SHARP ESTIMATE OF THE SPREADING SPEED DETERMINED BY NONLINEAR FREE BOUNDARY PROBLEMS

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ABSTRACT. We study nonlinear diffusion problems of the form $u_t = u_{xx} + f(u)$ with free boundaries. Such problems may be used to describe the spreading of a biological or chemical species, with the free boundaries representing the expanding fronts. For monostable, bistable and combustion types of nonlinearities, Du and Lou [7] obtained rather complete description of the long-time dynamical behavior of the problem and revealed sharp transition phenomena between spreading ($\lim_{t\to\infty} u(t,x) = 1$) and vanishing($\lim_{t\to\infty} u(t,x) = 0$). They also determined the asymptotic spreading speed of the fronts by making use of semi-waves when spreading happens. In this paper, we give a much sharper estimate for the spreading speed of the fronts than that in [7], and describe how the solution approaches the semi-wave when spreading happens.

1. Introduction and Main Results

We are interested in obtaining sharp estimates for the spreading speed determined by the following free boundary problem:

$$
(1.1)\qquad \begin{cases} u_t - u_{xx} = f(u), & t > 0,\ g(t) < x < h(t), \\ u(t, g(t)) = u(t, h(t)) = 0, & t > 0, \\ g'(t) = -\mu u_x(t, g(t)) & t > 0, \\ h'(t) = -\mu u_x(t, h(t)), & t > 0, \\ -g(0) = h(0) = h_0, u(0, x) = u_0(x), & -h_0 \le x \le h_0, \end{cases}
$$

where $x = h(t)$ and $x = g(t)$ are the moving boundaries to be determined together with $u(t, x)$, μ is a given positive constant, $f : [0, \infty) \to \mathbb{R}$ is C^1, $f(0) = 0$ and is of monotstable, or bistable, or of combustion type. The initial function u_0 belongs to $\mathscr{X}(h_0)$ for some $h_0 > 0$, where

$$
\mathscr{X}(h_0) := \{\phi \in C^2[-h_0, h_0] : \ \phi(-h_0) = \phi(h_0) = 0,\ \phi'(-h_0) > 0, \\ \phi'(h_0) < 0, \phi(x) > 0 \text{ in } (-h_0, h_0)\}.
$$

For any $h_0 > 0$ and $u_0 \in \mathscr{X}(h_0)$, a triple $(u(t, x), g(t), h(t))$ is a (classical) solution to (1.1) for $0 < t \le T$ if it belongs to $C^{1,2}(G_T) \times C^1[0, T] \times C^1[0, T]$ and all the identities in (1.1) are satisfied pointwisely, where

$$
G_T := \{(t, x)|t \in (0, T], x \in [g(t), h(t)]\}.
$$

Problem (1.1) with $f(u) = au - bu^2$ was introduced by Du and Lin [6] to describe the spreading of a new or invasive species. The free boundaries $x = g(t)$ and $x = h(t)$ represent the spreading

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fronts of the population whose density is represented by $u(t,x)$. The results in [6] were extended by Du and Guo [5] to higher dimensions in a radially symmetric setting. A deduction of the free boundary condition based on ecological assumptions can be found in [4].

Problem (1.1) with a rather general $f(u)$ was recently studied by Du and Lou [7]. In particular, if $f(u)$ is monostable, or bistable, or of combustion type, it was shown in [7] that (1.1) has a unique solution which is defined for all $t>0$, and as $t\to\infty$, the interval $(g(t),h(t))$ converges either to a finite interval (g_∞,h_∞), or to $(-\infty,+\infty)$. Moreover, in the former case, $u(t,x)\to 0$ uniformly in x, while in the latter case, $u(t,x)\to 1$ locally uniformly in $x\in(-\infty,+\infty)$ (except for a non-generic transition case when f is of bistable or combustion type). The situation that

$$u\to 0 \text{ and } (g,h)\to(g_\infty,h_\infty)$$

is called the **vanishing** case, and

$$u\to 1 \text{ and } (g,h)\to(-\infty,+\infty)$$

is called the **spreading** case.

When spreading happens, it is shown in [7] that there exists $c^*>0$ such that

$$\lim_{t\to\infty}\frac{-g(t)}{t}=\lim_{t\to\infty}\frac{h(t)}{t}=c^*.$$

The number c^* is therefore called the asymptotic spreading speed determined by (1.1).

The main purpose of this paper is to obtain a much better estimate for $g(t)$ and $h(t)$ for large t when spreading happens, and at the same time obtain a much better understanding of the behavior of $u(t,x)$ as $t\to\infty$.

Before describing our main results, let us be more precise about the three types of nonlinearities of f mentioned above:

$$(\mathrm{f_M})\ \text{ monostable case, }\ (\mathrm{f_B})\ \text{ bistable case, }\ (\mathrm{f_C})\ \text{ combustion case.}$$

In the monostable case $(\mathrm{f_M})$, we assume that f is C^1 and it satisfies

$$f(0)=f(1)=0,\ f'(0)>0,\ f'(1)<0,\ (1-u)f(u)>0 \text{ for } u>0, u\neq 1.$$

A typical example is $f(u)=u(1-u)$.

In the bistable case $(\mathrm{f_B})$, we assume that f is C^1 and it satisfies

$$\begin{cases} f(0)=f(\theta)=f(1)=0,\\ f(u)<0 \text{ in } (0,\theta),\ f(u)>0 \text{ in } (\theta,1),\ f(u)<0 \text{ in } (1,\infty),\end{cases}$$

for some $\theta\in(0,1)$, $f'(0)<0$, $f'(1)<0$ and

$$\int_0^1 f(s)ds>0.$$

A typical example is $f(u)=u(u-\theta)(1-u)$ with $\theta\in\left(0,\frac{1}{2}\right)$.

In the combustion case $(\mathrm{f_C})$, we assume that f is C^1 and it satisfies

$$f(u)=0 \text{ in } [0,\theta],\quad f(u)>0 \text{ in } (\theta,1),\ f'(1)<0,\ f(u)<0 \text{ in } [1,\infty)$$

for some $\theta\in(0,1)$, and there exists a small $\delta_0>0$ such that

$$f(u) \text{ is nondecreasing in } (\theta,\theta+\delta_0).$$

The asymptotic spreading speed c^* mentioned above is determined by the following problem,

$$\begin{cases} q''-cq'+f(q)=0 \ \text{ in } (0,\infty),\\ q(0)=0, q(\infty)=1, q(z)>0 \ \text{ in } \ (0,\infty).\end{cases} \tag{1.2}$$

Proposition 1.1 (Proposition 1.8 and Theorem 6.2 of [7])**.** *Suppose that f is of* (f_M)*, or* (f_B)*, or* (f_C) *type. Then for any $\mu > 0$ there exists a unique $c^* = c^*_\mu > 0$ and a unique solution q_{c^*} to* (1.2) *with $c = c^*$ such that $q'_{c^*}(0) = \frac{c^*}{\mu}$.*

We remark that this function q_{c^*} is shown in [7] to satisfy $q'_{c^*}(z) > 0$ for $z \geq 0$. We call q_{c^*} a *semi-wave with speed* c^*, since the function $v(t,x) := q_{c^*}(c^*t - x)$ satisfies

$$\begin{cases} v_t = v_{xx} + f(v) \text{ for } t \in \mathbb{R}^1, x < c^*t, \\ v(t, c^*t) = 0,\ v_x(t, c^*t) = c^*,\ v(t, -\infty) = 1. \end{cases}$$

Our main result is the following theorem.

Theorem 1.2. *Assume that f is of* (f_M)*, or* (f_B)*, or* (f_C) *type and (u, g, h) is the unique solution to* (1.1) *for which spreading happens. Let (c^*, q_{c^*}) be given by Proposition 1.1. Then there exist $\hat{H}, \hat{G} \in \mathbb{R}$ such that*

$$\lim_{t\to\infty} (h(t) - c^*t - \hat{H}) = 0,\ \lim_{t\to\infty} h'(t) = c^*,$$

$$\lim_{t\to\infty} (g(t) + c^*t - \hat{G}) = 0,\ \lim_{t\to\infty} g'(t) = -c^*,$$

and

$$\lim_{t\to\infty} \sup_{x\in[0,\, h(t)]} |u(t,x) - q_{c^*}(h(t) - x)| = 0, \tag{1.3}$$

$$\lim_{t\to\infty} \sup_{x\in[g(t),\, 0]} |u(t,x) - q_{c^*}(x - g(t))| = 0. \tag{1.4}$$

We would like to remark that while problem (1.1) is relatively new, the corresponding Cauchy problem

$$\begin{cases} u_t = u_{xx} + f(u), & x \in \mathbb{R}^1, t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^1 \end{cases} \tag{1.5}$$

has a long history and has been extensively studied. For example, the classical paper of Aronson and Weinberger [1] contains a systematic investigation of this problem (and [2] contains its higher dimensional extension). Various sufficient conditions for $\lim_{t\to\infty} u(t,x) = 1$ ("spreading" or "propagation") and for $\lim_{t\to\infty} u(t,x) = 0$ ("vanishing" or "extinction") are known, and when u_0 is nonnegative and has compact support, the way $u(t,x)$ approaches 1 as $t \to \infty$ has been used to describe the spreading of a (biological or chemical) species, which is characterized by certain travelling waves, and the speed of these traveling waves determines the asymptotic spreading speed of the species; see for example [1, 2, 8, 9, 11, 12, 14]. The relationship between the spreading speed determined by (1.1) and that determined by (1.5) is given in Theorem 6.2 of [7].

