International Journal of Bifurcation and Chaos, Vol. 18, No. 3 (2008) 827–834

AN ALGEBRAIC APPROACH FOR THE RECONSTRUCTION OF CHUA'S SYSTEM

CARLOS AGUILAR IBÁÑEZ*,†, HEBERTT SIRA-RAMÍREZ‡
and H. JORGE SANCHEZ*
*CIC-IPN, Av. Juan de Dios Bátiz s/n Esq. Manuel Othón de M.
Unidad Profesional Adolfo López Mateos,
Col. San Pedro Zacatenco, A.P. 75476,
México, D.F. 07700, México
†caguilar@cic.ipn.mx
‡CINVESTAV-IPN, Departamento de Ingeniería Eléctrica,
Av. IPN 2508, A.P. 14740, México, D.F. 07360, México
‡hsira@cinvestav.mx

Received February 23, 2007; Revised May 30, 2007

An algebraic approach is proposed for the fast, accurate, identification of the unknown parameters in Chua's chaotic oscillator. The proposed algorithm uses the availability of two measurable output voltage signals and produces an exact formula for the unknown parameters, which may be realized in terms of iterated convolutions. We show that Chua's system parameters are linearly identifiable, with respect to the two proposed measurable outputs, thus allowing us to obtain a linear system for the unknown parameters from where these unknowns are readily obtained. Suitable algebraic operations on the output differential equations make the proposed algorithm independent of the unavailable initial conditions of the underlying nonlinear dynamical system and robust with respect to high frequency output measurement noises. Suitable algorithm reinitialization, or resetting of the integrations, allow for the efficient computation of piecewise constant varying parameters. Convincing computer simulations are presented and discussed.

Keywords: Chua's circuit; chaos; reconstruction.

1. Introduction

In the last two decades, considerable attention has been paid to the reconstruction of chaotic attractors from one, or more, available output variables (see the pioneering works by [Takens *et al.*, 1981; Packard *et al.*, 1980; Sauer *et al.*, 1991]). Such inverse problem consists in recovering the underlying system state variables and the unknown parameters from partial knowledge of the given chaotic system. There are, generally speaking, two ways to approach the reconstruction problem: The first approach is based on embedding a time series of the observed variables into a phase space. Roughly speaking, the vector state is constructed with the time delayed values of the measured scalar quantity [Nayfeh & Mook, 1979; Crutchfield & McNamara, 1987; King & Stewart, 1992; Abarbanel, 1996; Parlitz *et al.*, 1994; Alligood *et al.*, 1997]. The second approach exploits a combination of control theoretic ideas, such as inverse system computation and, more notably, the vast discipline of system identification. The approach is characterized by the extensive use of Kalman's filters, Luenberger's observers and high gain, nonlinear, observers [Nijmeijer & Mareels, 1997; Diop *et al.*, 1987; Landau, 2000; Dabroom & Khalil, 1999].

In this article, we present an algebraic approach to the fast, nonasymptotic, estimation of unknown parameters in a chaotic system of the Chua type. The theoretical basis of this approach, which rests in the Automatic Control theory context and also in a Signal Processing context, may be found in the recent work of Fliess and his co-workers (see [Fliess & Sira-Ramirez, 2003; Fliess *et al.*, 2003]). We concentrate mainly on identifying the unknown parameters of a Chua's Chaotic Oscillator (**CCO**) using two independent noisy measured outputs. The system parameters are readily shown to be linearly identifiable. Via simple algebraic operations, we show how to compute these parameters without the influence of unknown initial conditions and, also, in a robust fashion with respect to unavoidable output measurement noises. The piecewise constant variation of the parameters to new unknown levels is handled via the combination of a detection procedure and a reinitialization procedure. The detection procedure is automatically performed from the fact that the constant, online, computed value of the parameter starts diverging once an abrupt, unmodeled, change of its constant value has taken place. The reinitialization of the involved integrations in the proposed algorithm represents a classical procedure in automatic control engineering since the early days of the PID (Proportional Integral Derivative) controllers in the chemical process industry, and the integral term used to receive the name of the "reset" term.

