric functions up to order four that will be enough for our purposes.

The simplest example of this is the cover $\pi : \mathcal{C} \to \mathcal{C}^*$ for which $\pi^*(\mu\,d\mu/\nu) = dx/y$ and $\pi^*(d\mu/\nu) = x\,dx/y^2$. One then finds for example that

$$\int_0^\alpha \frac{dx}{y} = \int_0^\infty \frac{\mu\,d\mu}{\nu}, \tag{7.30}$$

where both y and ν are evaluated on the first sheet. A change of variable shows that

$$\int_0^\infty \frac{\mu\,d\mu}{\nu} = \frac{2\pi}{3\sqrt{3}}\,(b-2\imath)^{-1/3}\,{}_2F_1\left(\frac{1}{2},\frac{1}{3};1;\frac{4\imath}{2\imath-b}\right).$$

Now the left-hand side of equation (7.30) is $-\mathcal{I}_1$ (the minus sign arising when we go to Wellstein variables $y \to -w$) and this has been evaluated in (7.2). Comparison of these two representations yields the hypergeometric equality

$$F\left(\frac{1}{2},\frac{1}{3};1;\frac{4\imath}{2\imath-b}\right) = \left(\frac{2(b-2\imath)}{b+\sqrt{b^2+4}}\right)^{\frac{1}{3}} F\left(\frac{1}{3},\frac{1}{3};1;\frac{b-\sqrt{b^2+4}}{b+\sqrt{b^2+4}}\right),$$

which is one of Goursat's quadratic equalities [Gou81]; see also [BE55, Sect. 2.11, Eq. (31)]. Further identities ensue from the coverings $\mathcal{C} \to \mathcal{E}_\pm$ and we shall describe these as needed below.

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[7]We thank Chris Eilbeck for this observation.

We remark that the curve (7.22) already appeared in Hutchinson's study [Hut02] of automorphic functions associated with singular, genus two, trigonal curves in which he developed earlier investigations of Burkhardt [Bur93]. These results were employed by Grava and one of the authors [EG04] to solve the Riemann-Hilbert problem and associated Schlesinger system for certain class of curves with Z_N-symmetry.

7.6. **Weierstrass reduction.** It is possible for the theta functions associated to a period matrix τ to simplify (or admit reduction) and be expressible in terms of lower dimensional theta functions. Such happens when the curve covers a curve of lower genus, but it may also occur without there being a covering. Reduction may be described purely in terms of the Riemann matrix of periods (see [Mar92b]; for more recent expositions and applications see [BE01],[BE02]). A $2g \times g$ Riemann matrix $\Pi = \begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix}$ is said to admit *reduction* if there exists a $g \times g_1$ matrix of complex numbers λ of maximal rank, a $2g_1 \times g_1$ matrix of complex numbers Π_1 and a $2g \times 2g_1$ matrix of integers M also of maximal rank such that

$$\Pi\lambda = \Pi_1 M, \tag{7.31}$$

where $1 \leq g_1 < g$. When a Riemann matrix admits reduction the corresponding period matrix may be put in the form

$$\tau = \begin{pmatrix} \tau_1 & Q \\ Q^T & \tau^{\#} \end{pmatrix}, \tag{7.32}$$

where Q is a $g_1 \times (g - g_1)$ matrix with rational entries and the matrices τ_1 and $\tau^{\#}$ have the properties of period matrices. Because Q here has rational entries there exists a diagonal $(g - g_1) \times (g - g_1)$ matrix $D = \mathrm{Diag}(d_1, \ldots, d_{g-g_1})$ with positive integer entries for which $(QD)_{jk} \in \mathbb{Z}$. With $(z, w) = (z_1, \ldots, z_{g_1}, w_1, \ldots, w_{g-g_1})$ the theta function associated with τ may then be expressed in terms of lower dimensional theta functions as

$$\theta((z, w); \tau) = \sum_{\substack{\mathbf{m}=(m_1,\ldots,m_{g-g_1}) \\ 0 \leq m_i \leq d_i - 1}} \theta(z + Q\mathbf{m}; \tau_1)\, \theta \begin{bmatrix} D^{-1}\mathbf{m} \\ 0 \end{bmatrix} (Dw; D\tau^{\#} D). \tag{7.33}$$

Our curve admits many reductions. Of itself this just means that the theta functions may be reduced to theta functions of fewer variables. It is only when the Ercolani-Sinha vector correspondingly reduces that we obtain real simplification. In the remainder of this section we shall describe these reductions and later see how dramatic simplifications occur.

First let us describe the Riemann matrix of periods. We may evaluate the remaining period integrals as follows. Let

$$\int_0^\alpha du_i = \mathcal{I}_i(\alpha), \quad \int_0^\beta du_i = \mathcal{J}_i(\alpha), \quad i = 1, \ldots, 4.$$

Then for $k = 1, 2$ we have that

$$\int_0^{\rho^k\alpha} du_{1,2} = \rho^k \mathcal{I}_{1,2}(\alpha), \quad \int_0^{\rho^k\beta} du_{1,2} = \rho^k \mathcal{J}_{1,2}(\alpha),$$

$$\int_0^{\rho^k\alpha} du_3 = \rho^{2k} \mathcal{I}_3(\alpha), \quad \int_0^{\rho^k\beta} du_3 = \rho^{2k} \mathcal{J}_3(\alpha),$$

$$\int_0^{\rho^k\alpha} du_4 = \mathcal{I}_4(\alpha), \qquad \int_0^{\rho^k\beta} du_4 = \mathcal{J}_4(\alpha),$$

where it is again supposed that the integrals $\mathcal{I}_*$ and $\mathcal{J}_*$ are computed on the first sheet. We have already computed $\mathcal{I}_1(\alpha)$ and $\mathcal{J}_1(\alpha)$. The integrals $\mathcal{I}_*$ and $\mathcal{J}_*$ are found to be

$$\begin{aligned}
\mathcal{I}_1(\alpha) &= -\frac{2\pi\alpha}{3\sqrt{3}}\,{}_2F_1\left(\frac13,\frac13;1;-\alpha^6\right) = -\frac{2\pi}{3\sqrt{3}}\,\frac{\alpha}{(1+\alpha^6)^{\frac13}}\;{}_2F_1(\frac13,\frac23;1,t),\\
\mathcal{J}_1(\alpha) &= \frac{2\pi}{3\sqrt{3}\alpha}\,{}_2F_1\left(\frac13,\frac13;1;-\frac{1}{\alpha^6}\right) = \frac{2\pi}{3\sqrt{3}}\,\frac{\alpha}{(1+\alpha^6)^{\frac13}}\;{}_2F_1(\frac13,\frac23;1,1-t),\\
\mathcal{I}_2(\alpha) &= \frac{4\pi^2}{9\Gamma\left(\frac23\right)^3}\frac{\alpha}{(1+\alpha^6)^{\frac13}},\\
\mathcal{J}_2(\alpha) &= -\frac{4\pi^2}{9\Gamma\left(\frac23\right)^3}\frac{\alpha}{(1+\alpha^6)^{\frac13}},\\
\mathcal{I}_3(\alpha) &= \frac{2\pi\alpha^2}{3\sqrt{3}}\,{}_2F_1\left(\frac23,\frac23;1;-\alpha^6\right) = \frac{2\pi}{3\sqrt{3}}\,\frac{\alpha^2}{(1+\alpha^6)^{\frac23}}\;{}_2F_1(\frac13,\frac23;1,t),\\
\mathcal{J}_3(\alpha) &= \frac{2\pi}{3\sqrt{3}\alpha^2}\,{}_2F_1\left(\frac23,\frac23;1;-\frac{1}{\alpha^6}\right) = \frac{2\pi}{3\sqrt{3}}\,\frac{\alpha^2}{(1+\alpha^6)^{\frac23}}\;{}_2F_1(\frac13,\frac23;1,1-t),\\
\mathcal{I}_4(\alpha) &= \alpha^3\,{}_2F_1\left(\frac23,1;\frac43;-\alpha^6\right),\\
\mathcal{J}_4(\alpha) &= -\frac{1}{\alpha^3}\,{}_2F_1\left(\frac23,1;\frac43;-\frac{1}{\alpha^6}\right),
\end{aligned}$$

with $t=\alpha^6/(1+\alpha^6)$.

We observe that the relations

$$\mathcal{R}\equiv\frac{\mathcal{I}_1(\alpha)}{\mathcal{J}_1(\alpha)} = -\frac{\mathcal{I}_3(\alpha)}{\mathcal{J}_3(\alpha)},\qquad \mathcal{I}_2(\alpha)+\mathcal{J}_2(\alpha)=0,\qquad \mathcal{I}_4(\alpha)-\mathcal{J}_4(\alpha)=\mathcal{I}_2(\alpha), \tag{7.34}$$

follow from the above formulae.