We remark that at the level of accuracy for the spreading speed considered here, the theory for the free boundary model (1.1) and that for the Cauchy problem (1.5) exhibit some sharp differences. While all three basic cases (f_M), (f_B) and (f_C) can by covered in a unified fashion for the free boundary model (see Theorem 1.2 above), this is not the case for the Cauchy problem.

A classical result of Fife and McLeod [8] shows that for f of type (f_B), and for appropriate initial function u_0, the solution u to (1.5) satisfies

$$|u(t,x) - U(x - ct - x_0)| < Ke^{-\omega t} \text{ for } x < 0,$$

$$|u(t,x) - U(-x - ct - x_1)| < Ke^{-\omega t} \text{ for } x > 0.$$

Here $U(x)$ is the unique traveling wave solution (with speed c, and $U(0) = 1/2$), $x_0, x_1 \in \mathbb{R}$, and K, ω are suitable positive constants. More precisely, (U, c) is the unique solution of

$$U'' - cU' + f(U) = 0 \text{ in } \mathbb{R},\ U(-\infty) = 1,\ U(+\infty) = 0,\ U(0) = 1/2. \tag{1.6}$$

In the monostable case, significant differences arise. First, for such f, (1.6) has multiple solutions: There exists $c_0 > 0$ such that (1.6) has a unique solution U_c for every $c \geq c_0$, and it has no solution for $c < c_0$ (see [2]). Second, there is an essential difference on how the solution of (1.5) approaches the traveling waves: When ($\mathrm{f_M}$) holds and furthermore $f(u) \leq f'(0)u$ for $u \in (0,1)$, there exist a constant $C > 0$ and functions $\xi_\pm : \mathbb{R} \to [-C, C]$ such that

$$\lim_{t\to\infty} \|u(t,\cdot) - U_{c_0}(\cdot - c_0 t + \frac{3}{c_0}\ln t + \xi_+(t))\|_{L^\infty(\mathbb{R}_+)} = 0,$$

and

$$\lim_{t\to\infty} \|u(t,\cdot) - U_{c_0}(c_0 t - \frac{3}{c_0}\ln t - \xi_-(t) - \cdot)\|_{L^\infty(\mathbb{R}_-)} = 0.$$

The term $\frac{3}{c_0}\ln t$ is known as the logarithmic Bramson correction; see [3, 10, 13, 14] for more details.

2. Some Basic and Known Results

In this section we give some basic and known results which will be frequently used later. The first two results are for $f(u)$ more general than the three types of nonlinearities in Theorem 1.2. They only require

$$f \text{ is } C^1 \text{ and } f(0) = 0. \tag{2.1}$$

Lemma 2.1 (Lemma 2.2 of [7]). *Suppose that* (2.1) *holds,* $T \in (0,\infty)$, $\overline{g}$, $\overline{h} \in C^1[0,T]$, $\overline{u} \in C(\overline{D}_T) \cap C^{1,2}(D_T)$ *with* $D_T = \{(t,x) \in \mathbb{R}^2 : 0 < t \leq T,\ \overline{g}(t) < x < \overline{h}(t)\}$, *and*

$$\begin{cases} \overline{u}_t \geq \overline{u}_{xx} + f(\overline{u}), & 0 < t \leq T, \overline{g}(t) < x < \overline{h}(t), \\ \overline{u} \geq u, & 0 < t \leq T, x = \overline{g}(t), \\ \overline{u} = 0,\ \overline{h}'(t) \geq -\mu \overline{u}_x, & 0 < t \leq T, x = \overline{h}(t). \end{cases}$$

If

$$\overline{g}(t) \geq g(t) \text{ in } [0,T], h_0 \leq \overline{h}(0),\ u_0(x) \leq \overline{u}(0,x) \text{ in } [\overline{g}(0), h_0],$$

where (u,g,h) *is a solution to* (1.1), *then*

$$\begin{aligned} &h(t) \leq \overline{h}(t) \text{ in } (0,T], \\ &u(t,x) \leq \overline{u}(t,x) \text{ for } t \in (0,T] \text{ and } \overline{g}(t) < x < \overline{h}(t). \end{aligned}$$

The function $\overline{u}$, or the triple $(\overline{u}, \overline{g}, \overline{h})$ in Lemma 2.1 is usually called an upper solution of (1.1). We can define a lower solution by reversing the inequalities in the obvious places. There is a symmetric version of Lemma 2.1, where the conditions on the left and right boundaries are interchanged. We also have corresponding comparison results for lower solutions in each case.

Lemma 2.2 (Lemma 2.6 of [7]). *Suppose that* (2.1) *holds,* (u,g,h) *is a solution to* (1.1) *defined for* $t \in [0,T_0)$ *for some* $T_0 \in (0,\infty)$, *and there exists* $C_1 > 0$ *such that*

$$u(t,x) \leq C_1 \text{ for } t \in [0,T_0) \text{ and } x \in [g(t), h(t)].$$

Then there exists C_2 *depending on* C_1 *but independent of* T_0 *such that*

$$-g'(t),\ h'(t) \in (0, C_2] \text{ for } t \in (0, T_0).$$

Moreover, the solution can be extended to some interval $(0,T)$ *with* $T > T_0$.

Lemma 2.3 (Lemma 6.5 of [7]). *Suppose that f is of* (f_M)*, or* (f_B)*, or* (f_C) *type. Let (u, g, h) be the unique solution of* (1.1) *for which spreading happens. For any $c \in (0, c^*)$ there exist $\delta \in (0, -f'(1))$, $T^* > 0$ and $M > 0$ such that for $t \geq T^*$,*

$$[g(t), h(t)] \supset [-ct, ct], \tag{2.2}$$

$$u(t,x) \geq 1 - Me^{-\delta t} \text{ for } x \in [-ct, ct], \tag{2.3}$$

$$u(t,x) \leq 1 + Me^{-\delta t} \text{ for } x \in [g(t), h(t)]. \tag{2.4}$$

3. Proof of Theorem 1.2

Throughout this section we assume that f is of type (f_M), or (f_B), or (f_C) and (u, g, h) is a solution to (1.1) for which spreading happens. Our proof is divided into three parts, each consisting of a subsection. In part 1, we show that $|g(t) + c^*t|$ and $|h(t) - c^*t|$ are both bounded for all $t > 0$. This is achieved by constructing suitable upper and lower solutions. In part 2, we show that along any sequence $t_n \to \infty$, there is a subsequence $\{\tilde{t}_n\}$ and a constant $\hat{H} \in \mathbb{R}$ such that $h(\tilde{t}_n + \cdot) - c^*(\tilde{t}_n + \cdot) \to \hat{H}$ in $C^1_{loc}(\mathbb{R})$ and $u(\tilde{t}_n, z + c^*\tilde{t}_n) \to q_{c^*}(\hat{H} - z)$. This is a crucial technical step and relies on an energy argument and various parabolic estimates. The proof of Theorem 1.2 is completed in part 3, by constructing finer upper and lower solutions based on the result in part 2. Our approach in parts 2 and 3 is motivated by the method of Fife and McLeod [8].

3.1. Bound for $|g(t) + c^*t|$ and $|h(t) - c^*t|$.

Proposition 3.1. *There exists $C > 0$ such that*

$$|g(t) + c^*t|, |h(t) - c^*t| \leq C \textit{ for all } t > 0. \tag{3.1}$$

We will show (3.1) for $h(t)$ only, since the proof for $g(t)$ is similar. Our arguments are based on the construction of suitable upper and lower solutions.

Fix $c \in (0, c^*)$. From Lemma 2.3, there exist $\delta \in (0, -f'(1))$, $M > 0$ and $T^* > 0$ such that for $t \geq T^*$, (2.2), (2.3) and (2.4) hold. Since $0 < \delta < -f'(1)$ we can find some $\eta > 0$ such that

$$\begin{cases} \delta \leq -f'(u) & \text{for } 1 - \eta \leq u \leq 1 + \eta, \\ f(u) \geq 0 & \text{for } 1 - \eta \leq u \leq 1. \end{cases} \tag{3.2}$$

By enlarging T^* we may assume that

$$Me^{-\delta T^*} < \eta/2.$$

We take $M' > M$ satisfying

$$M'e^{-\delta T^*} \leq \eta.$$

Since $q_{c^*}(z) \to 1$ as $z \to 1$, we can find $X_0 > 0$ such that

$$(1 + M'e^{-\delta T^*})q_{c^*}(X_0) \geq 1 + Me^{-\delta T^*}. \tag{3.3}$$

We now construct an upper solution $(\overline{u}, \overline{g}, \overline{h})$ to (1.1) as follows:

$$\begin{aligned} &\overline{g}(t) := g(t) \\ &\overline{h}(t) := c^*(t - T^*) + \sigma M'(e^{-\delta T^*} - e^{-\delta t}) + h(T^*) + X_0, \\ &\overline{u}(t,x) := (1 + M'e^{-\delta t})q_{c^*}(\overline{h}(t) - x), \end{aligned}$$

where $\sigma > 0$ is a positive constant to be determined.