2. Chua's Chaotic Oscillator

Consider the **CCO** system, shown in Fig. 1. The system is a third order system with four linear elements: two capacitors, one resistor and one inductor. It also includes a non-linear element, which is a piecewise linear resistor. The nonlinear model

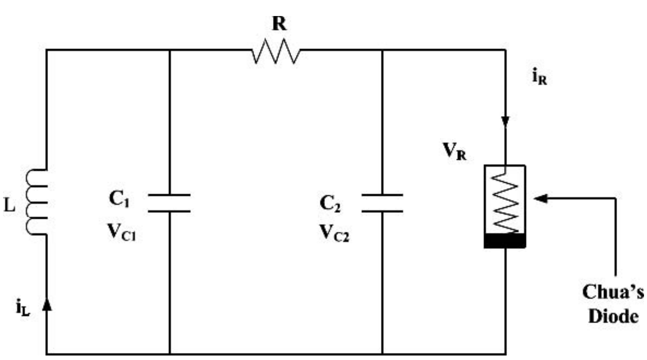

Fig. 1. Chua's chaotic oscillator system.

of this circuit, which is can be obtained by using Kirchoff's laws, is given by,

$$\begin{aligned} C_1\frac{dv_{c_1}}{dt} &= \frac{1}{R}(v_{c_2}-v_{c_1})-\phi(v_{c_1}),\\ C_2\frac{dv_{c_2}}{dt} &= \frac{1}{R}(v_{c_1}-v_{c_2})+i_l,\\ L\frac{di_l}{dt} &= -v_{c_2}, \end{aligned} \tag{1}$$

where v_{c_1} and v_{c_2} are, respectively, the voltages across the capacitors C_1 and C_2. The variable i_l is the current through the inductor L and the function $\phi(v_{c_1})$ is the current through the nonlinear resistor in terms of the voltage across the capacitor C_1. This nonlinear function is described by an odd-symmetric piecewise-linear function made up of three straight-line segments which has the following explicit representation:

$$\phi(x) = -\left(\overline{m}_1 v_{c_1} v_{c_1} + \frac{\overline{m}_0-\overline{m}_1}{2}(|v_{c_1}+B_p| - |v_{c_1}-B_p|)\right), \tag{2}$$

where $\overline{m}_0, \overline{m}_1$ and B_p are three fixed constants of a diode-based realization. The last equations can be rewritten, in dimensionless, or normalized form, as follows:

$$\begin{aligned} \dot{x}_1 &= \alpha(-x_1+x_2-\phi(x_1))\\ \dot{x}_2 &= x_1-x_2+x_3\\ \dot{x}_3 &= -\beta x_2, \end{aligned} \tag{3}$$

with

$$\phi(x) = ax + b(|x+1|-|x-1|) \tag{4}$$

where

$$x_1 = \frac{v_{c_1}}{B_p},\quad x_2 = \frac{v_{c_2}}{B_p},\quad x_3 = \frac{i_L R}{B_p},\quad \alpha = \frac{C_2}{C_1},$$

$$\beta = \frac{C_2R^2}{L},\quad a = \overline{m}_1 R,\quad b = \frac{\overline{m}_0 R - \overline{m}_1 R}{2}.$$

For fixed values of the system parameters, chosen in a neighborhood of $\{\beta = 27, \alpha = 15.6, a = -5/7, b = -3/14\}$, this system is known to exhibit the, so-called, *double scroll chaotic attractor*.

3. Problem Formulation

It is desired to identify the set of unknown parameters $\{\beta, \alpha, a, b\}$, and the state x_3, under the assumption that the two capacitor voltage output signals: $y_1(t) = x_1(t)$ and $y_2(t) = x_2(t)$ are available for measurement, for all $t > 0$. That is, the two capacitor voltages in the circuit are being continuously monitored.