The vectors $\boldsymbol{x},\ldots,\boldsymbol{d}$ are

$$\begin{aligned}
\boldsymbol{x} &= \begin{pmatrix} -(2\mathcal{J}_1+\mathcal{I}_1)\rho-2\mathcal{I}_1-\mathcal{J}_1\\ (\mathcal{J}_1-\mathcal{I}_1)\rho+\mathcal{I}_1+2\mathcal{J}_1\\ (\mathcal{J}_1+2\mathcal{I}_1)\rho+\mathcal{I}_1-\mathcal{J}_1\\ 3(\mathcal{J}_1-\mathcal{I}_1)\rho+3\mathcal{J}_1 \end{pmatrix},\quad \boldsymbol{b}=\mathcal{I}_2\begin{pmatrix}1+2\rho\\ -2-\rho\\ 1-\rho\\ 0\end{pmatrix},\\
\boldsymbol{c} &= \begin{pmatrix} (\mathcal{I}_3+2\mathcal{J}_3)\rho+\mathcal{J}_3-\mathcal{I}_3\\ (\mathcal{I}_3-\mathcal{J}_3)\rho+\mathcal{J}_3+2\mathcal{I}_3\\ -(2\mathcal{I}_3+\mathcal{J}_3)\rho-2\mathcal{J}_3-\mathcal{I}_3\\ 3(\mathcal{I}_3-\mathcal{J}_3)\rho+3\mathcal{I}_3 \end{pmatrix},\quad \boldsymbol{d}=(\rho-1)\mathcal{I}_2\begin{pmatrix}1\\1\\1\\0\end{pmatrix}.
\end{aligned} \tag{7.35}$$

One may can easily check that

$$\boldsymbol{x}^T H\boldsymbol{b} = \boldsymbol{x}^T H\boldsymbol{c} = \boldsymbol{x}^T H\boldsymbol{d} = 0.$$

We then have that

(7.36)

$$\mathcal{A} = \begin{pmatrix} -1-2\rho-(2+\rho)\mathcal{R} & 1+2\rho & 1+2\rho+(1-\rho)\mathcal{R} & -1+\rho \\ 2+\rho+(1-\rho)\mathcal{R} & -2-\rho & 1-\rho-(2+\rho)\mathcal{R} & -1+\rho \\ -1+\rho+(1+2\rho)\mathcal{R} & 1-\rho & -2-\rho+(1+2\rho)\mathcal{R} & -1+\rho \\ 3+3\rho-3\rho\mathcal{R} & 0 & -3\rho-3(1+\rho)\mathcal{R} & 0 \end{pmatrix} \begin{pmatrix} \mathcal{J}_1 & & & \\ & \mathcal{I}_2 & & \\ & & \mathcal{J}_3 & \\ & & & \mathcal{I}_2 \end{pmatrix},$$

$$\mathcal{B} = \begin{pmatrix} 2+\rho+(1-\rho)\mathcal{R} & 1-\rho & 1-\rho-(2+\rho)\mathcal{R} & 2+\rho \\ -1+\rho+(1+2\rho)\mathcal{R} & 1+2\rho & -2-\rho+(1+2\rho)\mathcal{R} & 2+\rho \\ -1-2\rho-(2+\rho)\mathcal{R} & -2-\rho & 1+2\rho+(1-\rho)\mathcal{R} & 2+\rho \\ 3-3(1+\rho)\mathcal{R} & 0 & 3-3\rho\mathcal{R} & 0 \end{pmatrix} \begin{pmatrix} \mathcal{J}_1 & & & \\ & \mathcal{I}_2 & & \\ & & \mathcal{J}_3 & \\ & & & \mathcal{I}_2 \end{pmatrix}.$$

The Ercolani-Sinha conditions, $\boldsymbol{n}^T\mathcal{A}+\boldsymbol{m}^T\mathcal{B} = 6\chi^{\frac{1}{3}}(1,0,0,0)$ written for the vectors

$$\boldsymbol{n} = \begin{pmatrix} n_1 \\ m_1-n_1 \\ -m_1 \\ 2n_1-m_1 \end{pmatrix}, \qquad \boldsymbol{m} = \begin{pmatrix} m_1 \\ -n_1 \\ n_1-m_1 \\ -3n_1 \end{pmatrix} \tag{7.37}$$

lead to the equations

$$\mathcal{R} = -\frac{2n_1-m_1}{m_1+n_1}, \qquad \mathcal{J}_1 = \frac{\chi^{\frac{1}{3}}}{m_1-2n_1}, \tag{7.38}$$

which were obtained earlier. A calculation also shows that the relation (5.20)

$$\mathcal{M}\begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix} = \overline{\begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix}} \cdot T$$

yielding a nontrivial check of our procedure.

The integrals between infinities may be reduced to our standard integrals by writing

$$\int_{\infty_i}^{\infty_j} d\boldsymbol{u} = \int_{\tau(0_{\tau(i)})}^{\tau(0_{\tau(j)})} d\boldsymbol{u} = \int_{0_{\tau(i)}}^{0_{\tau(j)}} \tau^*(d\boldsymbol{u}) = \overline{\int_{0_{\tau(i)}}^{0_{\tau(j)}} d\boldsymbol{u}} \cdot T = \left(\overline{\int_{0_{\tau(i)}}^{\lambda_*} d\boldsymbol{u}} - \overline{\int_{0_{\tau(j)}}^{\lambda_*} d\boldsymbol{u}} \right) \cdot T,$$

where we write $\tau(\infty_i) = 0_{\tau(i)}$ and λ_* is any of the branch points. These are then calculated to be

(7.39)

$$\int_{\infty_1}^{\infty_2} d\boldsymbol{u} = \begin{pmatrix} (\rho-1)\mathcal{J}_1 \\ -(\rho^2-1)\mathcal{J}_4 \\ (\rho^2-1)\mathcal{J}_3 \\ -(\rho^2-1)\mathcal{J}_2 \end{pmatrix}, \int_{\infty_1}^{\infty_3} d\boldsymbol{u} = \begin{pmatrix} (\rho^2-1)\mathcal{J}_1 \\ -(\rho-1)\mathcal{J}_4 \\ (\rho-1)\mathcal{J}_3 \\ -(\rho-1)\mathcal{J}_2 \end{pmatrix}, \int_{\infty_2}^{\infty_3} d\boldsymbol{u} = \begin{pmatrix} (\rho^2-\rho)\mathcal{J}_1 \\ -(\rho-\rho^2)\mathcal{J}_4 \\ (\rho-\rho^2)\mathcal{J}_3 \\ -(\rho-\rho^2)\mathcal{J}_2 \end{pmatrix}.$$

Our Riemann matrix admits a reduction with respect to any of it columns. We will exemplify this with the first column, a result we will use next; similar considerations apply to the other columns. Now from the above and (6.3) it follows that

(7.40)

$$\Pi\lambda = \begin{pmatrix} \mathcal{A} \\ \mathcal{B} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} = \begin{pmatrix} \oint_{\mathfrak{a}_i} du_1 \\ \oint_{\mathfrak{b}_i} du_1 \end{pmatrix} = \begin{pmatrix} \xi(H\boldsymbol{n}+\rho^2\boldsymbol{m}) \\ \xi(\rho\boldsymbol{n}+H\boldsymbol{m}) \end{pmatrix} = \xi\, M \begin{pmatrix} 1 \\ \rho \end{pmatrix},$$

where M is the $2g \times 2$ integral matrix

$$M^T = \begin{pmatrix} n_1-m_1 & n_2-m_2 & n_3-m_3 & -n_4-m_4 & m_1 & m_2 & m_3 & -m_4 \\ -m_1 & -m_2 & -m_3 & -m_4 & n_1 & n_2 & n_3 & n_4 \end{pmatrix}. \tag{7.41}$$

Then to every two Ercolani-Sinha vectors $\boldsymbol{n}$, $\boldsymbol{m}$ we have that

$$M^T J M = d\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad d = \boldsymbol{n}.H\boldsymbol{n} - \boldsymbol{m}.\boldsymbol{n} + \boldsymbol{m}.H\boldsymbol{m} = \sum_{j=1}^{4}(\varepsilon_j n_j^2 - n_j m_j + \varepsilon_j m_j^2). \tag{7.42}$$

The number d here is often called the Hopf number. In particular for $d \neq 0$ then M is of maximal rank and consequently our Riemann matrix admits reduction.

Let us now focus on the consequences of reduction for symmetric monopoles.

Theorem 7.4. *For the symmetric monopole we may reduce by the first column using the vector (7.40) whose elements are related by (7.6), with* $(n_1, m_1) = 1$. *Then*

$$d = 2(n_1 + m_1)(m_1 - 2n_1)$$

and for $d \neq 0$ *there exists an element* σ *of the symplectic group* $\mathrm{Sp}_{2g}(\mathbb{Z})$ *such that*

$$\tau'_{\mathfrak{b}} = \sigma \circ \tau_{\mathfrak{b}} = \begin{pmatrix} (\rho+2)/d & \alpha/d & 0 & \dots & 0 \\ \alpha/d & & & & \\ 0 & & \tau^{\#} & & \\ \vdots & & & & \\ 0 & & & & \end{pmatrix}. \tag{7.43}$$

Letting $p\,m_1 + q\,n_1 = 1$ *then*

$$\alpha = \gcd(m_1 + 4n_1 - q\,[m_1 - 2n_1], n_1 - 2m_1 - p\,[m_1 - 2n_1]). \tag{7.44}$$

When $\alpha = 1$ *a further symplectic transformation allows the simplification* $\tau'_{11} = \rho/d$.

Under σ *the Ercolani-Sinha vector transforms as*

$$\sigma \circ \boldsymbol{U} = \sigma \circ (\boldsymbol{m}^T + \boldsymbol{n}^T \tau_{\mathfrak{b}}) = (1/2, 0, 0, 0). \tag{7.45}$$

The proof of the theorem is constructive using work of Krazer, Weierstrass and Kowalewski. Martens [Mar92a, Mar92b] has given an algorithm for constructing σ which we have implemented using *Maple*. Because σ depends on number theoretic properties of n_1 and m_1 the form is rather unilluminating and we simply record the result (though an explicit example will be given in the following section). What is remarkable however is the simple universal form the Ercolani-Sinha vector takes under this transformation. This has great significance for us as we next describe.

