The Ramsey numbers for stripes and complete graphs 1

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Abstract


The Ramsey numbers \( r(mK_p, n_1P_2, \ldots, n_dP_2) \), \( p > 2 \), are calculated for \( d < p \) and \( n_j \geq m \) for each \( j \).

1. Introduction

This paper continues the programme of calculating the Ramsey numbers \( r(m_1K_{p_1}, \ldots, m_cK_{p_c}) \) outlined at the end of [3] by proving the following.

Theorem. If \( p > 2 \), \( d < p \) and \( n_j \geq m \) for \( j = 1, \ldots, d \) then

\[
r(mK_p, n_1P_2, \ldots, n_dP_2) = mp + \sum_{j=1}^{d} (2n_j - 1 - m).
\]

Here, if \( G_1, \ldots, G_c \) are graphs without loops or multiple edges, the Ramsey number \( r(G_1, \ldots, G_c) \) is the smallest integer such that if the edges of a complete graph \( K_n \), with \( n \geq r(G_1, \ldots, G_c) \), are painted arbitrarily with \( c \) colours the \( i \)th coloured subgraph contains \( G_i \) as a subgraph for at least one \( i \). Also, \( mK_p \) stands for \( m \) disjoint copies of the complete graph on \( p \) vertices and \( n_jP_2 \) stands for \( n_j \) disjoint copies of the path (or complete graph) with two vertices.

In previous papers [1-3] \( r(n_1P_2, \ldots, n_dP_2) \) and \( r(K_p, n_1P_2, \ldots, n_dP_2) \), \( p > 2 \), have been calculated and bounds have been established for \( r(m_1K_{p_1}, \ldots, m_cK_{p_c}) \) with \( p_j > 2 \) for each \( j = 1, \ldots, c \).
In Section 2 the number on the right in the theorem is established as a lower bound by constructing a counterexample of order one less. In Section 3 it is established as an upper bound.

Throughout we will be dealing with $1 + d$ colours. The one relevant to the $mK_p$ will be called red and the others will be called colour 1 to colour $d$.

2. The lower bound

Suppose $d < p$ and $n_j \geq m$ for $j = 1, \ldots, d$. For each $j = 1, \ldots, d$ let $A_j$ be a complete graph of order $2n_j - 1$ with all edges of colour $j$ and let $A_{d+1}$ be a complete graph of order $m(p - d) - 1$ with all edges coloured red. Form the disjoint union of $A_1, \ldots, A_{d+1}$ and colour all edges between two of these graphs red. Then the new graph $\Sigma$ has no red $mK_p$ and no $n_jP_2$ of colour $j$ for any $j$. As $\Sigma$ has order

$$m(p - d) - 1 + \sum_{j=1}^{d} (2n_j - 1),$$

the number

$$m(p - d) + \sum_{j=1}^{d} (2n_j - 1),$$

which is equal to

$$mp + \sum_{j=1}^{d} (2n_j - 1 - m)$$

is a lower bound for $r(mK_p, n_1P_2, \ldots, n_dP_2)$.

3. The upper bound

The more substantial part of the theorem will now be proved: if $d < p$ and $n_j \geq m$ for each $j = 1, \ldots$, then

$$r(mK_p, n_1P_2, \ldots, n_dP_2) \leq mp + \sum_{j=1}^{d} (2n_j - 1 - m).$$

The proof is by induction on $m$ and $\sum_{j=1}^{d} n_j$. The case $m = 1$ was proved in [3] so we take $m > 1$.