3.1. *Some algebraic properties*

We introduce two useful properties, which are satisfied by the **CCO** system (see [Diop & Fliess, 1991]). Consider a smooth nonlinear system, described by a state vector $X = \{x_i\}_1^{i=n} \in R^n$ and by the output vector $Y = \{y_i\}_1^{i=m} \in R^m$, of the form:

$$\dot{X} = f(X, P), \quad Y = h(X), \tag{5}$$

where $h(\cdot)$ is a smooth vector function and $P \in R^l$ is a constant parameter vector, with $l < n$. Let $Y^{(j)}$ denote the jth time derivative of the vector Y. We say that the vector state X is algebraically observable, if it can be uniquely expressed as

$$X = \Phi(Y, \ldots, Y^{(j)}, P), \tag{6}$$

for some integer j and for some smooth function Φ. Moreover, if the vector of parameters, P, satisfies the following linear relation:

$$\Omega_1(Y, \ldots, Y^{(j)}) = \Omega_2(Y, \ldots, Y^{(j)})P, \tag{7}$$

where $\Omega_1(\cdot)$ and $\Omega_2(\cdot)$ are, respectively, $n \times 1$ and $n \times n$ smooth matrices, then P is said to be algebraically linearly identifiable with respect to the output vector Y.

Evidently, system (3) is algebraically observable with respect to the output vector $Y = (y_1, y_2)^T$ with $y_1 = x_1$ and $y_2 = x_2$, since the system state variables can be rewritten as

$$x_1 = y_1, \quad x_2 = y_2, \quad x_3 = \dot{y}_2 + y_2 - y_1 \tag{8}$$

From the first of Eq. (3), we easily confirm that,

$$\dot{y}_1 = y_1 p_1 + y_2 p_2 + \phi_0(y_1) p_3 \tag{9}$$

where

$$p_1 = -\alpha(1 + a); \quad p_2 = \alpha; \quad p_3 = \alpha b \tag{10}$$

and

$$\phi_0(y_1) = |1 - y_1| - |y_1 + 1| \tag{11}$$

For the estimation of the parameter β, we observe that the third of Eq. (3) leads to

$$\ddot{y}_2 = -\dot{y}_2 - \beta y_2 + \dot{y}_1 \tag{12}$$

From the two differential relations, given in (9) and (12), we establish that the vector of unknown constant parameters, (p_1, p_2, p_3, β) is linearly identifiable from the knowledge of Y. It follows from the definitions of p_1, p_2 and p_3 above, that the vector of unknown constant parameters, $P = (\beta, \alpha, a, b)$ is *weakly linearly identifiable* [Fliess & Sira-Ramirez, 2003] with respect to the selected output vector $Y = (y_1, y_2)^T$.

Note that once the parameters p_1, p_2 and p_3 are estimated as p_{1e}, p_{2e}, p_{3e} and p_{4e} the original parameters $\{\alpha, a, b\}$ are estimated via

$$a_e = -\left(1 + \frac{p_{1e}}{p_{2e}}\right), \quad b_e = \frac{p_{3e}}{p_{2e}}, \quad \alpha_e = p_{2e} \tag{13}$$

4. Model Based Online Parameters Estimation

Having shown that the **CCO** is algebraically identifiable with respect to the output vector Y, we focus our attention on obtaining the unknown constant vector P. Intuitively, we attempt to build a set of a linear algebraic equations, free of initial conditions, for the components of P. To this end, the following assumptions are introduced:

A1. The set of parameters P belongs to some neighborhood in the parameter space, such that the **CCO** exhibits a chaotic behavior.

A2. The output vector $Y = (y_1, y_2)^T$ is available for measurement.

The nonasymptotic, online, algebraic identifier for the unknown parameters is obtained as follows: First multiply both sides of (9) by the time variable t. Secondly, integrate the resulting expression by parts where possible. Finally, generate by additional time integration as many equations as necessary to obtain a linear system for the unknown linearly identifiable parameters. These first two operations clearly eliminate the effect of the unknown initial conditions. We perform then the following algebraic manipulations:

$$\int_0^t \sigma \dot{y}_1(\sigma) d\sigma = p_1 \int_0^t \sigma y_1(\sigma) d\sigma + p_2 \int_0^t \sigma y_2(\sigma) d\sigma + p_3 \int_0^t \sigma \phi_0(y_1(\sigma)) d\sigma. \tag{14}$$