Lemma 3.2. *For sufficiently large $\sigma > 0$, $u(t,x)$ and $h(t)$ satisfy*

$$u(t,x) \le \overline{u}(t,x) \text{ for } x \in [g(t),h(t)],\ t \ge T^*,$$
$$h(t) \le \overline{h}(t) \text{ for } t \ge T^*.$$

Proof. We check that $(\overline{u},\overline{g},\overline{h})$ is an upper solution for $t > T^*$, that is,

$$\overline{u}_t - \overline{u}_{xx} \ge f(\overline{u}) \quad \text{for} \quad t > T^*, \overline{g}(t) < x < \overline{h}(t), \tag{3.4}$$

$$\overline{u} \ge u \text{ for } \ t \ge T^*,\ x = \overline{g}(t), \tag{3.5}$$

$$\overline{u} = 0,\ \overline{h}'(t) \ge -\mu \overline{u}_x(t,x) \quad \text{for} \quad t \ge T^*,\ x = \overline{h}(t), \tag{3.6}$$

$$h(T^*) \le \overline{h}(T^*), u(T^*,x) \le \overline{u}(T^*,x) \quad \text{for} \quad x \in [\overline{g}(T^*), h(T^*)]. \tag{3.7}$$

Clearly $\overline{u}$ satisfies (3.5) since $u(t,\overline{g}(t)) = u(t,g(t)) = 0$. We now show (3.6). It is obvious that $\overline{u}$ satisfies $\overline{u}(t,\overline{h}(t)) = 0$. Direct computations yield that

$$\overline{h}'(t) = c^* + \sigma M' \delta e^{-\delta t}$$

and

$$-\mu \overline{u}_x(t,\overline{h}(t)) = \mu(1 + M'e^{-\delta t}) q_{c^*}'(0) = \mu(1 + M'e^{-\delta t}) \frac{c^*}{\mu} = (1 + M'e^{-\delta t})c^*.$$

Hence (3.6) holds for $\sigma > 0$ satisfying $c^* \le \sigma\delta$.

Next we show (3.7). From the definition of $\overline{h}$ we see that $h(T^*) \le \overline{h}(T^*)$. By (2.4) and the choice of X_0 in (3.3) we have

$$\begin{aligned} \overline{u}(T^*,x) &= (1 + M'e^{-\delta T^*}) q_{c^*}(\overline{h}(T^*) - x) \\ &= (1 + M'e^{-\delta T^*}) q_{c^*}(h(T^*) + X_0 - x) \\ &\ge (1 + M'e^{-\delta T^*}) q_{c^*}(X_0) \\ &\ge 1 + Me^{-\delta T^*} \ge u(T^*,x) \end{aligned}$$

for $x \in [\overline{g}(T^*), h(T^*)]$. Thus (3.7) holds.

Finally we show that (3.4) holds for sufficiently large $\sigma > 0$. Put $z = \overline{h}(t) - x$. Since

$$\begin{aligned} \overline{u}_t &= -\delta M'e^{-\delta t} q_{c^*}(z) + (1 + M'e^{-\delta t}) \overline{h}'(t) q_{c^*}'(z) \\ &= -\delta M'e^{-\delta t} q_{c^*}(z) + (1 + M'e^{-\delta t})(c^* + \sigma M'\delta e^{-\delta t}) q_{c^*}'(z), \end{aligned}$$

and

$$\overline{u}_{xx} = (1 + M'e^{-\delta t}) q_{c^*}''(z),$$

we have

$$\begin{aligned} &\overline{u}_t - \overline{u}_{xx} - f(\overline{u}) \\ =& -\delta M'e^{-\delta t} q_{c^*}(z) + (1 + M'e^{-\delta t})(c^* + \sigma M'\delta e^{-\delta t}) q_{c^*}'(z) \\ &\quad - (1 + M'e^{-\delta t}) q_{c^*}''(z) - f((1 + M'e^{-\delta t}) q_{c^*}(z)) \\ =& -\delta M'e^{-\delta t} q_{c^*}(z) + (1 + M'e^{-\delta t}) \left\{ -q_{c^*}''(z) + c^* q_{c^*}'(z) \right\} \\ &\quad + \sigma M'\delta e^{-\delta t}(1 + M'e^{-\delta t}) q_{c^*}'(z) - f((1 + M'e^{-\delta t}) q_{c^*}(z)) \\ =& -\delta M'e^{-\delta t} q_{c^*}(z) + \sigma M'\delta e^{-\delta t}(1 + M'e^{-\delta t}) q_{c^*}'(z) \\ &\quad + (1 + M'e^{-\delta t}) f(q_{c^*}(z)) - f((1 + M'e^{-\delta t}) q_{c^*}(z)). \end{aligned}$$

Now we consider the term $(1+M'e^{-\delta t})f(q_{c^*}(z)) - f((1+M'e^{-\delta t})q_{c^*}(z))$. Denote

$$F(\xi,u) := (1+\xi)f(u) - f((1+\xi)u).$$

The mean value theorem yields

$$F(\xi,u) = \xi f(u) + f(u) - f((1+\xi)u) = \xi f(u) - \xi f'(u+\theta_{\xi,u}\xi u)u$$

for some $\theta_{\xi,u} \in (0,1)$. Since $q_{c^*}(z) \to 1$ as $z \to \infty$, there exists $z_\eta > 0$ such that $q_{c^*}(z) \geq 1-\eta$ for $z \geq z_\eta$.

For $\overline{h}(t) - x \geq z_\eta$, we have

$$\begin{aligned}
&\overline{u}_t - \overline{u}_{xx} - f(\overline{u})\\
=&\ -\delta M'e^{-\delta t}q_{c^*}(z) + \sigma M'\delta e^{-\delta t}(1+M'e^{-\delta t})q'_{c^*}(z) + F(M'e^{-\delta t}, q_{c^*}(z))\\
=&\ \sigma M'\delta e^{-\delta t}(1+M'e^{-\delta t})q'_{c^*}(z) + M'e^{-\delta t}f(q_{c^*}(z))\\
&\quad + M'e^{-\delta t}q_{c^*}(z)\Big\{-f'\big(q_{c^*}(z) + \theta' M'e^{-\delta t}q_{c^*}(z)\big) - \delta\Big\}\\
\geq&\ 0,
\end{aligned}$$

where $\theta' = \theta'(t,z) \in (0,1)$, and we have used $M'e^{-\delta t} \leq \eta$ for $t \geq T^*$ and (3.2).

On the other hand for $0 \leq \overline{h}(t) - x \leq z_\eta$, we obtain

$$\begin{aligned}
&\overline{u}_t - \overline{u}_{xx} - f(\overline{u})\\
=&\ M'e^{-\delta t}f(q_{c^*}(z)) + \sigma M'\delta e^{-\delta t}(1+M'e^{-\delta t})q'_{c^*}(z)\\
&\quad + M'e^{-\delta t}\Big\{-f'\big(q_{c^*}(z) + \theta' M'e^{-\delta t}q_{c^*}(z)\big) - \delta\Big\}q_{c^*}(z)\\
\geq&\ M'e^{-\delta t}\min_{0\leq s\leq 1} f(s) + \sigma\delta M'e^{-\delta t}Q_\eta - M'e^{-\delta t}\left(\max_{0\leq s\leq 1+\eta} f'(s) + \delta\right)\\
=&\ M'e^{-\delta t}\left\{\min_{0\leq s\leq 1} f(s) - \max_{0\leq s\leq 1+\eta} f'(s) - \delta + \sigma\delta Q_\eta\right\},
\end{aligned}$$

where $Q_\eta := \min_{0\leq z\leq z_\eta} q'_{c^*}(z) > 0$. Thus $\overline{u}_t - \overline{u}_{xx} - f(\overline{u}) \geq 0$ for sufficiently large $\sigma > 0$.

We may now apply Lemma 2.1 to conclude that

$$u(t,x) \leq \overline{u}(t,x),\quad h(t) \leq \overline{h}(t)\quad \text{for } t \geq T^* \text{ and } x \in [\overline{g}(t), h(t)].$$

This completes the proof of the lemma. □

Next we bound u and h from below by constructing a lower solution $(\underline{u}, \underline{g}, \underline{h})$ to (1.1). For η given in (3.2), we define constants $\zeta_\eta \in (0,\infty)$ and Q'_η as follows:

$$q_{c^*}(\zeta_\eta) = 1 - \frac{\eta}{2},\ Q'_\eta = \min_{0\leq\zeta\leq\zeta_\eta} q'_{c^*}(\zeta) > 0.$$

Then we take $T^{**} > T^*$ so that

$$Me^{-\delta t} \leq \frac{\eta}{2}\ \text{ for } t \geq T^{**}. \tag{3.8}$$

Let c, M and δ be as before. We now define $\underline{g}(t)$, $\underline{h}(t)$ and $\underline{u}(t,x)$ as follows:

$$\begin{aligned}
&\underline{g}(t) = -ct,\\
&\underline{h}(t) = c^*(t-T^{**}) + cT^{**} - \sigma M(e^{-\delta T^{**}} - e^{-\delta t}),\\
&\underline{u}(t,x) = (1-Me^{-\delta t})q_{c^*}(\underline{h}(t) - x).
\end{aligned}$$

Lemma 3.3. *For sufficiently large $\sigma > 0$, $u(t,x)$ and $h(t)$ satisfy*

$$\underline{u}(t,x) \le u(t,x) \text{ for } x \in [\underline{g}(t), \underline{h}(t)],\ t \ge T^{**},$$
$$\underline{h}(t) \le h(t) \text{ for } t \ge T^{**}.$$

Proof. We will check that $(\underline{u}, \underline{g}, \underline{h})$ is a lower solution to (1.1) for $t \ge T^{**}$. First, from (2.3) we can easily see that $\underline{u} \le u$ at $x = \underline{g}(t)$ since for $t \ge T^{**}$,

$$\underline{u}(t, \underline{g}(t)) = \underline{u}(t, -ct) \le 1 - Me^{-\delta t} \le u(t, -ct) = u(t, \underline{g}(t)).$$

Next we check that $\underline{h}$ and $\underline{u}$ satisfy the required conditions at $x = \underline{h}(t)$. It is obvious that $\underline{u}(t, \underline{h}(t)) = 0$. Direct computations yield that

$$\underline{h}'(t) = c^* - \sigma M \delta e^{-\delta t}$$

and

$$-\mu \underline{u}_x(t, \underline{h}(t)) = \mu(1 - Me^{-\delta t}) q_{c^*}'(0) = c^* - c^* M e^{-\delta t},$$

from which we see $\underline{h}'(t) \le -\mu \underline{u}_x(t, \underline{h}(t))$ for $\sigma > 0$ satisfying $c^* \le \sigma\delta$.