Suppose that $\Sigma$ is a counterexample to the inequality for the minimum value of $m$ possible and that, among the counterexamples for this value of $m$, $\Sigma$ is a counterexample with the minimum value of $\sum_{j=1}^{d} n_j$. Then $\Sigma$ has order

$$mp + \sum_{j=1}^{d} (2n_j - 1 - m),$$

has no red $mK_p$ and has no $n_jP_2$ of colour $j$, $1 \leq j \leq d$. 
Suppose that $n_i > m$ for some $i$. Put $m_i = n_i - 1$ and $m_j = n_j$ for $j \neq i$. Then the order of $\Sigma$ is

$$2 + mp + \sum_{j=1}^{d} (2m_j - 1 - m).$$

As $\sum_{j=1}^{d} m_j < \sum_{j=1}^{d} n_j$ it follows from the minimal property of $\Sigma$ that it has either a red $mK_p$ or an $m_iP_2$ of colour $j$ for some $j$: the only possibility is an $m_iP_2$ of colour $i$. As $m_i > m > 2$, $\Sigma$ has at least one edge of colour $i$. Remove an edge $e$ of this colour from $\Sigma$ to get a new graph $\Sigma_1$ of order

$$mp + \sum_{j=1}^{d} (2m_j - 1 - m).$$

The argument just used now shows that $\Sigma_1$ has an $m_iP_2$ of colour $i$. With $e$ this forms an $n_iP_2$ of colour $i$ in $\Sigma$, contrary to assumption. Thus $n_i = m$ for each value of $i$. As the order of $\Sigma$ is greater than $(m - 1)p + (m - 1)d$ and $m > 1$, the minimal property of $\Sigma$ implies that it has at least one red $K_p$ as a subgraph.

In summary, for a counterexample $\Sigma$ with minimum value of $m$ and then of $\sum_{j=1}^{d} n_j$, we have shown that:

(a) $n_i = m$ for each value of $i$,

(b) $\Sigma$ has order $mp + (m - 1)d$ but has no red $mK_p$ nor any $mP_2$ of another colour,

(c) $\Sigma$ has at least one red $K_p$ as a subgraph.

The proof continues by analyzing subgraphs of $\Sigma$ which have order $p + d$ and contain a red $K_p$. A subgraph like this which has edges of the maximum number of colours possible among the subgraphs of $\Sigma$ will be called a $C$-subgraph. A description of some of the properties of $C$-subgraphs is now given.

**Lemma 1.** Let $\Lambda$ be a $C$-subgraph of $\Sigma$. Then

(a) the edges of $\Lambda$ are coloured red and exactly $d - 1$ other colours, say $2, \ldots, d$;

(b) if $A$ is a red $K_p$ of $\Lambda$ then

(i) just one edge of $\Lambda - A$ is not red, say colour 2, and all edges of $\Lambda$ having colour 2 have a vertex in common with this edge;

(ii) for $i = 3, \ldots, d$, all the edges of $\Lambda$ of colour $i$ are incident with a single vertex of $\Lambda - A$;

(c) $\Sigma - \Lambda$ has a red $(m - 1)P_2$ of colour 1.

**Proof.** Let $\Lambda$ be a $C$-subgraph of $\Sigma$ and suppose that the edges of $\Lambda$ are coloured by red and $d$ other colours.

Removing $\Lambda$ from $\Sigma$ gives a graph $\Sigma - \Lambda$ of order $(m - 1)p + (m - 2)d$. By the minimal property of $m$, $\Sigma - \Lambda$ has either a red $(m - 1)K_p$ or an $(m - 1)P_2$ of some other colour. In the former case, adding the red $K_p$ in $\Lambda$ gives a red $mK_p$ in $\Sigma$, which is not possible. Hence the latter case is true: by renumbering the colours if
necessary, it may be assumed that \( \Sigma - A \) has an \((m - 1)P_2\) of colour 1. Then \( A \) can have no edge of colour 1, so that \( d_1 < d_2 \).

Suppose, first, that \( d = 1 \). Then \( d_1 = 0 \) so that every subgraph of order \( p + 1 \) which contains a red \( K_p \) has only red edges. As \( \Sigma \) has order \( mp + (m - 1) \) and has no red \( mK_p \) it has at least one edge, say \( uv \), of colour 1. Let \( A \) be a red \( K_p \) in \( \Sigma \). As \( d_1 = 0 \), neither \( u \) nor \( v \) are vertices of \( A \) and, by the above, both \( A \cup \{u\} \) and \( A \cup \{v\} \) are red \( K_{p+1} \). Let \( a \) be a vertex of \( A \) and let \( \Lambda = (A \setminus \{a\}) \cup \{u, v\} \). Then \( \Lambda \) has order \( p + 1 \), has \( (A \setminus \{a\}) \cup \{u\} \) as a red \( K_p \) and has an edge \( uv \) of colour 1. This contradicts the fact that \( d_1 = 0 \). Hence \( d > 1 \).