By integrating the above relation, one and two times with respect to t, we obtain three linearly independent expressions leading to the following linear system in the unknown parameters:

$$\Theta(t) \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{bmatrix} \tag{15}$$

where

$$q_1(t) = \int t\dot{y}_1(t) = ty_1 - \int y_1(t),$$

$$q_2(t) = \int^{(2)} t\dot{y}_1(t) = \int ty_1 - \int^{(2)} y_1(t) \quad (16)$$

$$q_3(t) = \int^{(3)} t\dot{y}_1(t) = \int^{(2)} ty_1 - \int^{(3)} y_1(t)$$

and

$$\theta_{11}(t) = \int ty_1(t) \quad \theta_{12}(t) = \int ty_2(t)$$

$$\theta_{13}(t) = \int t\phi_0(y_1(t))$$

$$\theta_{21}(t) = \int^{(2)} ty_1(t) \quad \theta_{22}(t) = \int^{(2)} ty_2(t)$$

$$\theta_{23}(t) = \int^{(2)} t\phi_0(y_1(t)) \quad (17)$$

$$\theta_{31}(t) = \int^{(3)} ty_1(t) \quad \theta_{23}(t) = \int^{(3)} ty_2(t)$$

$$\theta_{23}(t) = \int^{(3)} t\phi_0(y_1(t))$$

Here, we have used, for simplicity, the notation:

$$\int^{(m)} t^j x(t) = \int_0^t \int_0^{\sigma_1} \cdots \int_0^{\sigma_{m-1}} (\sigma_m)^j \times x(\sigma_m) d\sigma_m d\sigma_{m-1} \cdots d\sigma_1.$$

We find that the matrix $\Theta(t)$ is zero at time $t = 0$ and that the vector $q(t)$ on the right-hand side is also zero at $t = 0$. The system of equations is ill defined at this instant of time. Note, nevertheless, that, for any time t after a small open time interval of the form $[0, \epsilon)$ with $\epsilon > 0$, the matrix $\Theta(t)$ becomes invertible and $q(t)$ is nonzero. In fact, we can easily prove that $\Theta(t)$ cannot be singular on any open interval of time.

To prove the last statement, suppose that $\Theta(t)$ is singular on an open interval of the form (t_0, t_1) with $t_1 > t_0$. Then $\det \Theta(t) = 0$ for all $t \in (t_0, t_1)$. This means that the components of the vector Y satisfy a differential equation which is completely independent of the system parameters. This contradicts the essential nature of the chaotic system where the state components are independent of each other and they all depend on the numerical value of the system parameters in a crucial way.

In a similar form, multiplying both sides of (3) by t^2 and integrating twice, with respect to t, it follows that

$$\int^{(2)} t^2\ddot{y}_2 = -\int^{(2)} t^2\dot{y}_2 - \beta \int^{(2)} t^2 y_2 + \int^{(2)} t^2\dot{y}_1, \quad (18)$$

Now, integrating by parts, we have

$$\int^{(2)} t^2\ddot{y}_2(t) = t^2 y_2 - 4\int ty_2(t) + 2\int^{(2)} y_2(t)$$

$$\int^{(2)} t^2\dot{y}_2(t) = \int t^2 y_2(t) - 2\int^{(2)} ty_2(t) \quad (19)$$

$$\int^{(2)} t^2\dot{y}_1(t) = \int t^2 y_1(t) - 2\int^{(2)} ty_1(t)$$

Therefore, the parameter $p_4 = \beta$ may be directly obtained from (18), as

$$p_{4e} = \beta_e = \frac{S(t)}{\int^{(2)} t^2 y_2(t)} \quad (20)$$

where

$$S(t) = -t^2 y_2 + 4\int ty_2(t) - 2\int^{(2)} y_2(t) + 2\int^{(2)} ty_2(t) - \int t^2 y_2(t) + \int t^2 y_1(t) - 2\int^{(2)} ty_1(t) \quad (21)$$

Similarly to the previous computation, the numerator and denominator of (20) are zero at $t = 0$. However, this quotient is well defined outside an interval of the form $[0, \epsilon)$, for $\epsilon > 0$. Note that if the quotient is identically zero on any open time interval $[t_i, t_j]$, $t_j > t_i$ then, we must have that $y_2(t) = 0$ for all $t \in [t_i, t_j]$. This contradicts the assumption A1, since y_2 cannot be identically equal to zero in an open time interval. Clearly, the denominator trajectory may well pass through zero at isolated instants of time.