By (2.2) and (2.3),

$$\underline{h}(T^{**}) = cT^{**} \le h(T^{**})$$

and

$$\underline{u}(T^{**}, x) \le 1 - Me^{-\delta T^{**}} \le u(T^{**}, x) \quad \text{for } x \in \big[\underline{g}(T^{**}), \underline{h}(T^{**})\big].$$

Finally we will prove $\underline{u}_t - \underline{u}_{xx} - f(\underline{u}) \le 0$ for $t \ge T^{**}$. Put $\zeta = \underline{h}(t) - x$. Since

$$\underline{u}_t = \delta M e^{-\delta t} q_{c^*}(\zeta) + (1 - Me^{-\delta t}) \underline{h}'(t) q_{c^*}'(\zeta),$$
$$\underline{u}_{xx} = (1 - Me^{-\delta t}) q_{c^*}''(\zeta),$$

we have

$$\begin{aligned}
&\underline{u}_t - \underline{u}_{xx} - f(\underline{u}) \\
&= \delta M e^{-\delta t} q_{c^*}(\zeta) - \sigma M \delta e^{-\delta t}(1 - Me^{-\delta t}) q_{c^*}'(\zeta) \\
&\quad + (1 - Me^{-\delta t}) f(q_{c^*}(\zeta)) - f((1 - Me^{-\delta t}) q_{c^*}(\zeta)) \\
&= \delta M e^{-\delta t} q_{c^*}(\zeta) - \sigma M \delta e^{-\delta t}(1 - Me^{-\delta t}) q_{c^*}'(\zeta) + F(-Me^{-\delta t}, q_{c^*}(\zeta)).
\end{aligned}$$

For $\zeta \ge \zeta_\eta$ we can apply the mean value theorem to $F(\xi, u)$ as before to obtain

$$\begin{aligned}
&\underline{u}_t - \underline{u}_{xx} - f(\underline{u}) \\
&= \delta M e^{-\delta t} q_{c^*}(\zeta) - \sigma M \delta e^{-\delta t}(1 - Me^{-\delta t}) q_{c^*}'(\zeta) \\
&\quad - Me^{-\delta t} \Big\{ f(q_{c^*}(\zeta)) - f'\big(q_{c^*}(\zeta) - \theta'' M e^{-\delta t} q_{c^*}(\zeta)\big) q_{c^*}(\zeta) \Big\} \\
&= -Me^{-\delta t} f(q_{c^*}(\zeta)) - \sigma M \delta e^{-\delta t}(1 - Me^{-\delta t}) q_{c^*}'(\zeta) \\
&\quad + Me^{-\delta t} \Big\{ f'\big(q_{c^*}(\zeta) - \theta'' M e^{-\delta t} q_{c^*}(\zeta)\big) + \delta \Big\} q_{c^*}(\zeta) \ge 0,
\end{aligned}$$

for some $\theta'' = \theta''(t,z) \in (0,1)$. Here we have used the fact that for $\zeta \ge \zeta_\eta$, due to (3.8),

$$1 \ge q_{c^*}(\zeta) - \theta'' M e^{-\delta t} q_{c^*}(\zeta) \ge q_{c^*}(\zeta) - Me^{-\delta t} q_{c^*}(\zeta) \ge 1 - \eta,$$

and hence, by (3.2), $f'\big(q_{c^*}(\zeta) - \theta'' M e^{-\delta t} q_{c^*}(\zeta)\big) + \delta \le 0$.

For $0 \le \zeta \le \zeta_\eta$, we obtain

$$\begin{aligned}
&\underline{u}_t - \underline{u}_{xx} - f(\underline{u})\\
&= \delta M e^{-\delta t} q_{c^*}(\zeta) - \sigma M \delta e^{-\delta t}(1 - M' e^{-\delta t}) q'_{c^*}(\zeta)\\
&\quad - M e^{-\delta t} \Big\{ f(q_{c^*}(\zeta)) - f'\big(q_{c^*}(\zeta) - \theta'' M e^{-\delta t} q_{c^*}(\zeta)\big) q_{c^*}(\zeta) \Big\}\\
&= - M e^{-\delta t} f(q_{c^*}(\zeta)) - \sigma M \delta e^{-\delta t}(1 - M e^{-\delta t}) q'_{c^*}(\zeta)\\
&\quad + M e^{-\delta t} \Big\{ f'\big(q_{c^*}(\zeta) - \theta'' M e^{-\delta t} q_{c^*}(\zeta)\big) + \delta \Big\} q_{c^*}(\zeta)\\
&\le - M e^{-\delta t} \min_{0 \le s \le 1} f(s) - \sigma M \delta e^{-\delta t}(1 - M e^{-\delta t}) q'_{c^*}(\zeta)\\
&\qquad + M e^{-\delta t} \left(\max_{0 \le s \le 1} f'(s) + \delta \right)\\
&\le M e^{-\delta t}(1 - M e^{-\delta t}) \left\{ \frac{-\min_{0 \le s \le 1} f(s) + \max_{0 \le s \le 1} f'(s) + \delta}{1 - M e^{-\delta t}} - \sigma \delta Q'_\eta \right\}\\
&\le M e^{-\delta t}(1 - M e^{-\delta t}) \left\{ \frac{-\min_{0 \le s \le 1} f(s) + \max_{0 \le s \le 1} f'(s) + \delta}{1 - M e^{-\delta T^{**}}} - \sigma \delta Q'_\eta \right\}\\
&\le 0,
\end{aligned}$$

by taking $\sigma > 0$ sufficiently large. This completes the proof of the lemma. □

Proof of Proposition 3.1. From Lemmas 3.2 and 3.3, for $t \ge T^{**}$ we have

$$\begin{aligned}
(c - c^*)T^{**} - \sigma M(e^{-\delta T^{**}} - e^{-\delta t}) &\le h(t) - c^* t\\
&\le -c^* T^* + \sigma M'(e^{-\delta T^*} - e^{-\delta t}) + h(T^*) + X_0.
\end{aligned}$$

Hence if we define

$$\begin{aligned}
C := \max\Big\{ &- c^* T^* + \sigma M' e^{-\delta T^*} + h(T^*) + X_0,\\
&(c^* - c)T^{**} + \sigma M e^{-\delta T^{**}}, \max_{t \in [0, T^{**}]} |h(t) - c^* t| \Big\},
\end{aligned}$$

then

$$|h(t) - c^* t| \le C \text{ for all } t > 0.$$

This completes the proof of Proposition 3.1. □

3.2. **Convergence along a subsequence of $t_n \to \infty$.** Set

$$H(t) := h(t) - c^* t$$

and note that H and H' are bounded on $[0, \infty)$ by Proposition 3.1 and Lemma 2.2. We have the following technical result.

Proposition 3.4. *For any sequence $\{t_n\} \subset \mathbb{R}$ satisfying $\lim_{n\to\infty} t_n = \infty$, there exists a subsequence $\{\tilde{t}_n\} \subset \{t_n\}$ such that $\lim_{n\to\infty} H(\tilde{t}_n + \cdot) = \hat{H}$ in $C^1_{loc}(\mathbb{R})$ for some constant $\hat{H} \in \mathbb{R}$, and*

$$\lim_{n\to\infty} \sup_{z \in [-(c+c^*)\tilde{t}_n, \hat{H}]} |v(\tilde{t}_n, z) - q_{c^*}(\hat{H} - z)| = 0.$$

Here we have used the convention that $q_{c^*}(z) = 0$ for $z \le 0$ and $v(t, z) = 0$ for $z \ge H(t)$.

We will need the following energy functional

$$E(t) := \int_{g(t)-c^*t}^{h(t)-c^*t} e^{c^*z} \left\{ \frac{1}{2} v_z^2 - F(v) \right\} dz,$$

where

$$F(v) = \int_0^v f(s) ds.$$

Lemma 3.5. *The functional $E(t)$ is bounded from below and satisfies*

$$E'(t) = -\frac{h'(t)^2}{2\mu^2}(h'(t) - c^*)e^{c^*(h(t)-c^*t)} + \frac{g'(t)^2}{2\mu^2}(g'(t) - c^*)e^{c^*(g(t)-c^*t)} - \int_{g(t)-c^*t}^{h(t)-c^*t} e^{c^*z} \{v_{zz} + c^*v_z + f(v)\}^2 dz.$$

Proof. We first observe that $E(t)$ is bounded from below since $H(t)$ and $F(v(t,z))$ are bounded, and $\lim_{t\to\infty}(g(t) - c^*t) = -\infty$.