The argument in the next paragraph will be used again in this section.

Let \( u_1, v_1 \) be vertices of \( \Sigma - A \) for which the edge \( u_1v_1 \) is of colour 1 and let \( A \) be a red \( K_p \) in \( \Sigma \). Let \( u, v \) be any two vertices of \( \Lambda - A \). Let \( \Lambda_1 \) be the subgraph formed from \( \Lambda \) by replacing the vertices \( u, v \) by \( u_1, v_1 \):

\[
\Lambda_1 = (\Lambda \setminus \{u, v\}) \cup \{u_1, v_1\}.
\]

This subgraph has order \( p + d \), has \( A \) as a subgraph which is a red \( K_p \) and has an edge \( u_1v_1 \) of a colour that \( \Lambda \) does not have. As \( \Lambda \) is a \( C \)-subgraph, it must have edges of a colour that \( \Lambda_1 \) does not have, and they must all have \( u \) or \( v \) as a vertex. Hence, for all choices of \( u, v \) in \( \Lambda - A \), there is a colour which colours some edge of \( \Lambda \) and all edges of this colour in \( \Lambda \) are incident with either \( u \) or \( v \).

As \( \Lambda - A \) has \( d \) vertices and there are fewer than \( d \) colours available, it follows from the proposition in the appendix that one colour, say 2, has edges incident with only two vertices of \( \Lambda - A \) and that if \( 3 \leq j \leq d \) then colour \( j \) is used in \( \Lambda \) and colours edges incident with just one vertex of \( \Lambda - A \).

An easy consequence of the preceding paragraph is that every edge of \( \Lambda - A \) except one is red.

This proves Lemma 1. \( \square \)

Lemma 1 describes properties of all \( C \)-subgraphs of \( \Lambda \). The next lemma concentrates on a property that at least one of them must have.

**Lemma 2.** \( \Sigma \) has a \( C \)-subgraph having one edge of \( d - 1 \) of the colours 1, \ldots, \( d \) and every other edge coloured red.

**Proof.** Begin with any \( C \)-subgraph \( \Lambda \) of \( \Sigma \) and suppose that its edges are red and colours 2, \ldots, \( d \). Following Lemma 1, suppose that \( \Lambda \) is a red \( K_p \) of \( \Lambda \), \( a_2b_2 \) is an edge of \( \Lambda - A \) of colour 2, and that, for each \( i = 3, \ldots, d \), the vertex \( a_i \) of \( \Lambda - A \) is incident with each edge of \( \Lambda \) having colour \( i \).

The proof of the Lemma is in two parts, depending on whether there is an edge of colour 1 joining \( A \) to \( \Sigma - A \).

(1) Suppose that \( e \) is an edge of colour 1 joining \( A \) to \( \Sigma - A \). By Lemma 1, \( \Sigma - A \) has an \((m - 1)P_2\) of colour 1. As \( \Sigma \) has no \( mP_2 \) of colour 1, \( e \) must have a vertex, \( u_1 \) say, incident with an edge \( u_1v_1 \) of colour 1 in \( \Sigma - A \). Consider the
subgraph
\[ \Lambda_2 = (\Lambda - \{a_2, b_2\}) \cup \{u_1, v_1\}. \]

It has a subgraph as a red \( K_p \) and has edges of colours 1, 3, \ldots, \( d \). As it has order \( p + d \) it is a \( C \)-subgraph and, by Lemma 2, every edge of \( \Lambda_2 - A \) except \( u_1v_1 \) is red. Thus, the other end of \( e \) cannot be a vertex of \( \Lambda_2 - A \) and it must be \( a_2, b_2 \) or a vertex of \( A \). It will next be shown, in each of these cases, that \( A \) has exactly one edge of colour 2, namely \( a_2b_2 \).