Using (7.33),(7.43) and say $D = \mathrm{Diag}(d, 1, 1)$ we have that[8]

$$\begin{aligned}\theta((z,w);\tau'_{\mathfrak{b}}) &= \sum_{m=0}^{d-1} \theta(z + \frac{m\alpha}{d}; \frac{\rho+2}{d})\, \theta\begin{bmatrix} \frac{m}{d} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}(Dw; D\tau^{\#}D) \\ &= \sum_{m=0}^{d-1} \theta\begin{bmatrix} \frac{m}{d} \\ 0 \end{bmatrix}(dz; d(\rho+2))\, \theta((w_1 + \frac{m\alpha}{d}, w_2, w_3); \tau^{\#}),\end{aligned}$$

where we have genus one and three theta functions on the right hand-side here. Comparison of (3.8) and (7.45) then reveals that the theta function dependence of $Q_0(z)$ is given wholly by the genus one theta functions. Further simplifications ensue from the identity

$$\theta\begin{bmatrix} \frac{\epsilon}{d} \\ \epsilon' \end{bmatrix}(dz; d\tau) = \mu(\tau) \prod_{l=0}^{d-1} \theta\begin{bmatrix} \frac{\epsilon}{d} \\ \frac{\epsilon'}{d} + \frac{d-(1+2l)}{2d} \end{bmatrix}(z;\tau),$$

where $\mu(\tau)$ is a constant. We then have

[8] When $\gcd(\alpha, d) \neq 1$ a smaller multiple than $d_1 = d$ would suffice here with correspondingly fewer terms in the sums $0 \leq m \leq d_1 - 1$.

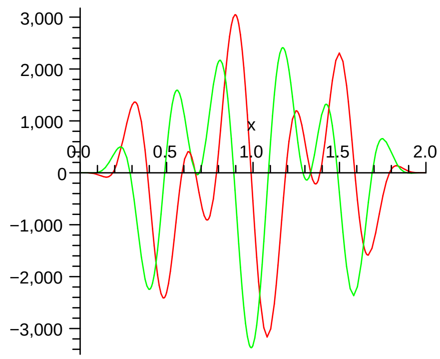

FIGURE 4. $n_1 = 2$, $m_1 = 1$.

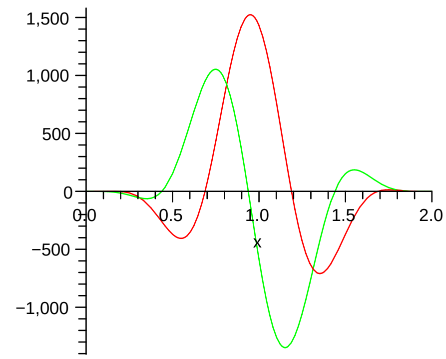

FIGURE 5. $n_1 = 1$, $m_1 = 1$.

Theorem 7.5. *For symmetric monopoles the theta function z-dependence of $Q_0(z)$is expressible in terms of elliptic functions.*

Thus far we have not discussed the final Hitchin constraint for symmetric monopoles. This theorem reduces the problem to one of the zeros of elliptic functions. The graph in Figure 4 shows the real and imaginary parts of the theta function denominator of $Q_0(z)$ for the $n_1 = 2$, $m_1 = 1$ symmetric monopole, the $b = 0$ Ramanjuan case. These vanish at $z = 0$ and $z = 2$ as desired, but additionally one finds vanishing at $z = 2/3$ and $z = 4/3$. Calculating the theta function with shifted argument in the numerator shows that there is no corresponding vanishing and consequently $Q_0(z)$ yields unwanted poles in $z \in (0, 2)$. Thus the $n_1 = 2$, $m_1 = 1$ curve does not yield a monopole.

A similar evaluation of the relevant $n_1 = 4$, $m_1 = -1$ and $n_1 = 5$, $m_1 = -2$ theta functions also reveals unwanted zeros and of the those cases from our table of symmetric 3-monopoles only the tetrahedrally symmetric case has the required vanishing. As yet we don't know whether there are further symmetric 3-monopoles with the required vanishing for a genuine monopole curve. Before turning to a more detailed examination of the tetrahedrally symmetric case in our next section we first describe how to calculate the remaining quantities appearing in our formula (3.8) for $Q_0(z)$.

7.7. **Calculating $\nu_i - \nu_j$.** Here we follow section §5.4. We calculate the $\mathfrak{a}$-periods of the differential dr_1 in a manner similar to the period integrals already calculated. Introduce integrals on the first sheet

$$\mathcal{K}_1(\alpha) = \int_0^\alpha \frac{z^4 dz}{3w^2}, \qquad \mathcal{L}_1(\beta) = \int_0^\beta \frac{z^4 dz}{3w^2}, \qquad \beta = -\frac{1}{\alpha}. \tag{7.46}$$

Evidently $\mathcal{K}_1(\rho^k\alpha) = \rho^{2k}\mathcal{K}(\alpha)$ and $\mathcal{L}_1(\rho^k\beta) = \rho^{2k}\mathcal{L}(\beta)$ and one finds that

$$\mathcal{K}_1 = -\frac{4\sqrt{3}\pi}{27}\alpha^5\, {}_2F_1\left(\frac{2}{3}, \frac{5}{3}; 2; -\alpha^6\right), \qquad \mathcal{L}_1 = \frac{4\sqrt{3}\pi}{27}\frac{1}{\alpha^5}\, {}_2F_1\left(\frac{2}{3}, \frac{5}{3}; 2; -\frac{1}{\alpha^6}\right). \tag{7.47}$$

We find, as before in the case of holomorphic differentials, that

$$y_1 = (\mathcal{K}_1 + 2\mathcal{L}_1)\rho - \mathcal{K}_1 + \mathcal{L}_1, \qquad y_2 = (\mathcal{K}_1 - \mathcal{L}_1)\rho + 2\mathcal{K}_1 + \mathcal{L}_1$$
$$y_3 = -(2\mathcal{K}_1 + \mathcal{L}_1)\rho - \mathcal{K}_1 - 2\mathcal{L}_1, \qquad y_4 = 3(\mathcal{K}_1 - \mathcal{L}_1)\rho + 3\mathcal{K}_1.$$

The Legendre relation (5.22) gives a non trivial consistency check of our calculations. This may be written in the form of the following hypergeometric equality

$$\frac{27}{4\sqrt{3}\pi} = \alpha^4\, {}_2F_1\left(\frac{1}{3},\frac{1}{3};1;-\frac{1}{\alpha^6}\right)\, {}_2F_1\left(\frac{2}{3},\frac{5}{3};2;-\alpha^6\right) + \frac{1}{\alpha^4}\, {}_2F_1\left(\frac{1}{3},\frac{1}{3};1;-\alpha^6\right)\, {}_2F_1\left(\frac{2}{3},\frac{5}{3};2;-\frac{1}{\alpha^6}\right)$$

and this may be established by standard means.

To calculating $\nu_i - \nu_j$ using (5.24) introduce the differential of the second kind,

$$s = d\left(\frac{w}{z}\right)(P) - 3dr_1(P) \equiv \frac{dz}{z^2w^2}, \tag{7.48}$$

with second order pole at 0 on all sheets,

$$\left.\frac{dz}{z^2w^2}\right|_{P=0_k} = \left\{\frac{1}{w(0_k)^2}\frac{1}{\xi^2} + \frac{2b}{3}\xi + \ldots\right\}d\xi = \left\{-\frac{w(0_k)}{\xi^2} + \frac{2b}{3}\xi + \ldots\right\}d\xi.$$

(Here we took into account $w(0_k)^3 = -1$ for $k = 1,2,3$.) Then

$$\nu_i - \nu_j = 3\boldsymbol{y}.\int_{\infty_j}^{\infty_j}\boldsymbol{v} + \int_{\infty_j}^{\infty_j}\frac{dz}{z^2w^2} \tag{7.49}$$

The last integral in (7.49) may also be expressed in terms of hypergeometric functions as follows. First we remark that

$$\int_{\infty_i}^{\infty_j}\frac{dz}{z^2w^2} = (\rho_i - \rho_j)\int_{\alpha}^{\infty_1}\frac{dz}{z^2w^2},$$

where $\rho_i = \rho^{i-1}$. Next, for the integrals on the first sheet we have

$$\int_{\alpha}^{\infty}\frac{dz}{z^2w^2} = \frac{4\sqrt{3}\pi}{27}\frac{1}{\alpha^5}F\left(\frac{2}{3},\frac{5}{3};2;-\frac{1}{\alpha^6}\right) = \mathcal{L}_1,$$
$$\int_{-\frac{1}{\alpha}}^{\infty}\frac{dz}{z^2w^2} = -\frac{4\sqrt{3}\pi}{27}\alpha^5F\left(\frac{2}{3},\frac{5}{3};2;-\alpha^6\right) = \mathcal{K}_1.$$

8. The tetrahedral 3-monopole

The curve of the tetrahedrally symmetric monopole is of the form

$$\eta^3 + \chi(\zeta^6 + 5\sqrt{2}\zeta^3 - 1) = 0. \tag{8.1}$$

In this case we may take

$$t = \frac{1}{2} - \frac{5\sqrt{3}}{18},\ \alpha = \frac{\sqrt{3}-1}{\sqrt{2}},\ \mathcal{J}_1(\alpha) = -2\mathcal{I}_1(\alpha).$$

For these values we may explicitly evaluate the various hypergeometric functions. Using Ramanujan's identity (7.15) together with the standard quadratic transformation of the hypergeometric function

$${}_2F_1\left(\frac{1}{2},\frac{1}{2},1,z\right) = (1+\sqrt{z})^{-1}\, {}_2F_1\left(\frac{1}{2},\frac{1}{2},1,\frac{4\sqrt{z}}{(1+\sqrt{z})^2}\right),$$