Direct calculation yields that

$$\begin{aligned} E'(t) = &\ (h'(t) - c^*)e^{c^*(h(t)-c^*t)} \left\{ \frac{1}{2} v_z^2(t, h(t) - c^*t) - F(v(t, h(t) - c^*t)) \right\} \\ &- (g'(t) - c^*)e^{c^*(g(t)-c^*t)} \left\{ \frac{1}{2} v_z^2(t, g(t) - c^*t) - F(v(t, g(t) - c^*t)) \right\} \\ &+ \int_{g(t)-c^*t}^{h(t)-c^*t} e^{c^*z} \{v_z v_{zt} - f(v)v_t\} dz \\ = &\ (h'(t) - c^*)e^{c^*(h(t)-c^*t)} \cdot \frac{h'(t)^2}{2\mu^2} - (g'(t) - c^*)e^{c^*(g(t)-c^*t)} \cdot \frac{g'(t)^2}{2\mu^2} \\ &+ \int_{g(t)-c^*t}^{h(t)-c^*t} e^{c^*z} \{v_z v_{zt} - f(v)v_t\} dz. \end{aligned}$$

Let us consider the term $\int_{g(t)-c^*t}^{h(t)-c^*t} e^{c^*z} v_z v_{zt} dz$. Integration by parts yields that

$$\begin{aligned} &\int_{g(t)-c^*t}^{h(t)-c^*t} e^{c^*z} v_z v_{zt} dz \\ &= \left[e^{c^*z} v_z v_t \right]_{g(t)-c^*t}^{h(t)-c^*t} - \int_{g(t)-c^*t}^{h(t)-c^*t} v_t \left(e^{c^*z} v_z \right)_z dz \\ &= e^{c^*(h(t)-c^*t)} v_t(t, h(t) - c^*t) v_z(t, h(t) - c^*t) \\ &\quad - e^{c^*(g(t)-c^*t)} v_t(t, g(t) - c^*t) v_z(t, g(t) - c^*t) - \int_{g(t)-c^*t}^{h(t)-c^*t} e^{c^*z} \left(v_{zz} + c^* v_z \right) v_t dz. \end{aligned}$$

Differentiating the identities $v(t, h(t) - c^*t) = 0$ and $v(t, g(t) - c^*t) = 0$ in t we obtain

$$\begin{aligned} v_t(t, h(t) - c^*t) + (h'(t) - c^*) v_z(t, h(t) - c^*t) = 0, \\ v_t(t, g(t) - c^*t) + (g'(t) - c^*) v_z(t, g(t) - c^*t) = 0. \end{aligned}$$

It follows that

$$v_t(t,h(t)-c^*t)v_z(t,h(t)-c^*t)=-(h'(t)-c^*)v_z^2(t,h(t)-c^*t)=-\frac{h'(t)^2}{\mu^2}(h'(t)-c^*),$$
$$v_t(t,g(t)-c^*t)v_z(t,g(t)-c^*t)=-(g'(t)-c^*)v_z^2(t,g(t)-c^*t)=-\frac{g'(t)^2}{\mu^2}(g'(t)-c^*).$$

Using these identities we obtain

$$\begin{aligned}&\int_{g(t)-c^*t}^{h(t)-c^*t}e^{c^*z}v_zv_{zt}dz\\ =&-e^{c^*(h(t)-c^*t)}\frac{h'(t)^2}{\mu^2}(h'(t)-c^*)+e^{c^*(g(t)-c^*t)}\frac{g'(t)^2}{\mu^2}(g'(t)-c^*)\\ &-\int_{g(t)-c^*t}^{h(t)-c^*t}e^{c^*z}v_t\big(v_{zz}+c^*v_z\big)dz.\end{aligned}$$

Hence

$$\begin{aligned}E'(t)=&-\frac{h'(t)^2}{2\mu^2}(h'(t)-c^*)e^{c^*(h(t)-c^*t)}+\frac{g'(t)^2}{2\mu^2}(g'(t)-c^*)e^{c^*(g(t)-c^*t)}\\ &-\int_{g(t)-c^*t}^{h(t)-c^*t}e^{c^*z}v_t\big\{v_{zz}+c^*v_z+f(v)\big\}dz\\ =&-\frac{h'(t)^2}{2\mu^2}(h'(t)-c^*)e^{c^*(h(t)-c^*t)}+\frac{g'(t)^2}{2\mu^2}(g'(t)-c^*)e^{c^*(g(t)-c^*t)}\\ &-\int_{g(t)-c^*t}^{h(t)-c^*t}e^{c^*z}\big\{v_{zz}+c^*v_z+f(v)\big\}^2dz.\end{aligned}$$

This completes the proof. □

Let us define

$$E_0(t):=\frac{1}{2\mu^2}\int_0^t e^{c^*(h(s)-c^*s)}h'(s)^2(h'(s)-c^*)ds$$

and

$$\tilde{E}(t):=E(t)+E_0(t).$$

Lemma 3.6. $\lim_{t\to\infty}\tilde{E}'(t)=0$.

Proof. It is easily seen that $\tilde{E}(t)$ satisfies

$$\begin{aligned}\tilde{E}'(t)=&\frac{g'(t)^2}{2\mu^2}(g'(t)-c^*)e^{c^*(g(t)-c^*t)}\\ &-\int_{g(t)-c^*t}^{h(t)-c^*t}e^{c^*z}\{v_{zz}+c^*v_z+f(v)\}^2dz\le 0.\end{aligned}$$

Since

$$h'(s)^2[h'(s)-c^*]-(c^*)^2[h'(s)-c^*]=[h'(s)+c^*][h'(s)-c^*]^2\ge 0,$$

we have

$$\begin{aligned} E_0(t) &\geq \frac{1}{2\mu^2}\int_0^t e^{c^*(h(s)-c^*s)}(c^*)^2(h'(s)-c^*)ds \\ &= \frac{c^*}{2\mu^2}\int_0^t \frac{d}{ds}\left\{e^{c^*(h(s)-c^*s)}\right\}ds \\ &= \frac{c^*}{2\mu^2}\left(e^{c^*H(t)}-e^{c^*h_0}\right) \geq -C_0 \end{aligned}$$

for some $C_0 > 0$ independent of t.

Now $\tilde{E}(t) = E(t) + E_0(t)$ is bounded from below with $\tilde{E}'(t) \leq 0$. Hence

$$\lim_{t\to\infty} \tilde{E}(t) = E_\infty > -\infty.$$

If $\tilde{E}'(t)$ is uniformly continuous, then we necessarily have $\lim_{t\to\infty} \tilde{E}'(t) = 0$, for otherwise there exist a sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = \infty$ and $\varepsilon > 0$ such that $\tilde{E}'(t_n) \leq -\varepsilon$. Since $\tilde{E}'(t)$ is uniformly continuous $\tilde{E}'(t) \leq -\varepsilon/2$ holds for $t \in [t_n, t_n+\delta]$ with some $\delta > 0$ independent of n. Then we have, by passing to a subsequence of $\{t_n\}$ if necessary,

$$\begin{aligned} \tilde{E}(\infty) - \tilde{E}(t_1) &= \int_{t_1}^\infty \tilde{E}'(t)dt \\ &\leq \int_{\cup_{n=1}^\infty [t_n, t_n+\delta]} \tilde{E}'(t)dt = -\infty. \end{aligned}$$

This contradiction confirms our claim.

Moreover, since

$$\lim_{t\to\infty} \frac{g'(t)^2}{2\mu^2}(g'(t)-c^*)e^{c^*(g(t)-c^*t)} = 0,$$

to show $\lim_{t\to\infty} \tilde{E}'(t) = 0$, by the above discussion, we actually only have to show that the second term in the expression of $\tilde{E}'(t)$ is uniformly continuous in t for all large t, and this will be the case if for any $L > 0$, v, v_z and v_{zz} are uniformly continuous in t for $z \in [-L, h(t) - c^*t]$.

We first consider these functions over the domain $[t_0-1, t_0+1]\times[-L-1, h(t_0)-c^*t_0-\eta/3] \subset \mathbb{R}^2$ for $t_0 \in \mathbb{R}$, $L > 0$ and $\eta > 0$. Since $\|v\|_\infty$ and $\|f(v)\|_{L^\infty}$ are bounded, we can apply the parabolic L^p estimate to obtain

$$\|v\|_{W_p^{1,2}([t_0-1/2,t_0+1]\times[-L-1/2,h(t_0)-c^*t_0-\eta/2])} \leq C$$

for some $C > 0$ which does not depend on t_0. Here we have used the fact that $H(t_0) = h(t_0)-c^*t_0$ has a bound independent of $t_0 > 0$. By Sobolev imbedding we have

$$\|v\|_{C^{\frac{1+\gamma}{2},1+\gamma}([t_0-1/2,t_0+1]\times[-L-1/2,h(t_0)-c^*t_0-\eta/2])} \leq C'.$$

Using this and Schauder estimate we obtain

$$\|v\|_{C^{1+\frac{\alpha}{2},2+\alpha}([t_0,t_0+1]\times[-L,h(t_0)-c^*t-\eta])} \leq C''. \tag{3.9}$$

for some $C'' > 0$ which does not depend on t_0.

Next we consider the domain $\{(t,x)|t \in [t_0-1, t_0+1], x \in [h(t)-c^*t-2L, h(t)-c^*t]\}$ for $L > 0$. We first straighten the boundary $z = H(t) = h(t) - c^*t$. Let

$$z = y + H(t) \text{ and } w(t,y) = v(t, y + H(t)).$$

Then w satisfies

$$\begin{cases} w_t = w_{yy} + (H'(t) + c^*)w_y + f(w), t > 0, \ y \in (g(t) - c^*t - H(t), 0), \\ w(t, 0) = 0, \ t > 0, \\ H'(t) = -\mu w_y(t, 0) + c^*, \ t > 0. \end{cases} \tag{3.10}$$

Since $\|w\|_\infty$, $\|f(w)\|_{L^\infty}$ and $\|H'\|_{L^\infty}$ are bounded we can apply the parabolic L^p estimate to obtain

$$\|w\|_{W_p^{1,2}([t_0-1,t_0+1]\times[-3L/2,0])} \leq C$$

for some $C > 0$ which does not depend on t_0. By Sobolev imbedding we have

$$\|w\|_{C^{\frac{1+\gamma}{2},1+\gamma}([t_0-1/2,t_0+1]\times[-3L/2,0])} \leq C'$$

for some $\gamma \in (0, 1)$ and $C' > 0$ which do not depend on t_0. This implies that H' and $f(w)$ are Hölder continuous and by the parabolic Schauder estimate we obtain

$$\|w\|_{C^{1+\frac{\alpha}{2},2+\alpha}([t_0,t_0+1]\times[-L,0])} \leq C'' \tag{3.11}$$

for some $\alpha \in (0, 1)$ and $C'' > 0$ which do not depend on t_0.