Suppose \( e = u_i c \) where \( c \) is a vertex of \( A \). Let \( \Lambda_3 = (\Lambda - \{a_2\}) \cup \{u_i\} \). Then \( \Lambda_3 \) has order \( p + d \), has \( A \) as a red \( K_p \) and has edges coloured 1, 3, \ldots, \( d \). Thus, by Lemma 1, it has no edges of colour 2 and, in particular, no edge joining \( b_2 \) to \( \Lambda - \{a_2\} \) has colour 2. The same argument with \( b_2 \) in place of \( u_i \) shows that there is no edge of colour 2 joining \( a_2 \) to \( \Lambda - \{b_2\} \). Thus \( a_2b_2 \) is the only edge of colour 2 in \( \Lambda \).

Suppose \( e = u_1a_2 \) and let \( \Lambda_4 = (\Lambda - \{b_2\}) \cup \{u_1\} \). Then \( \Lambda_4 \) has order \( p + d \), has \( A \) as a red \( K_p \) and has edges coloured 1, 3, \ldots, \( d \). Thus it has no edges of colour 2, which implies that \( a_2 \) is not joined to \( A \) by an edge of colour 2. Suppose that \( b_2 \) is joined to a vertex \( c \) of \( A \) by an edge of colour 2. Put \( B = (\Lambda - \{c\}) \cup \{a_2\} \). Then \( B \) is a red \( K_p \) of \( \Lambda \) and \( \Lambda - B \) has two vertices \( b_2 \) and \( c \) incident with edges of colour 2 in \( \Lambda \). Moreover, the edge \( u_1a_2 \) joins \( \Sigma - \Lambda \) and \( \Lambda \) by an edge of colour 1 having the vertex \( a_2 \) in \( B \). Hence, by the result in the preceding paragraph, \( \Lambda \) has just one edge of colour 2, namely \( b_2c \). This contradicts the fact that some edge of colour 2 in \( \Lambda \) has \( a_2 \) as a vertex. Thus \( b_2 \) is joined to no vertex of \( A \) by an edge of colour 2 in \( \Lambda \) and \( a_2b_2 \) is the only edge of colour 2 in \( \Lambda \).

Hence \( a_2 \) and \( b_2 \) are joined to \( A \) only by red edges and both \( A \cup \{a_2\} \) and \( A \cup \{b_2\} \) are red \( K_{p+1} \).

Consider one of the other colours, 3, \ldots, \( d \), say \( i \). There is an edge \( a_ic \) of colour \( i \) with \( c \) in \( A \). Put \( B = (\Lambda - \{c\}) \cup \{a_2\} \). Then \( B \) is a red \( K_p \) and \( \Lambda - B \) has two vertices \( a_i \) and \( c \), incident with edges of colour \( i \). Hence all the above is true with \( B \) in place of \( \Lambda \) and \( i \) in place of 2. In particular, \( \Lambda \) has just one edge of colour \( i \).

It has now been established that if there is an edge of colour 1 joining \( \Lambda \) to \( \Sigma - \Lambda \) then \( \Lambda \) has exactly one edge of each colour 2, \ldots, \( d \).

(2) Suppose there is no edge of colour 1 joining \( \Lambda \) to \( \Sigma - \Lambda \) and that \( \Lambda \) has more than one edge of one of the colours 2, \ldots, \( d \).

Suppose that \( \Lambda \) has more than one edge of colour 2 and consider
\[ \Lambda_5 = (\Lambda - \{a_i, b_i\}) \cup \{u_1, v_1\}. \]