(valid for $|z|<1$, $\arg z<\pi$) we find that

$$ {}_2F_1\left(\frac{1}{3},\frac{2}{3},1,t\right)=\frac{3^{\frac{5}{4}}}{4}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2},1,\frac{2-\sqrt{3}}{4}\right). $$

(In verifying this we note that $p=4+3\sqrt{3}-2\sqrt{6}-3\sqrt{2}$ is the relevant value leading to our t in (7.15).) Now this last hypergeometric function is related to an elliptic integral we may evaluate [Law89, p 86],

$$ K\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)=\frac{\pi}{2}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2},1,\frac{2-\sqrt{3}}{4}\right)=\frac{\Gamma(\frac{1}{6})\Gamma(\frac{1}{3})}{3^{\frac{1}{4}}\,4\sqrt{\pi}}. $$

Bringing these results together we finally obtain

$$ {}_2F_1\left(\frac{1}{3},\frac{2}{3},1,t\right)=\frac{3\,\Gamma(\frac{1}{6})\Gamma(\frac{1}{3})}{8\pi^{\frac{3}{2}}}. \tag{8.2} $$

Then from (7.12) we obtain that

$$ \chi^{\frac{1}{3}}=-2\,\frac{2\pi}{3\sqrt{3}}\;\frac{\alpha}{(1-\alpha^6)^{\frac{1}{3}}}\;{}_2F_1(\frac{1}{3},\frac{2}{3};1,t)=-\frac{1}{2^{\frac{1}{6}}\sqrt{3}}\,\frac{\Gamma(\frac{1}{6})\Gamma(\frac{1}{3})}{2\sqrt{3}\pi^{\frac{1}{2}}}. \tag{8.3} $$

This agrees with the result of [HMR00]. We also note that upon using Goursat's identity [Gou81, (39)]

$$ {}_2F_1\left(\frac{1}{3},\frac{2}{3},1,x\right)=(1-2x)^{\frac{-1}{3}}\,{}_2F_1\left(\frac{1}{6},\frac{2}{3},1,\frac{4x(x-1)}{(2x-1)^2}\right), $$

we may establish the result of [HMR00] based on numerical evaluation, that

$$ {}_2F_1\left(\frac{1}{6},\frac{2}{3},1,-\frac{2}{25}\right)=\frac{5^{\frac{1}{3}}\,3}{8}\,\frac{\Gamma(\frac{1}{6})\Gamma(\frac{1}{3})}{\sqrt{3}\,\pi^{\frac{3}{2}}}. $$

Using these results and those of the previous section we have,

Theorem 8.1. *The tetrahedral 3-monopole for which $b=5\sqrt{2}$ admits the τ-matrix of the form*

$$ \begin{aligned} \tau &= \frac{1}{98}\begin{pmatrix} -73+51\,\imath\sqrt{3} & 9-13\,\imath\sqrt{3} & 15+11\,\imath\sqrt{3} & 42-28\,\imath\sqrt{3}\\ 9-13\,\imath\sqrt{3} & -34+60\,\imath\sqrt{3} & 2\,\imath\sqrt{3}-24 & 21+35\,\imath\sqrt{3}\\ 15+11\,\imath\sqrt{3} & 2\,\imath\sqrt{3}-24 & -40+36\,\imath\sqrt{3} & -63-7\,\imath\sqrt{3}\\ 42-28\,\imath\sqrt{3} & 21+35\,\imath\sqrt{3} & -63-7\,\imath\sqrt{3} & 49+49\,\imath\sqrt{3} \end{pmatrix}\\ &= \begin{pmatrix} -\frac{11}{49}+\frac{51}{49}\rho & -\frac{2}{49}-\frac{13}{49}\rho & \frac{13}{49}+\frac{11}{49}\rho & \frac{1}{7}-\frac{4}{7}\rho\\ -\frac{2}{49}-\frac{13}{49}\rho & \frac{13}{49}+\frac{60}{49}\rho & -\frac{11}{49}+\frac{2}{49}\rho & \frac{4}{7}+\frac{5}{7}\rho\\ \frac{13}{49}+\frac{11}{49}\rho & -\frac{11}{49}+\frac{2}{49}\rho & -\frac{2}{49}+\frac{36}{49}\rho & -\frac{5}{7}-\frac{1}{7}\rho\\ \frac{1}{7}-\frac{4}{7}\rho & \frac{4}{7}+\frac{5}{7}\rho & -\frac{5}{7}-\frac{1}{7}\rho & 1+\rho \end{pmatrix}. \end{aligned} \tag{8.4} $$

We have already seen that the symmetric monopole curve $\mathcal{C}$ covers two equianharmonic torii $\mathcal{E}_{1,2}$. For the value of the parameter $b=5\sqrt{2}$ the curve covers three further equianharmonic elliptic curves. These may be described as follows. For $i=3$, 4, 5 let $\pi_i:\mathcal{C}\to\mathcal{E}_i$ be defined by the formulae

$$ \mu_3=-\frac{\imath}{2^{\frac{4}{3}}}\frac{(1+z\alpha)^4+(z-\alpha)^4}{\alpha^2 w^2}, \qquad \nu_3=\frac{1+\imath}{\sqrt{2}}\frac{(1+\alpha^2)(z^2+1)}{(z-\alpha)(z\alpha+1)}, $$

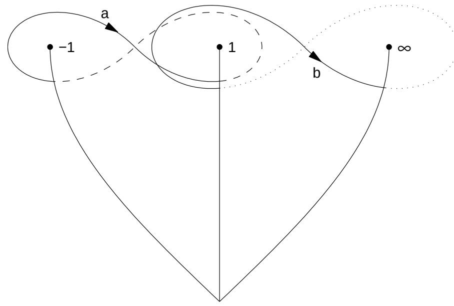

FIGURE 6. The elliptic curve homology basis

$$\mu_4 = -2^{\frac{2}{3}}\frac{(1+z\alpha)^4+(z-\alpha)^4}{2\alpha w(1+z\alpha)(z-\alpha)}, \qquad \nu_4 = 2^{\frac{2}{3}}\alpha w\frac{(1+z\alpha)^4-(z-\alpha)^4}{(z-\alpha)^3(z\alpha+1)^3},$$
$$\mu_5 = -\sqrt{3}\imath\frac{(z^2+1)(z^2-2\sqrt{2}z-1)}{(z^2+\sqrt{2}z-1)^2}, \quad \nu_5 = -4\sqrt{6}\imath\frac{w(z^4-\sqrt{2}z^3+3z^2+\sqrt{2}z+1)}{(z^2+\sqrt{2}z-1)^3}. \tag{8.5}$$

Then

$$\begin{aligned} \mathcal{E}_3: &\quad \{(\nu_3,\mu_3)\,|\,{\nu_3}^2-{\mu_3}^3-2\imath=0\},\\ \mathcal{E}_4: &\quad \{(\nu_4,\mu_4)\,|\,{\nu_4}^2-\mu_4({\mu_4}^3+4)=0\},\\ \mathcal{E}_5: &\quad \{(\nu_5,\mu_5)\,|\,{\nu_5}^3+24\sqrt{6}\imath({\mu_5}^2-1)^2=0\}, \end{aligned} \tag{8.6}$$

and we have the following relations between holomorphic differentials

$$du_2 = \frac{z\,dz}{w^2} = \frac{1}{2^{\frac{5}{3}}\sqrt{3}}\left\{(\imath-1)\pi_3^*\left(\frac{d\mu_3}{\nu_3}\right)+\pi_4^*\left(\frac{d\mu_4}{\nu_4}\right)\right\} \tag{8.7}$$

$$du_1 = \frac{dz}{w} = \pi_5^*\left(\frac{d\mu_5}{\nu_5}\right). \tag{8.8}$$

The final of these rational maps was introduced by [HMR00] and has the following significance.

Proposition 8.2. *Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be the $\mathfrak{a}$ and $\mathfrak{b}$-periods of the differential du_1 and denote by X, Y the $\mathfrak{a}$ and $\mathfrak{b}$-periods of the elliptic differential $d\mu_5/\nu_5$. Then*

$$\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix} = M_5\begin{pmatrix}X\\ Y\end{pmatrix}, \tag{8.9}$$

where M_5 is the matrix

$$M_5^T = \begin{pmatrix} -1 & 1 & 0 & 3 & 1 & 0 & -1 & 1\\ 0 & 1 & -1 & 2 & 1 & -1 & 0 & 3\end{pmatrix} \tag{8.10}$$

satisfying the condition

$$M_5^T\begin{pmatrix}0_4 & 1_4\\ -1_4 & 0_4\end{pmatrix}M_5 = 4\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \tag{8.11}$$

Proof. Introduce the homology basis for the elliptic curve as shown in Figure 6 and set

$$\mathcal{K}(\alpha) = \int_{-1}^{1}\frac{d\mu_5}{\nu_5}.$$

Then

$$X = (2+\rho)\mathcal{K}(\alpha), \quad Y = -(2\rho+1)\mathcal{K}(\alpha). \tag{8.12}$$

From the reduction formula (8.8) we next conclude that

$$\int_{\alpha\rho}^{\alpha\rho^2} \frac{dz}{w} = \mathcal{K}(\alpha) \tag{8.13}$$

and therefore have that

$$\begin{aligned} &2\mathcal{I}_1(\alpha) + \rho\mathcal{I}_1(\alpha) = \rho\mathcal{K}(\alpha), \\ &-2\rho\mathcal{I}_1(\alpha) - \mathcal{I}_1(\alpha) = \mathcal{K}(\alpha). \end{aligned} \tag{8.14}$$

Equations (8.12) and (8.14) permit us to express

$$\mathcal{I}_1(\alpha) = -\frac{Y}{3}, \qquad \rho\mathcal{I}_1(\alpha) = -\frac{X}{3} \tag{8.15}$$

and comparison with (7.36) yields the given M_5. The condition (8.11) is checked directly. The number 4 appearing in (8.11) means that the cover π_5 given in (8.5) be of degree 4.