From (3.9) and (3.11) we see that v, v_z and v_{zz} are uniformly continuous in t for $z \in [-L, h(t) - c^*t]$. This completes the proof. □

Lemma 3.7. *For any sequence* $\{t_n\}$ *satisfying* $\lim_{n\to\infty} t_n = \infty$ *and any* $K > 0$ *there exists a subsequence* $\{\tilde{t}_n\} \subset \{t_n\}$ *such that* $\lim_{n\to\infty} H(\tilde{t}_n + \cdot) = \hat{H}$ *in* $C^1_{loc}(\mathbb{R})$ *for some constant* $\hat{H} \in \mathbb{R}$*, and*

$$\lim_{n\to\infty} \sup_{z\in[-K,\hat{H}]} |v(\tilde{t}_n, z) - q_{c^*}(\hat{H} - z)| = 0$$

Proof. Without loss of generality we assume that $\{t_n\}$ is an increasing sequence of positive numbers satisfying $\lim_{n\to\infty} t_n = \infty$. Define

$$v_n(t, z) = v(t + t_n, z), \ w_n(t, y) = w(t + t_n, y),$$
$$H_n(t) = H(t + t_n), \ G_n(t) = g(t + t_n) - c^*(t + t_n) - H_n(t).$$

By (3.10) we have

$$\begin{cases} \dfrac{\partial w_n}{\partial t} = \dfrac{\partial^2 w_n}{\partial y^2} + (H_n'(t) + c^*)\dfrac{\partial w_n}{\partial y} + f(w_n), & t > -t_n, y \in (G_n(t), 0), \\ w_n(t, 0) = 0, & t > -t_n, \\ H_n'(t) = -\mu \dfrac{\partial w_n}{\partial y}(t, 0) - c^*, & t > -t_n. \end{cases}$$

By the same regularity consideration used to (3.10) above, $\{w_n\}$ is bounded in $C^{1+\frac{\alpha}{2},2+\alpha}([-R, R]\times[-R, 0])$ for any $R > 0$. Hence H_n' is uniformly bounded in $C^\alpha(I)$ for any bounded interval $I \subset \mathbb{R}$, and by passing to a subsequence, still dented by $\{t_n\}$, we have

$$H_n' \to \tilde{H} \ \text{ in } \ C^{\alpha'}_{\text{loc}}(\mathbb{R})$$

for some $\alpha' \in (0, \alpha)$. Similarly, subject to passing to a further subsequence,

$$w_n \to \hat{w} \ \text{ in } \ C^{1+\frac{\alpha'}{2},2+\alpha'}_{\text{loc}}(\mathbb{R} \times (-\infty, 0]), \tag{3.12}$$

and $\hat{w}$ satisfies

$$\begin{cases} \hat{w}_t = \hat{w}_{yy} + (\tilde{H}(t) + c^*)\hat{w}_y + f(\hat{w}), & t \in \mathbb{R}, y < 0, \\ \hat{w}(t, 0) = 0, & t \in \mathbb{R}, \\ \tilde{H}(t) = -\mu \hat{w}_y(t, 0) - c^*, & t \in \mathbb{R}. \end{cases}$$

Since

$$H_n(t) = H_n(0) + \int_0^t H_n'(s)ds$$

and $H_n' \to \tilde{H}$ in $C_{loc}^{\alpha'}(\mathbb{R})$, we obtain

$$H_n(t) \to \hat{H}(t) := \tilde{H}(0) + \int_0^t \tilde{H}(s)ds \text{ in } C_{loc}^{1,\alpha'}(\mathbb{R}).$$

Thus $\tilde{H}(t) = \hat{H}'(t)$ and

$$\begin{cases} \hat{w}_y = \hat{w}_{yy} + (\hat{H}'(t) + c^*)\hat{w}_y + f(\hat{w}), & t \in \mathbb{R},\ y < 0, \\ \hat{w}(t,0) = 0, & t \in \mathbb{R}, \\ \hat{H}'(t) = -\mu \hat{w}_y(t,0) - c^*, & t \in \mathbb{R}. \end{cases}$$

Now we examine v_n. It is easily checked that

$$\begin{cases} \dfrac{\partial v_n}{\partial t} = \dfrac{\partial^2 v_n}{\partial z^2} + c^* \dfrac{\partial v_n}{\partial z} + f(v_n), & t > -t_n, z < H_n(t), \\ v_n(t, H_n(t)) = 0, & t > -t_n, \\ H_n'(t) = -\mu \dfrac{\partial v_n}{\partial z}(t, H_n(t)) - c^*, & t > -t_n. \end{cases} \tag{3.13}$$

For any $\varepsilon > 0$ we consider (3.13) over

$$\Omega_\varepsilon := \left\{ (t,z) \mid t \in \left[-\varepsilon^{-1}, \varepsilon^{-1}\right], z \in \left[-\varepsilon^{-1}, \hat{H}(t) - \varepsilon\right] \right\}.$$

Applying the parabolic Schauder estimate we have by passing to a subsequence

$$v_n \to \hat{v} \ \text{ in } \ C^{1+\frac{\alpha'}{2}, 2+\alpha'}(\Omega_\varepsilon)$$

and $\hat{v}$ satisfies

$$\hat{v}_t = \hat{v}_{zz} + c^* \hat{v}_z + f(\hat{v}) \ \text{ in } \ \Omega_\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, by passing to a further subsequence we may assume $v_n \to \hat{v}$ in $C_{\rm loc}^{1+\frac{\alpha'}{2}, 2+\alpha'}(\Omega_0)$ with $\Omega_0 = \{(t,z) \mid t \in \mathbb{R},\ z < \hat{H}(t)\}$.

Next we show $\hat{v}_t \equiv 0$ and $\hat{v}(t,z) \equiv \hat{v}(z)$. By Lemma 3.6 we have

$$\begin{aligned} \tilde{E}'(t+t_n) = & \frac{g'(t+t_n)^2}{2\mu^2}(g'(t+t_n) - c^*) e^{c^*[g(t+t_n) - c^*(t+t_n)]} \\ & - \int_{g(t+t_n)-c^*(t+t_n)}^{H(t+t_n)} e^{c^* z} \{(v_n)_{zz} + c^*(v_n)_z + f(v_n)\}^2 dz \to 0 \end{aligned}$$

as $n \to \infty$. Since the first term on the right side of the above identity converges to 0, it follows that for any $K > 0$ and $\varepsilon > 0$,

$$\begin{aligned} 0 & \le \int_{-K}^{\hat{H}(t)-\varepsilon} e^{c^* z} \{\hat{v}_{zz} + c^* \hat{v}_z + f(\hat{v})\}^2 dz \\ & \le \lim_{n\to\infty} \int_{g(t+t_n)-c^*(t+t_n)}^{H(t+t_n)} e^{c^* z} \{(v_n)_{zz} + c^*(v_n)_z + f(v_n)\}^2 dz = 0. \end{aligned}$$

Since $\varepsilon, K > 0$ are arbitrarily, we obtain

$$\hat{v}_{zz} + c^* \hat{v}_z + f(\hat{v}) = 0 \ \text{ in } \Omega_0.$$

Hence $\hat{v}_t \equiv 0$ and $\hat{v}(t,z) \equiv \hat{v}(z)$.

To determine the boundary condition of $\hat{v}$ at $z = \hat{H}(t)$, we consider $\hat{v}$ on

$$\big\{(t,z) \mid t \in [-1/\varepsilon, 1/\varepsilon], z \in [\hat{H}(t)-\varepsilon, \hat{H}(t)]\big\}.$$

We observe that

$$v_n(t,z) = v(t+t_n, z) = w(t+t_n, z - H(t+t_n)) = w_n(t, z - H_n(t)),$$

and by (3.12),

$$\lim_{n\to\infty} \sup_{z\in[\hat{H}(t)-\varepsilon,\hat{H}(t)]} |w_n(t, z-H_n(t)) - \hat{w}(t, z-\hat{H}(t))| = 0$$

if we define $w_n(t,y) = 0$ for $y \ge 0$ and $\hat{w}(t,y) = 0$ for $y \ge 0$. It follows that $\hat{v}(t,z) \equiv \hat{w}(t, z-\hat{H}(t))$. Hence $\hat{v}(t, \hat{H}(t)) = 0$ and

$$\hat{H}'(t) = -\mu \hat{v}_z(t, \hat{H}(t)) - c^*, \quad t \in \mathbb{R}.$$

From $0 = \hat{v}(t,\hat{H}(t)) = \hat{v}(\hat{H}(t))$ and the fact that $\hat{v}(z) > 0$ for $z < \hat{H}(t)$ we obtain by the Hopf lemma that $\hat{v}_z(t,\hat{H}(t)) = \hat{v}_z(\hat{H}(t)) < 0$. On the other hand, from $0 = \hat{v}(\hat{H}(t))$ we deduce $0 = \hat{v}_z(\hat{H}(t))\hat{H}'(t)$. Therefore $\hat{H}'(t) \equiv 0$ and $\hat{H}(t) \equiv \hat{H}$. It follows that $c^* = -\mu\hat{v}_z(\hat{H})$. Together with

$$\hat{v}_{zz} + c^*\hat{v}_z + f(\hat{v}) = 0 \ \text{ in } \ (-\infty, \hat{H}), \ \hat{v}(\hat{H}) = 0,$$

this implies, by the uniqueness of q_{c^*}, that $\hat{v}(z) = q_{c^*}(\hat{H} - z)$. The proof is now complete. □

Proof of Proposition 3.4. By Lemmas 3.2 and 3.3 we have

$$(1 - Me^{-\delta t})q_{c^*}(\underline{h}(t) - x) \le u(t,x) \le (1 + M'e^{-\delta t})q_{c^*}(\overline{h}(t) - x)$$

for $t \in [T^{**}, \infty)$ and $x \in [-ct, h(t)]$, where we have assumed that $q_{c^*}(z) = 0$ for $z \le 0$.