This is a \( C \)-subgraph having \( A \) as a red \( K_p \), edges of colours 1, 3, \ldots, \( d \) and there is an edge of colour 2 joining \( \Lambda_5 \) to \( \Sigma - \Lambda_5 \). Thus, by (1) just proved, with \( \Lambda_5 \) in place of \( \Lambda \) and colour 2 in place of colour 1, \( \Lambda_5 \) has just one edge of each colour 1, 3, \ldots, \( d \). Thus \( \Lambda_5 \) satisfies the requirements of Lemma 2.
Alternatively, \( \Lambda \) has just one edge of colour 2, namely \( a,b_2 \), and has more than one edge of some colour \( i > 2 \). Suppose that \( a,c \) has colour \( i \) where \( c \) is in \( A \). Put \( B = (A - \{ c \}) \cup \{ a_2 \} \). Then \( B \) is a red \( K_p \) in \( \Lambda \) and \( \Lambda - B \) has an edge \( a,c \) of colour \( i \). The argument in the previous paragraph, with \( B \) in place of \( A \) and colour \( i \) in place of colour 2, now shows the existence of a \( C \)-subgraph \( \Lambda_0 \) satisfying the requirements of Lemma 2.

This proves Lemma 2. \( \square \)

The final contradiction will now be established. By Lemma 2, \( \Sigma \) has a \( C \)-subgraph which contains exactly one edge of \( d-1 \) of the colours 1, \ldots, \( d \).

After renaming, if necessary, call the subgraph \( \Lambda \) and the colours 2, \ldots, \( d \).

Moreover, \( \Lambda \) has a red subgraph \( A \) and the vertices of \( \Lambda - A \) can be labelled \( b_2, a_2, \ldots, a_d \) in such a way that \( a_2b_2 \) is the edge of colour 2, and for \( i = 3, \ldots, d \) the edge of colour \( i \) joins \( a_i \) to a vertex \( b_i \) of \( A \). (It has not been shown that \( b_3, \ldots, b_d \) are all different.) Let \( \Gamma \) be the subgraph of \( \Lambda \) having \( a_2, \ldots, a_d, b_2, \ldots, b_d \) as vertices. Then \( \Lambda - \Gamma \) is a complete red graph of order at least \( p + d - 2(d - 1) = p - d + 2 \), the vertices of which are joined to other vertices of \( \Lambda \) only by red edges. The vertices \( a_2, \ldots, a_d \) form a red \( K_{d-1} \). Join them to \( p - d + 2 \) of the vertices of \( \Lambda - \Gamma \) to form a red \( K_{p+1} \), call it \( D \).

Let \( E \) be a subgraph of order \( p \) of \( D \) having \( a_2, \ldots, a_d \) among its vertices. Then \( \Lambda \) is a \( C \)-subgraph of \( \Sigma \) having \( E \) as a red \( K_p \). Hence by Lemma 1, each vertex of \( \Lambda - E \) is incident with an edge of a colour not red. As this is not true of the vertex in \( D - E \), a contradiction is reached which establishes the upper bound for \( r(mK_p, n_1P_2, \ldots, n_dP_2) \).

**Appendix**

**Proposition.** Suppose a graph \( G \) of order \( d \geq 2 \) has some or all of its vertices labelled with one or more of the numbers 2, \ldots, \( d \) in such a way that removing any two vertices leaves a graph labelled with fewer numbers. Then every vertex is labelled with just one number, one number labels two vertices and each other number labels just one.

**Proof.** As the result is true for \( d = 2 \), suppose that \( d > 2 \) and the result is true for graphs of smaller order.

If a number labels more than two vertices it cannot be removed from \( G \) by removing two vertices. Thus it can be ignored in considering such reductions.

If a number labels just two vertices it can only be removed from \( G \) by removing just those two vertices. As there are \( \frac{d}{2}(d - 1) \) pairs of vertices, \( d - 1 \) numbers and \( d > 2 \), not all numbers can label two vertices. Hence at least one number, say \( d \), labels just one vertex.
Remove the vertex labelled by $d$ from $G$ to get a graph $G_1$ of order $d - 1$ labelled by $2, \ldots, d - 1$ which has the same property as $G$. Hence, by assumption, one of these numbers labels two vertices and each other labels one. The result will be proved if no number except $d$ labels the vertex of $G - G_1$. Suppose a number $i < d$ also labels this vertex. If $i = 2$ removing the two vertices of $G$ labelled by $i$ does not reduce the number of numbers used as labels and if $i > 2$ removing the vertex labelled $i$ and one of the vertices labelled 2 gives the same result. That proves the statement. \Box

References