□

We remark that the matrix M_5 of the proposition is obtained from the M of (7.41) by $M_5 = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M$, which simply reflects our choice of homology basis. Thus we are discussing the reduction of the previous section. Indeed with

$$\sigma = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & -1 & 1 & 1 & -1 & 0 & -3 \\ 5 & -1 & 0 & 3 & 0 & -1 & 1 & -2 \\ -6 & 0 & 0 & -3 & 0 & 0 & -1 & 2 \\ 7 & 0 & 0 & 3 & 0 & 0 & 0 & -2 \end{bmatrix} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

we find that the τ matrix (8.4) transforms to

$$\tau' = \sigma \circ \tau = (a\tau + b)(c\tau + d)^{-1} = \begin{bmatrix} \rho/4 & 1/4 & 0 & 0 \\ 1/4 & 5\rho/4 & \rho & 0 \\ 0 & \rho & 2\rho & \rho \\ 0 & 0 & \rho & 2/7 + 6\rho/7 \end{bmatrix}.$$

Combined with Theorem 7.4 we may reduce our expression for the tetrahedral monopoles $Q_0(z)$ to one built out of Jacobi elliptic theta functions. To compare with the Nahm data of [HMM95] we must solve for $C(z)$. This will be done elsewhere.

Part 3. End Matters

9. Conclusions

Although monopoles have been studied now for many years and from various perspectives, relatively few analytic solutions are known. This paper has sought to make effective the connection with integrable systems to construct such solutions. It is nevertheless only early steps upon this road.

The paper had two thrusts: an examination of the general construction and then a focus on a (new) class of charge three monopoles. In our general considerations we gave a further constraint on the Ercolani-Sinha vector (Lemma 3.1) and presented a new solution to the matrix Q_0 (Theorem 3.8), from which the Nahm data is reconstructed by solving a first order matrix differential equation. This latter step will be considered elsewhere. Our construction of the matrix Q_0 has been cast solely in terms of data built out of the spectral curve. Previous expressions for this matrix in terms of Baker-Akhiezer functions involve the choice of a non-special divisor which we relate to a gauge choice. Our analysis clearly identifies each of the ingredients necessary for the construction of this matrix and we showed how the fundamental bi-differential may be used in calculating this. Nearly all of the ingredients hinge on being able to integrate explicitly on the curve.

To apply our general construction beyond the known case of charge two we considered the restricted class of charge three monopoles (1.1) which includes the tetrahedrally symmetric monopole. This family of curves has many arithmetic properties that facilitates analytic integration. In particular the period matrix may be explicitly expressed in terms of just four integrals. Using this we were able to explicitly solve the Ercolani-Sinha constraints that are equivalent to Hitchin's transcendental condition (A2) of the triviality of a certain line bundle over the spectral curve (Proposition 6.1). Our approach reduces the problem to that of determining certain rationality properties of the (four) relevant periods. (Our result also admits another approach to seeking monopole curves: we may solve the Ercolani-Sinha constraints and then seek to impose Hitchin's reality conditions on the resulting curves. Results from this approach will be explored elsewhere.) To proceed further in this rather uncharted territory we further restricted our attention to what we have referred to as "symmetric 3-monopoles" whose spectral curve has the form (1.2). This reduced the required independent integrals from four to two, each of which were hypergeometric in form, and the rationality requirement is now for the ratio of these (Proposition 7.1). Extensions of work by Ramanujan mean this latter question may be replaced by number theory and of seeking solutions of various algebraic equations (depending on the primes involved in the rational ratio). Examples of such solutions were given (again including the tetrahedral case). We further examined the symmetries and coverings of these symmetric curves and their relation to higher Goursat hypergeometric identities. Having at hand now many putative spectral curves we proceeded to evaluate the remaining integrals needed in our construction. Remarkably we discovered that application of Weierstrass reduction theory showed that the Ercolani-Sinha vector transformed to a universal form and that all of the theta function z-dependence for symmetric 3-monopoles was expressible in terms of elliptic functions (Theorems 7.4,7.5). The final selection of permissible spectral curves at last reduced to the question of zeros of these elliptic functions. Unfortunately, of the symmetric 3-monopoles we have examined only the tetrahedral monopole has the required zeros. Further investigation is required to ascertain whether this is a general result.

Our final section then was devoted to the charge three tetrahedrally symmetric monopole. Here we were able to substantially simplify known expressions for the period matrix of the spectral curve as well as prove a conjectured identity of earlier workers. Again an explicit map was given and we have been able to reduce entirely to elliptic functions. The final comparison with the Nahm data of [HMM95] requires the next stage of the reconstruction, solving for the matrix $C(z)$. This and other matters will be left for a subsequent work.

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Appendix A. Theta Functions

For $r \in \mathbb{N}$ the canonical Riemann θ-function is given by

$$\theta(\boldsymbol{z};\tau) = \sum_{\boldsymbol{n}\in\mathbb{Z}^r} \exp(\imath\pi \boldsymbol{n}^T\tau\boldsymbol{n} + 2\imath\pi\boldsymbol{z}^T\boldsymbol{n}). \tag{A.1}$$

The θ-function is holomorphic on $\mathbb{C}^r \times \mathbb{S}^r$ and satisfies

$$\theta(\boldsymbol{z}+\boldsymbol{p}\,;\tau) = \theta(\boldsymbol{z};\tau), \quad \theta(\boldsymbol{z}+\boldsymbol{p}\tau;\tau) = \exp\{-\imath\pi(\boldsymbol{p}^T\tau\boldsymbol{p} + 2\boldsymbol{z}^T\boldsymbol{p})\}\,\theta(\boldsymbol{z};\tau), \tag{A.2}$$

where $\boldsymbol{p} \in \mathbb{Z}^r$.

The Riemann θ-function $\theta_{\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{z};\tau)$ with characteristics $\boldsymbol{a},\boldsymbol{b} \in \mathbb{Q}$ is defined by

$$\begin{aligned}\theta_{\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{z};\tau) &= \exp\left\{\imath\pi(\boldsymbol{a}^T\tau\boldsymbol{a} + 2\boldsymbol{a}^T(\boldsymbol{z}+\boldsymbol{b})))\right\}\theta(\boldsymbol{z}+\tau\boldsymbol{a}+\boldsymbol{b};\tau)\\ &= \sum_{\boldsymbol{n}\in\mathbb{Z}^r}\exp\left\{\imath\pi(\boldsymbol{n}+\boldsymbol{a})^T\tau(\boldsymbol{n}+\boldsymbol{a}) + 2\imath\pi(\boldsymbol{n}+\boldsymbol{a})^T(\boldsymbol{z}+\boldsymbol{b})\right\},\end{aligned}$$

where $\boldsymbol{a},\boldsymbol{b} \in \mathbb{Q}^r$. This is also written as

$$\theta_{\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{z};\tau) = \theta\begin{bmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{bmatrix}(\boldsymbol{z};\tau).$$

For arbitrary $\boldsymbol{a},\boldsymbol{b} \in \mathbb{Q}^r$ and $\boldsymbol{a}',\boldsymbol{b}' \in \mathbb{Q}^r$ the following formula is valid

(A.3)

$$\theta_{\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{z}+\boldsymbol{a}'\tau+\boldsymbol{b}';\tau) = \exp\left\{-\imath\pi\boldsymbol{a}'^T\tau\boldsymbol{a}' - 2\imath\pi\boldsymbol{a}'^T\boldsymbol{z} - 2\imath\pi(\boldsymbol{b}+\boldsymbol{b}')^T\boldsymbol{a}'\right\} \times \theta_{\boldsymbol{a}+\boldsymbol{a}',\boldsymbol{b}+\boldsymbol{b}'}(\boldsymbol{z};\tau).$$

The function $\theta_{\boldsymbol{a},\boldsymbol{b}}(\tau) = \theta_{\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{0};\tau)$ is called the θ-constant with characteristic $\boldsymbol{a},\boldsymbol{b}$. We have

$$\begin{aligned}&\theta_{-\boldsymbol{a},-\boldsymbol{b}}(\boldsymbol{z};\tau) = \theta_{\boldsymbol{a},\boldsymbol{b}}(-\boldsymbol{z};\tau)\\ &\theta_{\boldsymbol{a}+\boldsymbol{p},\boldsymbol{b}+\boldsymbol{q}}(\boldsymbol{z};\tau) = \exp(2\pi\imath\boldsymbol{a}^T\boldsymbol{q})\theta_{\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{z};\tau)\end{aligned}$$

The following transformation formula is given in [Igu72, p85, p176].