5. Invariant Filtering

The computation of an unknown parameter, denote it by p_j, in the above formulae, has the general form given by a quotient of the following kind

$$p_j = \frac{n_j(t)}{d_j(t)}, \quad t > \epsilon > 0 \quad (22)$$

where the numerator $n_j(t)$ and the denominator $d_j(t)$ are time functions depending on the available measured output signals constituting the output vector Y. Since such measurements are usually subject to noise processes whose statistics may be rather difficult to establish without extensive experimentation, we propose, as already advocated and justified in [Fliess & Sira-Ramirez, 2003] (see also the thought provoking articles by [Fliess, 2006, 2005]), the use of a filtering process. As customarily done in the control systems literature, we indicate this filtering process by combining frequency domain expressions with time domain expressions. Thus, let s be the standard complex variable and $G(s)$ be the rational transfer function, in the complex domain, of a low pass filter, preferably constituted by an iterated integration of the form $1/s^{\beta}$, for some integer $\beta > 0$. Then the operation:

$$p_j = \frac{G(s)n_j(t)}{G(s)d_j(t)}, \quad t > \epsilon > 0 \tag{23}$$

suitably enhances the signal to noise ratios that can be present independently in the numerator and the denominator. The justification of the name invariant filtering stems from the fact that in the noise-free case the relation $d_j(t)p_j = n_j(t)$ is "equivalent" to the relation $[G(s)d_j(t)]p_j = [G(s)n_j(t)]$.

6. Algebraic State Estimation

Finally, computation of the parameter β, allows for the state x_3 to be easily estimated. To accomplish this, multiply both sides of (3) by t and integrate once with respect to t. We obtain

$$\int t\dot{x}_3(t) = tx_3(t) - \int x_3(t) = -\beta \int ty_2(t).$$

Recalling that x_3 can be obtained from the third of Eq. (8), then the above relation leads to

$$\hat{x}_3(t) = \frac{1}{t}\left(-\hat{\beta}\int ty_2(t) + \int y_2 - \int y_1 + y_2 - y_2(0)\right)$$

where $\hat{\beta}$ is obtained directly from (20), which is well defined for all $t > 0$.

Remark 1. It should be noticed that the definite integration starting from $t = 0$, can be changed to any new initial value $t_0 > 0$. That is, the integration process may be restarted, at any instant of time, as desired, or needed.

7. Numerical Simulations

Computer simulations have been carried out in order to test the effectiveness of the proposed algebraic identification procedure for recovering the unknown parameters of the **CCO**. The program uses the Runge–Kutta integration algorithm, with an integration step equal to 0.001.

In the first simulation, the values of the parameters of the system were fixed at $\beta = 27, \alpha = 16$,