Since $f'(1) < 0$, by standard argument we have

$$|1 - q_{c^*}(z)| \le Ce^{-\beta z} \text{ for some } C > 0 \text{ and } \beta > 0.$$

Using this and the boundedness of the functions $\underline{h}(t) - c^*t$ and $\overline{h}(t) - c^*t$, we easily see that there exists some $C' > 0$ such that

$$v(t,z) := u(t, z + c^*t)$$

satisfies

$$|1 - v(t,z)| \le C'(e^{\beta z} + e^{-\delta t}) \tag{3.14}$$

for $t \in [T^{**}, \infty)$ and $z \in [-(c+c^*)t, h(t) - c^*t]$.

By (3.14) we have

$$|v(t_n, z) - 1| \le C'(e^{\beta z} + e^{-\delta t_n}) \ \text{ for } \ z \in [-(c+c^*)t_n, H(t_n)].$$

Therefore, for any $\varepsilon > 0$, there exists $K > 0$ and $T > 0$ such that

$$\sup_{z\in[-(c+c^*)t_n, -K]} |v(t_n, z) - q_{c^*}(\hat{H} - z)| < \varepsilon$$

for $t_n > T$. On the other hand from Lemma 3.7, for all large t_n,

$$\sup_{z\in[-K,\hat{H}]} |v(t_n,z) - q_{c^*}(\hat{H} - z)| < \varepsilon$$

and

$$|h(t_n) - c^*t_n - \hat{H}| < \varepsilon. \tag{3.15}$$

Hence we have

$$\sup_{z\in[-(c+c^*)t_n,\hat{H}]} |v(t_n,z) - q_{c^*}(\hat{H}-z)| < \varepsilon \tag{3.16}$$

for all large n. This completes the proof of the proposition. □

3.3. **Completion of the proof of Theorem 1.2.** With the help of Proposition 3.4, we are now able to refine the upper and lower solutions used in proving Proposition 3.1, which will lead to the required estimates in Theorem 1.2.

First we construct an upper solution. Take an arbitrary $\varepsilon > 0$, and fix t_n such that (3.15), (3.16) hold and $e^{-\delta t_n} \le \varepsilon$. From (3.16) and (3.15) we have

$$\begin{aligned} &v(t_n,z) \le q_{c^*}(\hat{H}-z) + \varepsilon \quad \text{for} \quad z \in [-(c+c^*)t_n, \hat{H}], \\ &H(t_n) = h(t_n) - c^* t_n \le \hat{H} + \varepsilon. \end{aligned}$$

Hence we have

$$v(t_n,z) \le q_{c^*}(\hat{H}+\varepsilon - z) + \varepsilon \quad \text{for} \quad z \in [-(c+c^*)t_n, \hat{H}+\varepsilon].$$

We note that we can find $N > 1$ independent of $\varepsilon > 0$ such that

$$(1+N\varepsilon)q_{c^*}(\hat{H}+N\varepsilon - z) \ge q_{c^*}(\hat{H}+\varepsilon-z)+\varepsilon \text{ for } z \le \hat{H}+\varepsilon.$$

Indeed, if $q_{c^*}(\hat{H}+N\varepsilon-z) \ge 1/2$, then the required inequality holds provided that $N \ge 2$. If $q_{c^*}(\hat{H}+N\varepsilon-z) < 1/2$, then due to $\delta_0 := \min_{\zeta\in[0,\zeta_0]} q'_{c^*}(\zeta) > 0$, where $q_{c^*}(\zeta_0) = 1/2$, we have

$$q_{c^*}(\hat{H}+N\varepsilon-z) - q_{c^*}(\hat{H}+\varepsilon-z) \ge \delta_0(N-1)\varepsilon,$$

and hence the required inequality holds when $\delta_0(N-1) \ge 1$.

Now we define an upper solution $(\overline{u}, \overline{g}, \overline{h})$ as follows:

$$\begin{aligned} &\overline{u}(t,x) = (1+N\varepsilon e^{-\delta(t-t_n)})q_{c^*}(\overline{h}(t)-x), \\ &\overline{h}(t) = \hat{H} + c^*t + N\varepsilon + N\varepsilon\sigma(1-e^{-\delta(t-t_n)}), \\ &\overline{g}(t) = g(t). \end{aligned}$$

We will check $(\overline{u}, \overline{g}, \overline{h})$ satisfies the conditions in Lemma 2.1 for $t \ge t_n$, that is,

$$\overline{u}_t - \overline{u}_{xx} \ge f(\overline{u}) \quad \text{for} \quad t > t_n, \overline{g}(t) < x < \overline{h}(t), \tag{3.17}$$

$$\overline{u} \ge u \text{ for } \ t \ge t_n, \ x = \overline{g}(t), \tag{3.18}$$

$$\overline{u} = 0, \ \overline{h}'(t) \ge -\mu \overline{u}_x(t,x) \quad \text{for} \quad t \ge t_n, \ x = \overline{h}(t), \tag{3.19}$$

$$h(t_n) \le \overline{h}(t_n), u(t_n,x) \le \overline{u}(t_n,x) \quad \text{for} \quad x \in [\overline{g}(t_n), h(t_n)]. \tag{3.20}$$

From (3.15) we have

$$h(t_n) - c^*t_n = H(t_n) \le \hat{H}+\varepsilon \le \hat{H}+N\varepsilon = \overline{h}(t_n) - c^*t_n,$$

and so $h(t_n) \le \overline{h}(t_n)$. We also have

$$\begin{aligned} \overline{u}(t_n,x) &= (1+\varepsilon N)q_{c^*}(\hat{H}+\varepsilon N - (x-c^*t_n)) \\ &\ge q_{c^*}\big(\hat{H}+\varepsilon-(x-c^*t_n)\big)+\varepsilon \\ &\ge v(t_n, x-c^*t_n) = u(t_n,x) \end{aligned}$$

for $x \in [-ct_n, h(t_n)]$.

By Lemma 2.8 in [7], we have

$$u_x(t_n,x) \ge 0 \quad \text{for} \quad x \in [g(t_n), -h_0].$$

Therefore due to

$$\overline{u}_x(t_n,x)\le 0 \quad \text{for} \quad x\in(-\infty,\overline{h}(t)],$$

from $\overline{u}(t_n,z)\ge u(t_n,z)$ with $z=-ct_n$ we deduce

$$\overline{u}(t_n,x)\ge u(t_n,x) \text{ for } x\in[g(t_n),-ct_n].$$

Thus (3.20) holds.

We next show (3.19). By definition $\overline{u}(t,\overline{h}(t))=0$, and direct calculation gives

$$\begin{aligned}\overline{h}'(t)&=c^*+N\varepsilon\sigma\delta e^{-\delta(t-t_n)},\\ -\mu\overline{u}_x(t,\overline{h}(t))&=c^*+N\varepsilon c^* e^{-\delta(t-t_n)}.\end{aligned}$$

Hence if we take $\sigma>0$ so that $c^*\le\sigma\delta$ then

$$\overline{h}'(t)\ge -\mu\overline{u}_x(t,\overline{h}(t)).$$

Clearly (3.18) holds. Finally (3.17) can be proved in the same way that (3.4) is proved, the only point we should note is that by shrinking ε we can guarantee that $N\varepsilon<\eta$ and so $1+N\varepsilon e^{-\delta(t-t_n)}\le 1+\eta$ for all $t\ge t_n$. We further note that (3.17) holds for $\sigma>\sigma_0$ where σ_0 depends only on f and δ.

Thus the constructed triple is indeed an upper solution and we obtain

$$u(t,x)\le q_{c^*}(\hat{H}+N\varepsilon(1+\sigma)+c^*t-x)+\varepsilon N e^{-\delta(t-t_n)}, \tag{3.21}$$

$$h(t)-c^*t-\hat{H}\le N\varepsilon(1+\sigma) \tag{3.22}$$

for $t\ge t_n$ and $x\in[\overline{g}(t),h(t)]=[g(t),h(t)]$.