Proposition A.1. *For any* $\mathfrak{g} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g,\mathbb{Z})$ *and* $(\boldsymbol{a},\boldsymbol{b}) \in \mathbb{Q}^{2g}$ *we put*

$$\mathfrak{g}\cdot(\boldsymbol{a},\boldsymbol{b}) = (\boldsymbol{a},\boldsymbol{b})\mathfrak{g}^{-1} + \frac{1}{2}(\mathrm{diag}(CD^T), \mathrm{diag}(AB^T))$$
$$\boldsymbol{\phi}_{\boldsymbol{a},\boldsymbol{b}}(\mathfrak{g}) = -\frac{1}{2}(\boldsymbol{a}D^TB\boldsymbol{a}^T - 2\boldsymbol{a}B^TC\boldsymbol{b}^T + \boldsymbol{b}C^TA\boldsymbol{b}^T) + \frac{1}{2}(\boldsymbol{a}D^T - \boldsymbol{b}C^T)^T\mathrm{diag}(AB^T),$$

where $\mathrm{diag}(A)$ *is the row vector consisting of the diagonal components of* A*. Then for every* $\mathfrak{g} \in \mathrm{Sp}(2g,\mathbb{Z})$ *we have*

(A.4)
$$\theta_{\mathfrak{g}\cdot(\boldsymbol{a},\boldsymbol{b})}(0; (A\tau_{\mathfrak{b}} + B)(C\tau_{\mathfrak{b}} + D)^{-1}) = \kappa(\mathfrak{g})\exp(2\pi\imath\boldsymbol{\phi}_{\boldsymbol{a},\boldsymbol{b}}(\mathfrak{g}))\det(C\tau_{\mathfrak{b}} + D)^{\frac{1}{2}}\theta_{(\boldsymbol{a},\boldsymbol{b})}(0;\tau_{\mathfrak{b}})$$

in which $\kappa(\mathfrak{g})^2$ *is a 4-th root of unity depending only on* $\mathfrak{g}$ *while*

(A.5)
$$\theta_{\mathfrak{g}\cdot(\boldsymbol{a},\boldsymbol{b})}(z(C\tau_{\mathfrak{b}} + D)^{-1}; (A\tau_{\mathfrak{b}} + B)(C\tau_{\mathfrak{b}} + D)^{-1}) = \mu\,\exp\left(i\pi z(C\tau_{\mathfrak{b}} + D)^{-1}Cz^T\right)\,\det(C\tau_{\mathfrak{b}} + D)^{\frac{1}{2}} \times \theta_{(\boldsymbol{a},\boldsymbol{b})}(z;\tau_{\mathfrak{b}})$$

and μ *is a complex number independent of* τ *and* z *such that* $|\mu| = 1$.

A.1. **The Vector of Riemann Constants.** The convention we adopt for our vector of Riemann constants is

$$\theta\left(\boldsymbol{\phi}(P) - \boldsymbol{\phi}(\sum_{i=1}^{g} Q_i) - K\right) = 0$$

in the Jacobi inversion. This is the convention used by Farkas and Kra and the negative of that of Mumford; the choice of signs appears in the actual construction of K, such as (2.4.1) of Farkas and Kra. Then

(A.6)
$$\begin{aligned}(K_Q)_j &= \frac{1}{2}\tau_{jj} - \sum_k \oint_{\mathfrak{a}_k} \omega_k(P)\int_Q^P \omega_j,\\ &= \frac{1}{2}\left(\tau_{jj} + 1\right) - \sum_{k\neq j}\oint_{\mathfrak{a}_k}\omega_k(P)\int_Q^P \omega_j.\end{aligned}$$

The vector of Riemann constants depends on the homology basis and base point Q. If we change base points of the Abel map $\boldsymbol{\phi}_Q \to \boldsymbol{\phi}_{Q'}$ then $K_Q = K_{Q'} + \boldsymbol{\phi}_{Q'}(Q^{g-1})$. With this convention

(A.7)
$$\boldsymbol{\phi}_Q(\mathrm{Div}(K_{\mathcal{C}})) = -2K_Q.$$

A.2. **Theta Characteristics.** The set Σ of divisor classes D such that $2D = K_{\mathcal{C}}$, the canonical class, is called the set of *theta characteristics* of $\mathcal{C}$. The set Σ is a principal homogeneous space for the group J_2, the group of 2-torsion points of the group $\mathrm{Pic}^0(\mathcal{C})$ of degree zero line bundles on $\mathcal{C}$. Equivalently this may be viewed as the 2-torsion points of the Jacobian, $J_2 = \frac{1}{2}\Lambda/\Lambda$. Geometrically if ξ is a holomorphic line bundle on $\mathcal{C}$ such that ξ^2 is holomorphically equivalent to $K_{\mathcal{C}}$ then the divisor of ξ is a theta characteristic. If L is a holomorphic line bundle of order 2, that is L^2 is holomorphically trivial, then the divisor of $\xi \otimes L$ is also a theta characteristic. Thus there are $|J_2| = 2^{2g}$ theta characteristics.

We may view $J_2 = \{v \in \mathrm{Pic}^0(\mathcal{C}) | 2v = 0\}$ as a vector space of dimension $2g$ over $\mathbb{F}_2$. This vector space has a nondegenerate symplectic (and hence symmetric as the field is $\mathbb{F}_2$) form defined by the Weil pairing. If D and E are divisors with disjoint support in the classes of u and v respectively, and $2D = \mathrm{div}(f)$, $2E = \mathrm{div}(g)$ then the Weil Pairing is

$$\lambda_2 : J_2 \times J_2 \to \mathbb{F}_2, \qquad \lambda_2(u,v) = \frac{g(D)}{f(E)},$$

where if $D = \sum_j n_j x_j$ then $g(D) = \prod_j g(x_j)^{n_j}$. Mumford identifies $\mathbb{F}_2$ with ± 1 by sending 0 to 1 and 1 to -1. (In general we may consider J_r, the r-torsion points of $\mathrm{Pic}^0(\mathcal{C})$, and the Weil pairing gives us a nondegenerate antisymmetric map $\lambda_2 : J_r \times J_r \to \mu_r$ where μ_r are r-th roots of unity.) The $\mathbb{F}_2$ vector space J_2 may be identified with $H^1(\mathcal{C}, \mathbb{F}_2)$ and with this identification λ_2 is simply the cup product.

Define $\omega_\xi : J_2 \to \mathbb{F}_2$ by

$$\omega_\xi(u) = \mathrm{Dim}\, H^0(\mathcal{C}, \xi \otimes u) - \mathrm{Dim}\, H^0(\mathcal{C}, \xi) \pmod 2, \tag{A.8}$$

where $u = L_D$ is the line bundle with divisor D. Then

$$\lambda_2(u,v) = \omega_\xi(u \otimes v) - \omega_\xi(u) - \omega_\xi(v).$$

Any function ω_ξ satisfying this identity is known as an Arf function, and any Arf function is given by ω_ξ for some theta characteristic with corresponding line bundle ξ. Thus the space of theta characteristics may be identified with the space of quadratic forms (A.8).

Appendix B. Integrals between branch points

We shall now describe how to integrate holomorphic differentials between branch points. We use the fact that for non-invariant holomorphic differentials (as we have)

$$\sum_{i=1}^{3} \int_{\gamma_i(\lambda_A,\lambda_B)} \omega = \int_{\lambda_A}^{\lambda_B} (\omega + R_*\omega + R_*^2\omega) = 0.$$

Indeed, if ω is any holomorphic differential on a compact Riemann surface which is an N-fold branched cover of $\mathbb{CP}^1$ then $\sum_{j=1}^{N} \omega(P^{(j)}) = 0$, where $P^{(j)}$ are the preimages of $P \in \mathbb{CP}^1$. Then

$$\oint_{\mathfrak{a}_1 - \mathfrak{b}_1} \omega = 3 \int_{\gamma_1(\lambda_1,\lambda_2)} \omega, \qquad \oint_{\mathfrak{a}_2 - \mathfrak{b}_2} \omega = 3 \int_{\gamma_1(\lambda_3,\lambda_4)} \omega, \qquad \oint_{\mathfrak{a}_3 - \mathfrak{b}_3} \omega = 3 \int_{\gamma_1(\lambda_5,\lambda_6)} \omega,$$

and consequently

$$\begin{aligned}
\int_{\gamma_1(\lambda_1,\lambda_2)} \omega &= \frac{1}{3} \oint_{\mathfrak{a}_1 - \mathfrak{b}_1} \omega, \\
\int_{\gamma_2(\lambda_1,\lambda_2)} \omega &= \int_{\gamma_1(\lambda_1,\lambda_2) - \mathfrak{a}_1} \omega = \frac{1}{3} \oint_{-2\mathfrak{a}_1 - \mathfrak{b}_1} \omega, \\
\int_{\gamma_3(\lambda_1,\lambda_2)} \omega &= \frac{1}{3} \oint_{2\mathfrak{b}_1 + \mathfrak{a}_1} \omega,
\end{aligned}$$

with similar expressions obtained for $\gamma_i(\lambda_3, \lambda_4)$ and $\gamma_i(\lambda_5, \lambda_6)$.