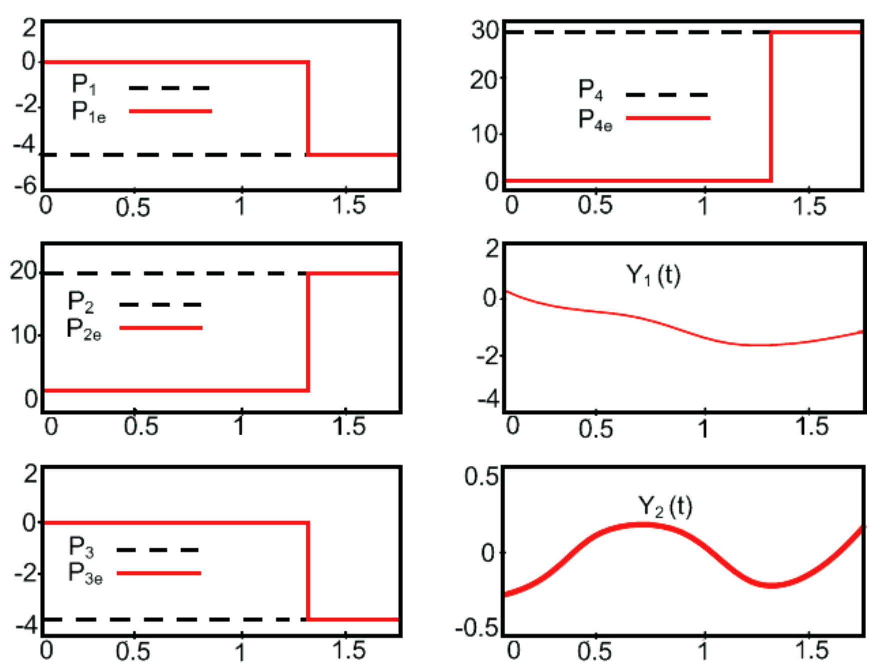

Fig. 2. Estimation of the parameters p_1, p_2, p_3 and p_4, respectively.

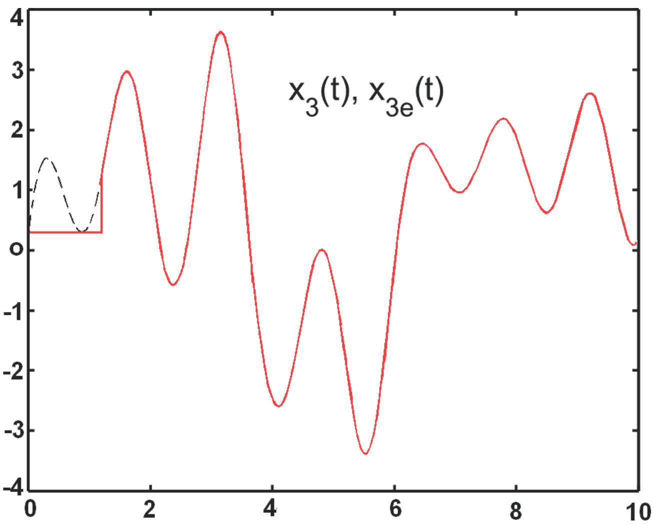

Fig. 3. Estimate of the unmeasured state x_3.

$a = -5/7$ and $b = -3/14$. The estimation process was started after $t \geq 0.5$ seconds. The initial conditions of the system were set to be: $x_1(0) = 0.25$, $x_2(0) = -0.2$ and $x_3(0) = 0.3$. For simplicity we use the symbols p_{ie} to denote the estimated values of p_i. Figure 2 displays the actual and the estimated values of the parameters $\{p_1, \ldots, p_4\}$ in terms of the time evolution of the computed values as well as the noisy measured outputs $y_1(t), y_2(t)$. As we can see, the obtained numerical estimations of the parameters are recovered, almost perfectly, after $t > 1.2$ [seconds]. We stress that in these computations the measured signals y_1 and y_2 were subject to additive computer generated stochastic noise processes constituted by sequences of random variables characterized by piecewise constant random values exhibiting rectangular probability density functions in the interval $[0, 1]$. The invariant filtering was achieved with the use of a second order integration of both numerator and denominator expressions for each computed parameter. Figure 3 depicts the actual and the estimated values of the evolution of the unmeasured state $x_3(t)$.