Next we construct a lower solution. From (3.16) and (3.15) we have

$$\begin{aligned}&q_{c^*}(\hat{H}-z)-\varepsilon\le v(t_n,z) \quad \text{for} \quad z\in[-(c+c^*)t_n,\hat{H}],\\ &\hat{H}-\varepsilon\le H(t_n)=h(t_n)-c^*t_n.\end{aligned}$$

Hence we have

$$q_{c^*}(\hat{H}-\varepsilon-z)\le v(t_n,z) \quad \text{for} \quad z\in[-(c+c^*)t_n,\hat{H}+\varepsilon].$$

We note that we can find $N>1$ which does not depend on $\varepsilon>0$ such that

$$(1-N\varepsilon)q_{c^*}(\hat{H}-N\varepsilon-z)\le q_{c^*}(\hat{H}+\varepsilon-z)-\varepsilon \text{ for } z\le\hat{H}-\varepsilon.$$

Now we define a lower solution $(\underline{u},\underline{g},\underline{h})$ as follows:

$$\begin{aligned}&\underline{u}(t,x)=(1-N\varepsilon e^{-\delta(t-t_n)})q_{c^*}(\underline{h}(t)-x),\\ &\underline{h}(t)=\hat{H}+c^*t-N\varepsilon-N\varepsilon\sigma(1-e^{-\delta(t-t_n)}),\\ &\underline{g}(t)=-ct.\end{aligned}$$

We will check that $(\underline{u},\underline{g},\underline{h})$ satisfies the inequalities for a lower solution for $t\ge t_n$, namely

$$\underline{u}_t-\underline{u}_{xx}\le f(\underline{u}) \qquad \text{for} \quad t>t_n, \underline{g}(t)<x<\underline{h}(t), \tag{3.23}$$

$$\underline{u}\le u \qquad \text{for} \quad t\ge t_n,\ x=\underline{g}(t), \tag{3.24}$$

$$\underline{u}=0,\ \underline{h}'(t)\le-\mu\underline{u}_x(t,x) \qquad \text{for} \quad t\ge t_n,\ x=\underline{h}(t), \tag{3.25}$$

$$\underline{h}(t_n)\le h(t_n), \underline{u}(t_n,x)\le u(t_n,x) \qquad \text{for} \quad x\in[\underline{g}(t_n),\underline{h}(t_n)]. \tag{3.26}$$

From (3.15) we have

$$\underline{h}(t_n)-c^*t_n=\hat{H}-N\varepsilon\le\hat{H}-\varepsilon\le H(t_n)=h(t_n)-c^*t_n,$$

and so $\underline{h}(t_n) \le h(t_n)$. Moreover,

$$\begin{aligned}\underline{u}(t_n, x) &= (1-\varepsilon N)q_{c^*}(\hat{H} - \varepsilon N - (x - c^*t_n))\\ &\le q_{c^*}\big(\hat{H} - \varepsilon - (x - c^*t_n)\big) - \varepsilon\\ &\le v(t_n, x - c^*t_n) = u(t_n, x)\end{aligned}$$

for $x \in [-ct_n, \underline{h}(t_n)]$. This proves (3.26).

Next we show that (3.24) holds. We have

$$\begin{aligned}\underline{u}(t, -ct) &= (1 - \varepsilon N e^{-\delta(t-t_n)})q_{c^*}(\hat{H} + (c+c^*)t - N\varepsilon)\\ &\le 1 - \varepsilon N e^{-\delta(t-t_n)}.\end{aligned}$$

On the other hand from Lemma 2.3 we have

$$\begin{aligned}u(t, -ct) &\ge 1 - Me^{-\delta t}\\ &= 1 - Me^{-\delta t_n} e^{-\delta(t-t_n)} \ge 1 - M\varepsilon e^{-\delta(t-t_n)}.\end{aligned}$$

Since we may assume $N \ge M$, we obtain $\underline{u}(t, \underline{g}(t)) \le u(t, \underline{g}(t))$, and (3.24) is proved.

We now show (3.25). By the definition of $\underline{u}$, we have $\underline{u}(t, \underline{h}(t)) = 0$, and direct calculation gives

$$\begin{aligned}\underline{h}'(t) &= c^* - N\varepsilon\sigma\delta e^{-\delta(t-t_n)},\\ -\mu \underline{u}_x(t, \underline{h}(t)) &= c^* - N\varepsilon c^* e^{-\delta(t-t_n)}.\end{aligned}$$

Hence if we take $\sigma > 0$ so that $c^* \le \sigma\delta$ then

$$\underline{h}'(t) \le -\mu \underline{u}_x(t, \underline{h}(t)).$$

Finally we can show (3.23) in the same way as in the proof of Lemma 3.3. We also note that (3.23) holds for $\sigma > \sigma_0$ where σ_0 depends only on f and δ.

We may now apply the comparison principle to obtain

$$q_{c^*}(\hat{H} - N\varepsilon(1+\sigma) + c^*t - x) - \varepsilon N e^{-\delta(t-t_n)} \le u(t, x), \tag{3.27}$$

$$-N\varepsilon(1+\sigma) \le h(t) - c^*t - \hat{H} \tag{3.28}$$

for $t \ge t_n$ and $x \in [\underline{g}(t), \underline{h}(t)] = [-ct, \underline{h}(t)]$.

By (3.22), (3.28) and the arbitrariness of ε, we find

$$\lim_{t\to\infty} (h(t) - c^*t - \hat{H}) = 0.$$

In the following, we will obtain an estimate for $\sup_{x\in[-ct,h(t)]} |u(t,x) - q_{c^*}(h(t) - x)|$. From (3.21) and (3.27), we have

$$\begin{aligned}q_{c^*}(\hat{H} - N\varepsilon(1+\sigma) + c^*t - x) - \varepsilon N e^{-\delta(t-t_n)} &\le u(t, x)\\ &\le q_{c^*}(\hat{H} + N\varepsilon(1+\sigma) + c^*t - x) + \varepsilon N e^{-\delta(t-t_n)}\end{aligned}$$

for $-ct \le x \le \underline{h}(t)$. Hence

$$\begin{aligned}&|u(t,x) - q_{c^*}(h(t) - x)|\\ &\quad \le \varepsilon N e^{-\delta(t-t_n)} + \max\Big\{q_{c^*}\big(\hat{H} + N\varepsilon(1+\sigma) + c^*t - x\big) - q_{c^*}(h(t) - x),\\ &\qquad\qquad q_{c^*}\big(h(t) - x\big) - q_{c^*}\big(\hat{H} - \varepsilon N(1+\sigma) + c^*t - x\big)\Big\}.\end{aligned}$$

By the mean value theorem and the monotonicity of $q_{c^*}(z)$ we have

$$\begin{aligned} 0 &\le q_{c^*}(\hat{H} + N\varepsilon(1+\sigma) + c^*t - x) - q_{c^*}(h(t) - x) \\ &\le \|q'_{c^*}\|_\infty \Big\{ N\varepsilon(1+\sigma) + [\hat{H} + c^*t - h(t)] \Big\}, \end{aligned}$$

and

$$\begin{aligned} 0 &\le q_{c^*}(h(t) - x) - q_{c^*}(\hat{H} - N\varepsilon(1+\sigma) + c^*t - x) \\ &\le \|q'_{c^*}\|_\infty \Big\{ [h(t) - c^*t - \hat{H}] + N\varepsilon(1+\sigma) \Big\}. \end{aligned}$$

It follows that

$$\limsup_{t\to\infty} \sup_{x\in[-ct, \underline{h}(t)]} |u(t,x) - q_{c^*}(h(t)-x)| \le C\varepsilon,$$

where $C = N(1+\sigma)\|q'_{c^*}\|_\infty$ is independent of ε.

For $x \in [\underline{h}(t), h(t)]$ we have

$$0 \le u(t,x) \le \overline{u}(t,x) \le q_{c^*}(\overline{h}(t) - x) + N\varepsilon e^{-\delta(t-t_n)}$$

and

$$\begin{aligned} -q_{c^*}(h(t)-x) &\le u(t,x) - q_{c^*}(h(t)-x) \\ &\qquad \le q_{c^*}(\overline{h}(t) - x) - q_{c^*}(h(t)-x) + N\varepsilon e^{-\delta(t-t_n)}. \end{aligned}$$

By the monotonicity of q_{c^*},

$$q_{c^*}(h(t)-x) \le q_{c^*}(\overline{h}(t) - \underline{h}(t)) \text{ for } \underline{h}(t) \le x \le h(t).$$

Because

$$0 \le q_{c^*}(\overline{h}(t) - x) - q_{c^*}(h(t)-x) \le \|q'_{c^*}\|_\infty[\overline{h}(t) - \underline{h}(t)] \le \|q'_{c^*}\|_\infty 2N(1+\sigma)\varepsilon,$$

and

$$0 \le q_{c^*}(\overline{h}(t) - \underline{h}(t)) \le \|q'_{c^*}\|_\infty[\overline{h}(t) - \underline{h}(t)] \le \|q'_{c^*}\|_\infty 2N(1+\sigma)\varepsilon,$$

we conclude that, for $t \ge t_n$,

$$\sup_{x\in[\underline{h}(t), h(t)]} |u(t,x) - q_{c^*}(h(t)-x)| \le C\varepsilon$$

where $C > 0$ does not depend on ε. Therefore

$$\limsup_{t\to\infty} \sup_{x\in[-ct,\, h(t)]} |u(t,x) - q_{c^*}(h(t)-x)| \le C\varepsilon.$$

Letting $\varepsilon \to 0$ we deduce

$$\lim_{t\to\infty} \sup_{x\in[-ct,\, h(t)]} |u(t,x) - q_{c^*}(h(t)-x)| = 0.$$

One can similarly show that

$$\lim_{t\to\infty} (g(t) + c^*t - \hat{G}) = 0$$

for some constant $\hat{G}$, and

$$\lim_{t\to\infty} \sup_{x\in[g(t),\, ct]} |u(t,x) - q_{c^*}(x - g(t))| = 0.$$

Finally from $H(t) \to \hat{H}$ as $t \to \infty$ we conclude that in Lemma 3.7, the conclusion $H(\tilde{t}_n + \cdot) \to \hat{H}$ in $C^1_{loc}(\mathbb{R})$ as $n \to \infty$ can be strengthened to $H(t + \cdot) \to \hat{H}$ in $C^1_{loc}(\mathbb{R})$ as $t \to \infty$. This implies that $H'(t) \to 0$ as $t \to \infty$, and hence

$$h'(t) \to c^* \text{ as } t \to \infty.$$

The proof for $g'(t) \to -c^*$ as $t \to \infty$ is similar. Theorem 1.2 is now proved.

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