Further utilising $\gamma_1(\lambda_2,\lambda_6) = \gamma_1(\lambda_2,\lambda_1) + \gamma_1(\lambda_1,\lambda_6)$ and $\gamma_2(\lambda_5,\lambda_1) = \gamma_2(\lambda_5,\lambda_6) + \gamma_2(\lambda_6,\lambda_1)$ we may write

$$\begin{aligned}
\mathfrak{a}_4 &= \mathfrak{b}_1 - \mathfrak{b}_3 - \mathfrak{a}_3 + \gamma_1(\lambda_1,\lambda_6) + \gamma_2(\lambda_6,\lambda_1), \\
\mathfrak{b}_4 &= \mathfrak{a}_1 + \mathfrak{b}_1 - \mathfrak{a}_3 + \gamma_1(\lambda_1,\lambda_6) + \gamma_3(\lambda_6,\lambda_1).
\end{aligned}$$

Appropriate linear combinations of these yield $\int_{\gamma_i(\lambda_1,\lambda_6)}\omega$ for $i=1,2,3$. For example

$$\int_{\gamma_1(\lambda_1,\lambda_6)}\omega=\frac{1}{3}\oint_{2\mathfrak{a}_3-2\mathfrak{b}_1-\mathfrak{a}_1+\mathfrak{b}_3+\mathfrak{a}_4+\mathfrak{b}_4}\omega.$$

In order to be able to integrate a holomorphic differential between any branch point we must show how we may integrate such between λ_4 and λ_5 on any branch. Now we use that there exist meromorphic functions $f=w/(z-\lambda_1)^2$ and $g=(z-\lambda_i)/(z-\lambda_j)$ (for each i, j) with (respective) divisors

$$(f)=\lambda_2+\lambda_3+\lambda_4+\lambda_5+\lambda_6-5\lambda_1,\qquad (g)=3(\lambda_i-\lambda_j).$$

Thus for any normalized holomorphic differential $\boldsymbol{v}$

$$\Lambda\ni\int_{\lambda_1}^{\lambda_2}\boldsymbol{v}+\int_{\lambda_1}^{\lambda_3}\boldsymbol{v}+\int_{\lambda_1}^{\lambda_4}\boldsymbol{v}+\int_{\lambda_1}^{\lambda_5}\boldsymbol{v}+\int_{\lambda_1}^{\lambda_6}\boldsymbol{v}=4\int_{\lambda_1}^{\lambda_2}\boldsymbol{v}+3\int_{\lambda_2}^{\lambda_3}\boldsymbol{v}+2\int_{\lambda_3}^{\lambda_4}\boldsymbol{v}+\int_{\lambda_4}^{\lambda_5}\boldsymbol{v}+\int_{\lambda_1}^{\lambda_6}\boldsymbol{v},$$

and $3\int_{\lambda_j}^{\lambda_i}\boldsymbol{v}\in\Lambda$, where Λ is the period lattice. These equalities hold (modulo a lattice vector) for a path of integration on any branch and so, for example,

$$\int_{\gamma_1(\lambda_4,\lambda_5)}\boldsymbol{v}\equiv\int_{\gamma_1(\lambda_3,\lambda_4)}\boldsymbol{v}-\int_{\gamma_1(\lambda_1,\lambda_2)}\boldsymbol{v}-\int_{\gamma_1(\lambda_1,\lambda_6)}\boldsymbol{v}\qquad \bmod \Lambda.$$

Appendix C. Möbius Transformations

We wish to determine when there is a Möbius transformation between the sets $H=\{\alpha_1,-1/\overline{\alpha}_1,\alpha_2,-1/\overline{\alpha}_2,\alpha_3,-1/\overline{\alpha}_3\}$ and $S=\{0,1,\infty,\Lambda_1,\Lambda_2,\Lambda_3\}$. The former corresponds to reality constraints on our data arising from $(H1)$ while the latter may be constructed from the period matrix of the curve in terms of various theta constants. If we have a period matrix satisfying $(H2)$ then we must satisfy $(H1)$.

At the outset we note that the Möbius transformation M sending $a\to 0, b\to 1, c\to\infty$ and its inverse M^{-1}

$$\begin{array}{cc} M(a)=0 & M^{-1}(0)=a\\ M(b)=1 & M^{-1}(1)=b\\ M(c)=\infty & M^{-1}(\infty)=c\end{array}$$

are given by

$$M(z)=\frac{b-c}{b-a}\frac{z-a}{z-c}\qquad M^{-1}(z)=\frac{z\,c(b-a)-a(b-c)}{z(b-a)-(b-c)}. \tag{C.1}$$

The transformation

$$M(z)=\lambda\cdot\frac{z-a}{z-c}\qquad=\frac{\alpha z+\beta}{\gamma z+\delta}$$

may be represented by the $SL(2,\mathbb{Z})$ matrix

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}=\begin{pmatrix}\frac{i\sqrt{\lambda}}{\sqrt{a+c}} & -\frac{ia\sqrt{\lambda}}{\sqrt{a+c}}\\ \frac{i}{\sqrt{\lambda}\sqrt{a+c}} & -\frac{ic}{\sqrt{\lambda}\sqrt{a+c}}\end{pmatrix} \tag{C.2}$$

and upon setting $\lambda=(b-c)/(b-a)$ we may determine a $SL(2,\mathbb{Z})$ representation of (C.1).

A Möbius transformation is conjugate to a rotation if and only if it is of the form $M(z)=\dfrac{(\alpha z+\beta)}{(-\overline{\beta}Z+\overline{\alpha})}$. In terms of (C.2) this means

$$a\overline{c}=-1\qquad\text{and}\qquad \lambda\overline{\lambda}=\frac{1}{a\overline{a}}.$$

Then $M(0)\overline{M(\infty)}=-1$.

The rotation $\begin{pmatrix} \dfrac{\overline{\alpha}_1}{\sqrt{1+|\alpha_1|^2}} & \dfrac{1}{\sqrt{1+|\alpha_1|^2}} \\ \dfrac{-1}{\sqrt{1+|\alpha_1|^2}} & \dfrac{\alpha_1}{\sqrt{1+|\alpha_1|^2}} \end{pmatrix}$ transforms the set H to one of the form $\{0, \infty, \tilde{\alpha}_2, -1/\tilde{\alpha}_2, \tilde{\alpha}_3, -1/\tilde{\alpha}_3\}$ where $\tilde{\alpha}_r = M(\alpha_r) = (1+\overline{\alpha}_1\alpha_r)/(\alpha_1 - \alpha_r)$ $(r = 2, 3)$. Upon setting $\tilde{\alpha}_2 = ae^{i\theta}$, $a = |\tilde{\alpha}_2|$ the rotation $\begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}$ will transform the latter set to one of the form $\{0, \infty, a, -1/a, w, -1/\overline{w}\}$. Finally the scaling $z \to z/a$ given by $\begin{pmatrix} \dfrac{1}{\sqrt{a}} & 0 \\ 0 & \sqrt{a} \end{pmatrix}$ transforms H to $H_s = \{0, 1, \infty, -1/a^2, w/a, -1/(a\overline{w})\}$. Such a set is of the desired form S and is characterised by 3 (real) parameters. With $\Lambda_1 = -1/a^2$, $\Lambda_2 = w/a$, $\Lambda_3 = -1/a\overline{w}$ we see we have $\Lambda_1 \in \mathbb{R}$, $\Lambda_1 < 0$, $\Lambda_2\overline{\Lambda}_3 = \Lambda_1$. From a set H_s and a choice of θ and α_1 (equivalently, a rotation) we may reconstruct H.

More generally, let us consider images $M(H)$ under Möbius transformations. Up to a relabelling of roots we have four possibilities of those roots we map to $\{0, 1, \infty\}$:

$$\begin{array}{lccc} \text{a.} & \alpha_1 \to 0 & \alpha_2 \to 1 & -1/\overline{\alpha}_1 \to \infty, \\ \text{b.} & \alpha_1 \to 0 & -1/\overline{\alpha}_1 \to 1 & \alpha_2 \to \infty, \\ \text{c.} & \alpha_1 \to 0 & -1/\overline{\alpha}_2 \to 1 & \alpha_2 \to \infty, \\ \text{d.} & \alpha_1 \to 0 & \alpha_3 \to 1 & \alpha_2 \to \infty. \end{array}$$

We have already considered (a) in the previous paragraph. For completeness let us give $\Lambda_1, \Lambda_2, \Lambda_3$ for the various cases and the various restrictions arising

a.

$$\begin{aligned} \Lambda_1 = M(\frac{-1}{\overline{\alpha}_2}) &= -\frac{(1+\overline{\alpha}_1\alpha_2)(1+\overline{\alpha}_2\alpha_1)}{(\alpha_1-\alpha_2)(\overline{\alpha}_1-\overline{\alpha}_2)} < 0, \\ \Lambda_2 = M(\alpha_3) &= \frac{\alpha_1-\alpha_3}{\alpha_1-\alpha_2}\frac{1+\overline{\alpha}_1\alpha_2}{1+\overline{\alpha}_1\alpha_3}, \\ \Lambda_3 = M(\frac{-1}{\overline{\alpha}_3}) &= -\frac{1+\alpha_1\overline{\alpha}_3}{\overline{\alpha}_1-\overline{\alpha}_3}\frac{1+\alpha_2\overline{\alpha}_1}{\alpha_1-\alpha_2}, \\ \Lambda_2\overline{\Lambda}_3 &= \Lambda_1; \end{aligned} \tag{C.3}$$

b.

$$\begin{aligned} \Lambda_1 = M(\frac{-1}{\overline{\alpha}_2}) &= -\frac{1+\overline{\alpha}_1\alpha_2}{1+\alpha_1\overline{\alpha}_1}\cdot\frac{1+\alpha_1\overline{\alpha}_2}{1+\alpha_2\overline{\alpha}_2} \in \mathbb{R} \quad 0 < \Lambda_1 < 1, \\ \Lambda_2 = M(\alpha_3) &= \frac{1+\overline{\alpha}_1\alpha_2}{1+\alpha_1\overline{\alpha}_1}\frac{\alpha_3-\alpha_1}{\alpha_3-\alpha_2}, \\ \Lambda_3 = M(\frac{-1}{\overline{\alpha}_3}) &= \frac{1+\overline{\alpha}_1\alpha_2}{1+\alpha_1\overline{\alpha}_1}\frac{1+\alpha_1\overline{\alpha}_3}{1+\alpha_2\overline{\alpha}_3}, \\ \frac{\Lambda_2}{\Lambda_2-1}\overline{\left(\frac{\Lambda_3}{\Lambda_3-1}\right)} &= \frac{\Lambda_1}{\Lambda_1-1}; \end{aligned} \tag{C.4}$$

c.