To show the robustness and flexibility of the proposed parameter estimation method, we have subjected their values to piecewise constant variations. We only changed rather slightly the parameter value for β while an abrupt variation was

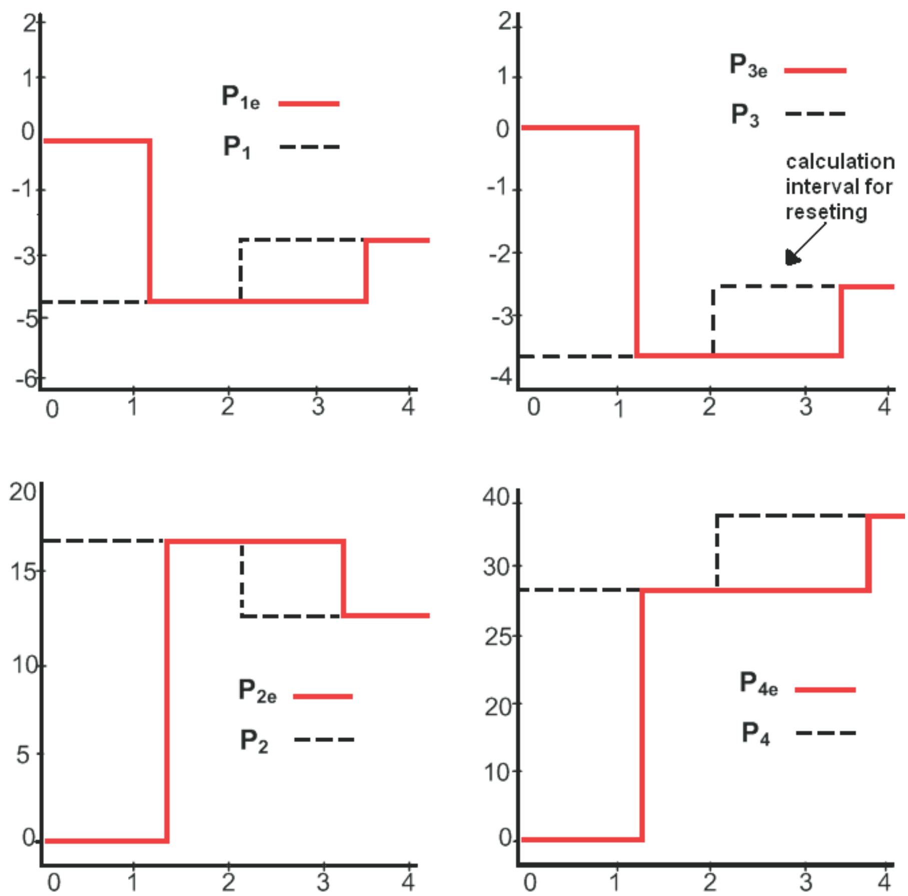

Fig. 4. Estimates of all four parameters when abrupt parametric variations in β and α are present in the system.

allowed for the parameter α, as specified below:

$$\{\beta = 27, \alpha = 16\}$$
$$\text{if } t \leq 2 \text{ else } \{\beta = 35, \alpha = 12.8\}. \tag{24}$$

The parameters a and b were taken as initially specified. The initial conditions of the **CCO** were set as before. Notice that the algorithm detects the abrupt variations in the parameters β and α, since the computation formula becomes invalid when such abrupt variations occur. When the parameter change is sensed this is an indication for an immediate reinitialization of the estimation process. That is, the time integration process has to immediately reset or restart. Figure 4 shows the corresponding estimated parameters. The simulation results show that even when abrupt variations are introduced at $t = 2$ [s], in the parameters β and α, respectively, the unknown parameters can still be estimated rather accurately after the instant $t = 3.2$ [s].

8. Conclusions

In this article, we have proposed an algebraic approach for the nonasymptotic, but fast and accurate, identification of the constant unknown parameters defining a Chua's chaotic oscillator (CCO). Under the assumption of two independent measurable output voltage signals, the proposed algorithm establishes an exact formula for the exact computation of the unknown parameters. The formula entitles iterated integral convolutions of the measured signals and the straightforward avoidance of an initial computational singularity. We show that Chua's system parameters are linearly identifiable and a linear system is generated for the online calculation of the unknown parameters. We stress that suitable algebraic operations on the output differential equations make the proposed algorithm independent of the initial conditions of the underlying nonlinear dynamical system. Through a technique, here called invariant filtering, we also obtain parameter value estimates which are robust with respect to output measurement noises represented by computer generated stochastic processes. Statistics of such noises become unnecessary to obtain rather accurate estimation values. Resorting to the reinitializations, or resettings, of the integrations in the algorithm, allows for the efficient computation of piecewise constant varying parameters. Convincing computer simulations are presented and discussed.

Acknowledgments

This research was supported by the Coordinación de Posgrado e Investigación (CGPI-IPN) under research grants 20061128 and 20071088.

References

Abarbanel, H. D. I. [1996] *Analysis of Observed Chaotic Data* (Springer-Verlag).

Alligood, K. T., Sauer, T. D. & Yorke, J. A. [1997] *Chaos: An Introduction to Dynamical Systems* (Springer-Verlag, NY).

Baker, G. L., Gollub, J. P. & Blackburn, J. A. [1996] "Inverting chaos: Extracting system parameters from experimental data," *Chaos* **6**, 528–533.

Crutchfield, J. P. & McNamara, B. S. [1987] "Equations of motions a data series," *Compt. Syst.* **1**, 417–452.

Dabroom, A. M. & Khalil, H. K. [1999] "Discrete-time implementation of high-gain observers for numerical differentiation," *Int. J. Contr.* **72**, 1523–1537.

Diop, S., Grizzle, J. W., Moral, P. E. & Stefanopoulou, A. [1987] "Interpolation and numeric differentiation for observer design," *ACC* **2**, 1329–1333.

Diop, S. & Fliess, M. [1991] "Nonlinear observability, identifiablity and persistent trajectories," *36th IEEE CDC*, Brighton, pp. 714–719.

Feldmann, U., Hasler, M. & Schwarz, W. [1996] "Communication by chaotic signals: The inverse system approach," *Int. J. Circuit Th. Appl.* **24**, 551–579.

Fliess, M., Boup, M., Mounier, H. & Sira-Ramírez, H. [2003] "Questioning some paradigms of signal processing via concrete examples," in *Algebraic Methods in Flatness, Signal Processing and State Estimation*, eds. Sira-Ramírez, H. & Silva-Navarro, G. (Editorial Lagares, México), pp. 1–16.

Fliess, M. & Sira-Ramirez, H. [2003] "An algebraic framework for linea identification," *ESAIM* **9**, 151–168.

Fliess, M. [2005] "Réflexions sur la question fréquentielle en traitement du signal," Inria 00000461, Version 1–20.

Fliess, M. [2006] "Analyse non standard du bruit," *C.R. Acad. Sci. Paris, Ser. I* **342**, 797–802.

King, G. P. & Stewart, I. [1992] "Phase space reconstruction for symmetric dynamical systems," *Physica D* **58**, 216–228.

Lainscsek, C., Letellier, C. & Gorodnitsky, I. [2003] "Global modeling of the Rössler system from the z-variable," *Phys. Lett. A* **314**, 409–427.

Landau, I. D. [2000] *System Identification and Control Design* (Prentice Hall).

Mosekilde, E. [2003] *Topics in Nonlinear Dynamics: Applications to Physics, Biology and Economic Systems* (World Scientific, Singapore).

Nayfeh, A. H. & Mook, D. T. [1979] *Non-Linear Oscillations* (Wiley, Chichester).

Nijmeijer, H. & Mareels, I. [1997] "An observer look to synchronization," *IEEE Trans. Circuits Syst.* **36**, 882–890.

Packard, N. H., Crutchfield, J. P., Farmer, J. D. & Shaw, R. S. [1980] "Geometry from a time series," *Phys. Rev. Lett.* **45**, 712–716.

Parlitz, U., Zöller, R., Holzfuss, J. & Lauterborn, W. [1994] "Reconstructing physical variables and parameters from dynamical systems," *Int. J. Bifurcation and Chaos* **4**, 1715–1719.

Sauer, T., Yorke, J. & Casdagli, M. [1991] "Embedology," *J. Stat. Phys.* **65**, 579–616.

Soderstrom, P. & Stoica, P. [1989] *System Identification* (Prentice Hall).

Takens, F. [1981] *Dynamical Systems and Turbulence*, eds. Rand, D. A. & Young, L.-S. (Springer-Verlag, Berlin), pp. 366–381.