$$\begin{aligned} \Lambda_1 = M(\frac{-1}{\overline{\alpha}_1}) &= -\frac{1+\alpha_2\overline{\alpha}_2}{1+\alpha_1\overline{\alpha}_2}\cdot\frac{1+\alpha_1\overline{\alpha}_1}{1+\alpha_2\overline{\alpha}_1} \in \mathbb{R} \quad 1 < \Lambda_1 < \infty, \\ \Lambda_2 = M(\alpha_3) &= \frac{1+\alpha_2\overline{\alpha}_2}{1+\alpha_1\overline{\alpha}_2}\frac{\alpha_3-\alpha_1}{\alpha_3-\alpha_2}, \end{aligned}$$

$$\Lambda_3 = M(\frac{-1}{\overline{\alpha}_3}) = \frac{1+\alpha_2\overline{\alpha}_2}{1+\alpha_1\overline{\alpha}_2}\frac{1+\alpha_1\overline{\alpha}_3}{1+\alpha_2\overline{\alpha}_3},$$

$$(1-\Lambda_2)\overline{(1-\Lambda_3)} = 1-\Lambda_1; \tag{C.5}$$

d.

$$\Lambda_r = M(-\frac{1}{\overline{\alpha}_r}) = \frac{\alpha_3-\alpha_2}{\alpha_3-\alpha_1}\frac{1+\alpha_1\overline{\alpha}_r}{1+\alpha_2\overline{\alpha}_r}, \qquad r=1,2,3,$$

$$0<\Lambda_1\overline{\Lambda}_2\in\mathbb{R},\ 1<\frac{\Lambda_1}{\Lambda_2}\in\mathbb{R}, \qquad \Lambda_3 = \Lambda_2\frac{(1-\overline{\Lambda}_1)}{1-\overline{\Lambda}_2}. \tag{C.6}$$

The constraints (C.4) for case (b) may be obtained as follows. Further composing the Möbius transformation leading to (b) with that giving $0\to 0$, $1\to\infty$, $\infty\to 1$ gives us case (a) for which we know the constraint. This second Möbius transformation is given by $M(z) = M^{-1}(z) = z/(z-1)$ and we may transfer the constraint of (a) to (b). Similarly composing (c) with $M(z) = 1-z$ yields case (a) up to a relabelling of roots. Geometrically cases (a), (b), (c) consist of the following. A circle passes through $\{\alpha_1, -1/\overline{\alpha}_1, \alpha_2, -1/\overline{\alpha}_2\}$. Under a Möbius transformation to the set $\{0,1,\infty,\mu\}$ the circle becomes the real axis and so $\mu\in\mathbb{R}$. This is the real parameter appearing in each of these cases. A similar argument composing (d) with $M(z) = z/(z-\Lambda_1)$ will give the constraints (C.6).

In each case, given α, and a choice of θ (a rotation) we can construct S from H.

References

[Bak95] H.F. Baker, *Abel's theorem and the allied theory of theta functions*, Cambridge University Press, Cambridge, 1897, reprinted 1995.

[BE55] H. Bateman and A. Erdelyi, *Higher Transcendental Functions*, vol. 1, McGraw-Hill, New York, 1955.

[Bel60] G. Belardinelli, *Fonctions hypergéométriques de plusieurs variables et résolution analytique des équations algébraiques générales*, Mémorial des sciences mathématiques **CXLV** (1960), 1–74.

[BE01] E. D. Belokolos and V. Z. Enolskii, *Reduction of Abelian Functions and Algebraically Integrable Systems, part I*, Journal of Mathematical Sciences **106** (2001), no. 6, 3395 –3486.

[BE02] ———, *Reduction of Abelian Functions and Algebraically Integrable Systems, part II*, Journal of Mathematical Sciences **108** (2002), no. 3, 295–374.

[BBG95] B. C. Berndt, S. Bhargava, and F. G. Garvan, *Rananujans's theories of elliptic functions to alternative bases*, Trans. Amer. Math. Soc. **347** (1995), no. 11, 4163–4244.

[Ber98] Bruce C. Berndt, *Ramanujan's Notebooks Part V*, Springer-Verlag, New York, 1998.

[BB87] J. M. Borwein and P. B. Borwein, *Explicit Ramanujan-type approximations to pi of high order*, Proc. Indian Acad. Sci. (Math. Sci.) **97** (1987), 53–59.

[Bur93] H. Burkhardt, *Ueber die Darstellung einiger Falle der automorphen Primformendurch specielle Thetareihen*, Math. Ann. **42** (1893), 185–214.

[Cha98] Heng Huat Chan, *On Ramanujan's cubic transformation formula for* ${}_2F_1(\frac{1}{3},\frac{2}{3};1;z)$, Math. Proc. Camb. Phil. Soc. **124** (1998), 193–204.

[ClG66] A. Clebsch and P. Gordan, *Theorie der abelschen Funktionen*, Teubner, Leipzig, 1866.

[CG81] E. Corrigan and P. Goddard, *An n- Monopole Solution with* $4n-1$ *Degrees of Freedom*, Commun. Math. Phys. **80** (1981), 575–587.

[Die91] G. G. Diez, *Loci of curves which are prime Galois coverings of* $\mathbb{P}^1$, Proc. London Math. Soc. **62** (1991), 469–489.

[EG04] V. Z. Enolski and T. Grava, *Singular* Z_N*-Curves and the Riemann-Hilbert Problem*, Int. Math. Res. Notices **32** (2004), 1619–1683.

[ES89] N. Ercolani and A. Sinha, *Monopoles and Baker Functions*, Commun. Math. Phys. **125** (1989), 385–416.

[FK80] H. M. Farkas and I. Kra, *Riemann Surfaces*, Springer-Verlag, New York, 1980.

[Fay73] J. D. Fay, *Theta functions on Riemann surfaces*, Lectures Notes in Mathematics (Berlin), vol. 352, Springer, 1973.

[Fed99] Yu. Fedorov, *Classical Integrable Systems and Billiards Related to Generalized Jacobians*, Acta Applicandae Mathematicae **55** (1999), 251–301.

[Gou81] E. Goursat, *Sur L'équation différentielle linéaire qui admet pour intégrale la série hyperegéométrique*, Ann. Sci. École Norm. Sup. **2(10)** (1881), 3–142.

[Hit82] N. J. Hitchin, *Monopoles and Geodesics*, Commun. Math. Phys. **83** (1982), 579–602.

[Hit83] ———, *On the Construction of Monopoles*, Commun. Math. Phys. **89** (1983), 145–190.

[HMM95] N. J. Hitchin, N. S. Manton and M. K. Murray, *Symmetric monopoles*, Nonlinearity **8** (1995), 661–692.

[Hop31] H. Hopf, *Beiträge zur Klassifisierung der Flächenabbildungen*, J. reine angew. Math. **165** (1931), 225–236.

[HMR00] C. J. Houghton, N. S. Manton, and N. M. Romão, *On the constraints defining BPS monopoles*, Commun. Math. Phys. **212** (2000), 219–243. arXiv: hep-th/9909168, 1999.

[Hur83] Jacques Hurtubise, SU(2) *monopoles of charge* 2, Commun. Math. Phys. **92** (1983), 195–202.

[Hut02] J. I. Hutchinson, *On a class of automorphic functions*, Trans. Amer. Math. Soc. **3** (1902), 1–11.

[Igu72] J. Igusa, *Theta Functions, Grund. Math. Wiss.*, vol. 194, Springer, Berlin, 1972.

[Kr77] I. M. Krichever, *Integration of non-linear equations by algebraic-geometrical methods*, Func. Anal. Pril. **11** (1977) 15-31.

[Law89] D. F. Lawden, *Elliptic Functions and applications*, Applied Mathematical Sciences, vol. 80, Springer, New York, 1989.

[MS04] Nicholas Manton and Paul Sutcliffe, *Topological Solitons*, Cambridge University Press, Cambridge 2004.

[Mar92a] H. H. Martens, *A footnote to the Poincaré complete reducibility theorem*, Publications Matematiques **36** (1992), 111–129.

[Mar92b] H. H. Martens, *On the Reduction of Abelian Integrals and a Problem of H.Hopf*, Contemporary Mathematics **136** (1992), 287–296.

[Mat01] K. Matsumoto, *Theta constants associated with the cyclic triple coverings of the complex projective line branching at six points*, Publ. Res. Inst. Math. Sci **37** (2001), 419–440, arXiv: math.AG/0008025.

[Nah82] W. Nahm, *The construction of all self-dual multimonopoles by the ADHM method*, in Monopoles in Quantum Field Theory, edited by N.S. Craigie, P. Goddard and W. Nahm (World Scientific, Singapore 1982).

[Pic83] E. Picard, *Sur les fonctions de deux variables indépendentes analogues aux fonctions modulaires*, Acta Math. **2** (1883), 114–135.

[Shi88] H. Shiga, *On the representation of the Picard modular function by θ constants I-II*, Publ. RIMS, Kyoto Univ. **24** (1988), 311–360.

[TT84] C. L. Tretkoff and M. D. Tretkoff, *Combinatorial Group Theory, Riemann Surfaces and Differential Equations*, Contemporary Mathematics **33** (1984) 467-519.

[Wel99] J. Wellstein, *Zur Theorie der Functionenclasse* $s^3 = (z-\alpha_1)(z-\alpha_2)\dots(z-\alpha_6)$, Math. Ann. **52** (1899), 440–448.

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